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# Almost classical solutions of Hamilton-Jacobi equations

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### Abstract

We study the existence of everywhere differentiable functions which are almost everywhere solutions of quite general Hamilton-Jacobi equations on open subsets of $\mathbb{R}^d$ or on $d$-dimensional manifolds whenever $d \geq 2$. In particular, when $M$ is a Riemannian manifold, we prove the existence of a differentiable function $u$ on $M$ which satisfies the Eikonal equation $\|\nabla u(x)\|_x = 1$ almost everywhere on $M$.

## 1. Introduction

It has been proved by Z. Buczolich [4] that if $d \geq 2$, there exists $u : \mathbb{R}^d \to \mathbb{R}$, differentiable at every point, such that $\nabla u(0) = 0$ and $\|\nabla u(x)\| \geq 1$ almost everywhere, thus giving a negative answer to the gradient problem of C. E. Weil [10]. Malý and Zelený [8] gave an elegant proof of this result using a new mathematical game. Then Deville and Matheron [6], refining the methods introduced by the above authors, proved that if $\Omega$ is a bounded open subset of $\mathbb{R}^d$ with $d \geq 2$, there exists a function $u : \overline{\Omega} \to \mathbb{R}$, continuous on $\overline{\Omega}$, differentiable at every point of $\Omega$, such that $u(x) = 0$ for all $x \in \partial\Omega$, and such that $\|\nabla u(x)\| = 1$ almost everywhere on $\Omega$. Notice that because of Rolle's theorem, there exists $x_0 \in \Omega$ such that $\nabla u(x_0) = 0$, so the function $u$ cannot be $\mathcal{C}^1$-smooth. We shall call $u$ an almost-classical solution of the Eikonal equation $\|\nabla u\| = 1$. This equation has also a unique viscosity solution, which is the function $x \mapsto dist(x, \partial\Omega)$, where $dist(x, \partial\Omega) = \inf\{\|x - y\|; y \in \partial\Omega\}$. The viscosity solution is not everywhere differentiable on $\Omega$. Therefore, an almost classical solution of the Eikonal equation is not equal to the viscosity solution of the Eikonal equation. Nevertheless in optimal control, where

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this equation arises naturally, the viscosity solution is the “right” solution of the Eikonal equation. We refer to [2] and [5] for an account on viscosity solutions of Hamilton-Jacobi equations.

The contents of the paper are as follows. In Section 2, we recall some technical results from [6] which will be needed in this paper. In Sections 3 and 4, we study the existence of almost-classical solutions for more general Hamilton-Jacobi equations on open subsets of $\mathbb{R}^d$. Finally, Section 5 is devoted to Hamilton-Jacobi equations on manifolds, and in particular we will consider the Eikonal equation $\|\nabla u(x)\|_x = 1$ on a Riemannian manifold. See e.g. [1], [7] and [9] for further information about Hamilton-Jacobi equations on Riemannian manifolds.

Now we introduce some terminology. Let $\Omega$ be an open subset of $\mathbb{R}^d$, and let $F : \mathbb{R}\times\Omega\times\mathbb{R}^d \to \mathbb{R}$ and $u_0 : \partial\Omega \to \mathbb{R}$ be continuous. As usual, we say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a *classical solution* of $F\big(u(x), x, \nabla u(x)\big) = 0$ with Dirichlet condition $u\,|_{\partial\Omega}= u_0$ if for all $x \in \partial\Omega$, $u(x) = u_0(x)$, and for all $x \in \Omega$, $u$ is differentiable at $x$ and $F(u(x), x, \nabla u(x)) = 0$.

We say that $u$ is a *classical subsolution* of $F\big(u(x), x, \nabla u(x)\big) = 0$ if for all $x \in \Omega$, $u$ is differentiable at $x$ and $F\big(u(x), x, \nabla u(x)\big) \leq 0$.

**Definition 1.1.** We say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ is an *almost classical solution* of $F(u(x), x, \nabla u(x)) = 0$ with Dirichlet condition $u\,|_{\partial\Omega}= u_0$ if:

- $u(x) = u_0(x)$ for all $x \in \partial\Omega$,
- $u$ is a classical subsolution of $F(u(x), x, \nabla u(x)) = 0$,
- and $u$ satisfies $F(u(x), x, \nabla u(x)) = 0$ for almost every $x \in \Omega$ (in the sense of Lebesgue measure on $\mathbb{R}^d$).

Notice that a classical solution is an almost classical solution, and that if $u$ is an almost classical solution, then $u$ is continuous on $\overline{\Omega}$ and differentiable at every point of $\Omega$. In many natural examples, classical solutions of the Hamilton-Jacobi equation $F(u(x), x, \nabla u(x)) = 0$ exist only under very restrictive conditions on $F$. We prove the existence of almost classical solutions under quite general hypotheses on $F$. Observe that our results imply the existence of an almost classical solution $u$ of the Eikonal equation satisfying the boundary condition $u\,|_{\partial\Omega}= 0$. In particular, we have:

**Theorem 1.2.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(*A*) *There exists a continuous function $u_0 : \overline{\Omega} \to \mathbb{R}$, which is $\mathcal{C}^1$-smooth on $\Omega$ and such that $F\big(x, \nabla u_0(x)\big) \leq 0$, for every $x \in \Omega$.*

($B$) *For each compact subset $K \subset \Omega$, there exists $M_K > 0$ such that*

$$\inf\left\{F(x,p) : x \in K,\, p \in \mathbb{R}^d,\, \|p\| \geq M_K\right\} > 0.$$

*Then there exists an almost classical solution of $F\big(x, \nabla u(x)\big) = 0$, with Dirichlet condition $u|_{\partial\Omega} = u_0$.*

The above result will actually follow from the more general Theorem 3.1 that will also provide other existence results of almost classical solutions of Hamilton-Jacobi equations. The proof of Theorem 3.1 will be given in Section 4.

In the last section, we consider Hamilton-Jacobi equations defined on a smooth manifold $M$ of dimension $d \geq 2$, which always will be assumed to be Hausdorff and second countable. As usual, $TM$ denotes the tangent bundle of $M$. A point in $TM$ will be $(x, v)$, where $x \in M$ and $v$ belongs to the tangent space $T_xM$. In the same way, $T^*M$ denotes the cotangent bundle of $M$. A point in $T^*M$ will be $(x, \xi)$, where $x \in M$ and $\xi \in T^*_xM$ is a linear form on the tangent space $T_xM$. If $u : M \to \mathbb{R}$ is differentiable at $x \in M$ we denote its differential at $x$ by $du(x)$. Under suitable hypotheses on $F : T^*M \to \mathbb{R}$, we obtain the existence of almost classical solutions of an equation of the form $F\big(x, du(x)\big) = 0$. In particular, we obtain:

**Theorem 1.3.** *Let $M$ be a smooth manifold of dimension $d \geq 2$, and let $F : T^*M \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function. Suppose that the following conditions hold:*

($A$) *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$ such that $F\big(x, du_0(x)\big) \leq 0$, for every $x \in M$.*

($B$) *For each $x \in M$, the set $B(x) = \{\xi \in T^*_xM : F(x,\xi) \leq 0\}$ is compact, the set $S(x) = \{\xi \in T^*_xM : F(x,\xi) = 0\}$ is connected, and the function $F(x,\cdot)$ has maximal rank on the set $S(x)$.*

*Then there exists a differentiable function $u : M \to \mathbb{R}$ such that $F\big(x, du(x)\big) = 0$ for almost every $x \in M$.*

If now we have a Riemannian manifold $(M, g)$ and $u : M \to \mathbb{R}$ is differentiable, for every $x \in M$ we identify in the usual way the differential $du(x)$ with the gradient $\nabla u(x)$ by means of the scalar product $g_x(\cdot,\cdot)$ on the tangent space $T_xM$. In this case we obtain the following analogue of Theorem 1.2:

**Theorem 1.4.** *Let $(M, g)$ be a Riemannian manifold of dimension $d \geq 2$, and let $F : TM \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

($A$) *There exists a* $\mathcal{C}^1$ *function* $u_0 : M \to \mathbb{R}$, *such that* $F\big(x, \nabla u_0(x)\big) \le 0$, *for every* $x \in M$.

($B$) *There exists a locally bounded function* $\rho : M \to (0, \infty)$ *such that, for every* $x \in M$, *the set* $B(x) = \{v \in T_x M : F(x, v) \le 0\}$ *is contained in the ball of center* 0 *and radius* $\rho(x)$ *in* $T_x M$.

*Then there exists a differentiable function* $u : M \to \mathbb{R}$ *such that* $F\big(x, \nabla u(x)\big) = 0$ *for almost every* $x \in M$.

Thus if for a Riemannian manifold $(M, g)$ we consider the function $F : TM \to \mathbb{R}$ given by

$$F(x, v) = \|v\|_x - 1 = \big(g_x(v, v)\big)^{1/2} - 1,$$

it is clear that the constant functions $u_0 \equiv 0$ and $\rho \equiv 1$ satisfy the above requirements. Therefore we obtain that there exists a differentiable function $u$ on $M$ which satisfies the Eikonal equation $\|\nabla u(x)\|_x = 1$ almost everywhere on $M$. Whenever the manifold $M$ is compact, there exists a point $x_0 \in M$ such that $\nabla u(x_0) = 0$. Therefore, there is no classical solution of this equation, and an almost classical solution $u$ of this equation cannot be $\mathcal{C}^1$-smooth. So almost classical solutions of Hamilton-Jacobi equations are often exotic.

## 2. Preliminary results

We recall three lemmas from [6] that we shall use here. The first lemma is a criterium of differentiability for the sum of a series of $\mathcal{C}^1$-smooth functions. We shall use the following notation: if $X$ and $Z$ are Banach spaces and $f : X \to Z$, then the oscillation of $f$ with respect to $\delta > 0$ is defined by

$$osc(f, \delta) = \sup\big\{\|f(x_1) - f(x_2)\| : x_1, x_2 \in X, \|x_1 - x_2\| \le \delta\big\}$$

**Lemma 2.1.** *Let* $(u_n)_{n \ge 1}$ *be a sequence of* $\mathcal{C}^1$ *functions between two Banach spaces* $X$ *and* $Y$. *Assume that:*

(a) *the series* $\big(\sum \nabla u_n(x)\big)$ *is pointwise convergent;*

(b) *the sequence* $\big(\nabla u_n\big)$ *converges uniformly to* 0*;*

(c) $\|u_{n+1}\|_\infty = o(\|u_n\|_\infty)$*;*

(d) $\displaystyle\lim_{n\to\infty} osc\left(\sum_{k=1}^{n} \nabla u_k, \|u_{n+1}\|_\infty\right) = 0.$

*Then the series* $\big(\sum u_n\big)$ *is uniformly convergent, the function* $u := \sum_{n=1}^\infty u_n$ *is everywhere differentiable, and* $\nabla u(x) = \sum_{n=1}^\infty \nabla u_n(x)$ *for all* $x \in X$.

We say that a subset $Q$ of $\mathbb{R}^d$ is a cube if $Q = \prod_{i=1}^d [a_i, b_i[$, where each $[a_i, b_i]$ is a closed and bounded interval of $\mathbb{R}$. And we say that $Q$ is a closed cube if $Q = \prod_{i=1}^d [a_i, b_i]$. A function $v$ defined on a cube $Q$ is said to be *piecewise constant* if there is a finite partition $\mathcal{Q}$ of $Q$ into cubes such that $v$ is constant on every cube of the partition $\mathcal{Q}$. The following result gives the existence of a $\mathcal{C}^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$, which vanishes in a neighbourhood of the exterior of a cube $Q$ and such that its derivative is equal to $a$ or $-a$ (where $a$ is a given non zero vector in $\mathbb{R}^d$) on a subset of $Q$ of measure almost equal to the measure of $Q$. The Lebesgue measure on $\mathbb{R}^d$ will be denoted $\lambda_d$.

**Lemma 2.2.** *Let $a \in \mathbb{R}^d$ be a non zero vector, let $Q$ be a cube in $\mathbb{R}^d$, and let $\varepsilon > 0$. Then, there exists a bounded, $\mathcal{C}^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the following properties:*

- (a) *$u$ vanishes in a neighbourhood of $\partial Q$ and $\|u\|_\infty \leq \varepsilon$;*
- (b) *$\lambda_d\big(\{x \in Q : \nabla u(x) = -a \text{ or } \nabla u(x) = a\}\big) \geq (1-\varepsilon)\lambda_d(Q)$;*
- (c) *one can write $\nabla u = v + w$ with $\|w\|_\infty < \varepsilon$; the set $\{v(x) : x \in Q\}$ is included in the segment $[-a, a]$, and the function $v$ is piecewise constant on $Q$.*

The last lemma relies on ideas due to J. Maly and M. Zeleny [8], and is also from [6]. The mapping $\mathbf{t}$ is defined using that a suitable game has a winning strategy.

**Lemma 2.3.** *Let $B$ be a closed ball of $\mathbb{R}^d$. Then, there exists a map $\mathbf{t} : B \to \mathbb{R}^d$ such that if a sequence $(\sigma_n) \in B$ satisfies $\langle \mathbf{t}(\sigma_n), \sigma_{n+1} - \sigma_n \rangle \geq 0$ for all $n$, then $(\sigma_n)$ converges.*

## 3. Almost classical solutions on open subsets of $\mathbb{R}^d$

For a wide class of Hamilton-Jacobi equations, we give an existence theorem of almost classical solutions defined on the closure of on an open subset of $\mathbb{R}^d$, and satisfying an homogeneous Dirichlet condition.

**Theorem 3.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and $F : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

- $(A)$ *$F(0, x, 0) \leq 0$, for every $x \in \Omega$; that is, the function $u_0$ identically equal to $0$ is a classical subsolution of $F\big(u(x), x, \nabla u(x)\big) = 0$.*
- $(B)$ *For each compact subset $K \subset \Omega$, there exist $\alpha_K > 0$ and $M_K > 0$ such that for all $x \in K$, for all $u \in [0, \alpha_K]$ and for all $p \in \mathbb{R}^d$ satisfying $\|p\| \geq M_K$, we have $F(u, x, p) > 0$.*

*Then there exists a function* $u \geq 0$ *on* $\overline{\Omega}$ *which is an almost classical solution of* $F\big(u(x), x, \nabla u(x)\big) = 0$*, with Dirichlet condition* $u\mid_{\partial\Omega} = 0$*. Moreover, the extension* $\widetilde{u}$ *of* $u$ *to* $\mathbb{R}^d$ *satisfying* $\widetilde{u}(x) = 0$ *if* $x \notin \overline{\Omega}$ *is differentiable at every point of* $\mathbb{R}^d$*.*

The proof of Theorem 3.1 will be postponed until Section 4. Along this section, we will obtain several consequences of this result.

**Remark 3.2.** It will be useful to note that condition (B) in Theorem 3.1 is equivalent to condition(B') and also to condition (B") below:

(B') *For each compact subset* $K \subset \Omega$*, there exists* $\alpha_K > 0$ *such that the set*

$$B(K;\alpha_K) = \big\{(u, x, p) \in [0, \alpha_K] \times K \times \mathbb{R}^d :\ F(x, u, p) \leq 0\big\}$$

*is compact in* $\mathbb{R} \times \Omega \times \mathbb{R}^d$*.*

(B") *For each* $x_0 \in \Omega$*, there exist a compact neighborhood* $V^{x_0}$ *and* $\alpha > 0$*, such that the set*

$$B(V^{x_0};\alpha) = \big\{(u, x, p) \in [0, \alpha] \times V^{x_0} \times \mathbb{R}^d : F(u, x, p) \leq 0\big\}$$

*is compact in* $\mathbb{R} \times \Omega \times \mathbb{R}^d$*.*

We now consider the case of general Dirichlet conditions.

**Corollary 3.3.** *Let* $\Omega$ *be an open subset of* $\mathbb{R}^d$ *with* $d \geq 2$*, and* $F : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ *be a continuous function. Suppose that the following conditions hold:*

($A$) *There exists a continuous function* $u_0 : \overline{\Omega} \to \mathbb{R}$*, which is* $\mathcal{C}^1$*-smooth on* $\Omega$ *and such that* $F\big(u_0(x), x, \nabla u_0(x)\big) \leq 0$*, for every* $x \in \Omega$*.*

($B$) *For each compact subset* $K \subset \Omega$*, there exist* $M_K > 0$ *and* $\alpha_K > 0$ *such that for all* $x \in K$*, for all* $u \in [0, \alpha_K]$ *and for all* $p \in \mathbb{R}^d$ *satisfying* $\|p\| \geq M_K$*, we have* $F(u_0(x) + u, x, p) > 0$*.*

*Then there exists an almost classical solution* $u$ *of* $F\big(u(x), x, \nabla u(x)\big) = 0$*, with Dirichlet condition* $u\mid_{\partial\Omega} = u_0$*. Moreover, if* $u_0$ *is* $\mathcal{C}^1$*-smooth on* $\mathbb{R}^d$*, the function* $u$ *can be extended to a differentiable function on* $\mathbb{R}^d$*.*

**Proof .** Define $G(u, x, p) = F\big(u + u_0(x), x, p + \nabla u_0(x)\big)$. Conditions ($A$) and ($B$) of Theorem 3.1 are satisfied for $G$. Thus, there exists an almost classical solution $v$ of $G(v(x), x, \nabla v(x)) = 0$, with Dirichlet condition $v\mid_{\partial\Omega} = 0$, and furthermore $v$ can be extended to a differentiable function on $\mathbb{R}^d$. The function $u : \overline{\Omega} \to \mathbb{R}$ defined by $u(x) = u_0(x) + v(x)$ is then an almost classical solution of $F\big(u(x), x, \nabla u(x)\big) = 0$, with Dirichlet condition $u\mid_{\partial\Omega} = u_0$. ■

Notice that Theorem 1.2 is a straightforward consequence of Corollary 3.3. Another easy consequence of Corollary 3.3 is the following existence result of almost classical solutions for stationary Hamilton-Jacobi equations:

**Corollary 3.4.** *Let* $\Omega$ *be an open subset of* $\mathbb{R}^d$ *with* $d \geq 2$*, and let* $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *be a continuous function. Suppose that the following conditions hold:*

- $(A)$ *There exists a continuous function* $u_0 : \overline{\Omega} \to \mathbb{R}$*, which is* $\mathcal{C}^1$*-smooth on* $\Omega$ *and such that* $u_0(x) + F\big(x, \nabla u_0(x)\big) \leq 0$*, for every* $x \in \Omega$*.*
- $(B)$ *For each compact* $K \subset \Omega$*, there exists* $M_K > 0$ *such that*

$$\inf \big\{ u_0(x) + F(x,p);\, x \in K,\, p \in \mathbb{R}^d,\, \|p\| \geq M_K \big\} > 0$$

*Then there exists an almost classical solution of* $u(x) + F\big(x, \nabla u(x)\big) = 0$*, with Dirichlet condition* $u \mid_{\partial\Omega} = u_0$*.*

Next we give a further application of Theorem 3.1. We shall need the following notions. If $A$ is a subset of $\mathbb{R}^d$, we denote its complement by $A^c = \mathbb{R}^d \setminus A$. Let us recall the definition of the *Hausdorff distance* between closed sets of a metric space. If $X$ is a metric space, for each $A \subset X$ and $r > 0$ we denote $B(A,r) = \{x \in X : dist(x,A) < r\}$. We denote $\mathcal{C}(X)$ the set of all closed bounded subsets of $X$. If $C$ and $D$ are in $\mathcal{C}(X)$, the Hausdorff distance between them is

$$d_H(C,D) = \inf \big\{ r \in (0,\infty] : C \subset B(D,r) \text{ and } D \subset B(C,r) \big\}.$$

**Theorem 3.5.** *Let* $\Omega$ *be an open subset of* $\mathbb{R}^d$ *with* $d \geq 2$*. For each* $x \in \Omega$ *let* $U(x)$ *be an open bounded subset of* $\mathbb{R}^d$ *containing* $0$*. Assume that the set-valued mapping* $x \mapsto \partial U(x)$ *from* $\Omega$ *into* $(\mathcal{C}(\mathbb{R}^d), d_H)$ *is continuous on* $\Omega$*. Then there exists a differentiable function* $u : \mathbb{R}^d \to \mathbb{R}$ *such that:*

1. $u \mid_{\Omega^c} \equiv 0$ *and* $\nabla u \mid_{\Omega^c} \equiv 0$*.*
2. $\nabla u(x) \in \overline{U(x)}$ *for every* $x \in \mathbb{R}^d$*.*
3. $\nabla u(x) \in \partial U(x)$ *for almost every* $x \in \Omega$*.*

**Proof**. Consider the function $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by $F(u,x,p) = -dist\big(p,\, \partial U(x)\big)$ if $x \in U(x)$, and $F(u,x,p) = dist\big(p,\, \partial U(x)\big)$ otherwise. Since the mapping $x \mapsto \partial U(x)$ from $\Omega$ into $(\mathcal{C}(\mathbb{R}^d), d_H)$ is continuous on $\Omega$, is is easy to see that $F$ is continuous on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. The function identically equal to 0 is a classical subsolution of $F(u,x,p)=0$. On the other hand, for each compact subset $K\subset\mathbb{R}^d$, there exists $R>0$ such that

$$\bigcup_{x\in K}\partial U(x)\subset B(0,R),$$

and therefore

$$\bigcup_{x\in K}\overline{U(x)}\subset B(0,R).$$

Hence for all $u\in\mathbb{R}$, for all $p\in\mathbb{R}^d$ satisfying $\|p\,\|\ge 2R$ and for all $x\in K$, we have $F(u,x,p)>0$. The two hypothesis of Theorem 3.1 are then satisfied. The almost classical solution of $F(u,x,p)=0$ given by Theorem 3.1 satisfies the required properties. ■

## 4. Proof of Theorem 3.1

In order to prove Theorem 3.1, we first consider the case of a cube, on which almost classical solutions will be obtained as the sum of a series of $\mathcal{C}^\infty$-smooth functions, and the general case will then follow easily.

**Lemma 4.1.** *Assume that the hypotheses of Theorem 3.1 are satisfied, and let $C$ be a cube such that $\overline{C}$ is contained in $\Omega$. Then there exists a differentiable function $u_C:\mathbb{R}^d\to\mathbb{R}$ such that:*

1. $u_C\ge 0$ *and* $u_C(x)=0$ *for all $x$ which is not in the interior of $C$.*
2. $F\big(u_C(x),x,\nabla u_C(x)\big)\le 0$ *for every $x\in C$.*
3. $F\big(u_C(x),x,\nabla u_C(x)\big)=0$ *for almost every $x\in C$.*

**Proof of Theorem 3.1.** We first fix an increasing sequence $(K_n)_{n\ge1}$ of compact subsets of $\Omega$ such that the union of all $K_n$'s is equal to $\Omega$. We also assume that each $K_n$ is the closure of a finite union of cubes. By assumption $(B)$, for each $n\ge1$ there exist $M_n>0$ and $\alpha_n>0$ such that, for all $x\in K_n$, for all $u\in[0,\alpha_n]$ and for all $p\in\mathbb{R}^d$ satisfying $\|p\|\ge M_n$, we have $F(u,x,p)>0$. We consider a decomposition

$$\Omega=\bigcup_{j=1}^{\infty}C_j,$$

where $(C_j)_{j\ge1}$ is a locally finite family of cubes such that:

(a) $C_j\cap C_k=\emptyset$ if $j\ne k$.

(b) for each $j$, there exists $n$ such that $C_j\subset\overline{K_n\backslash K_{n-1}}$.

Refining if necessary this decomposition, we can also assume:

(c) $\operatorname{diam}(C_j) \le \frac{1}{2^n M_n}\, d_H(K_n, \partial\Omega)$ whenever $C_j \subset \overline{K_n \backslash K_{n-1}}$.

By Lemma 4.1, for each $j \geq 1$ there exists a differentiable function $u_j : \mathbb{R}^d \to \mathbb{R}$ such that:

1. $u_j \geq 0$ and $u_j(x) = 0$ for all $x$ which is not in the interior of $C_j$.
2. $F\big(u_j(x), x, \nabla u_j(x)\big) \leq 0$ for every $x \in C_j$.
3. $F\big(u_j(x), x, \nabla u_j(x)\big) = 0$ for almost every $x \in C_j$.

Then we define $u : \mathbb{R}^d \to \mathbb{R}$ by setting

$$u = \sup_{j \geq 1} u_j.$$

By property (1) above, $u = u_j$ on each $C_j$. Then it is easy to see that $u$ is differentiable on $\Omega$, identically equal to 0 on $\mathbb{R}^d \backslash \Omega$, satisfies $F\big(u(x), x, \nabla u(x)\big) \leq 0$ for every $x \in \Omega$ and $F\big(u(x), x, \nabla u(x)\big) = 0$ for almost every $x \in \Omega$. It remains to check that $u$ is differentiable at each point of $\partial\Omega$. Fix $n \geq 1$. We know that $u$ vanishes on the boundary of each cube $C_j$, so, by the mean value theorem,

$$\sup\big\{u(x) : x \in C_j\big\} \leq \sup\big\{\nabla u(x) : x \in C_j\big\} \cdot \operatorname{diam}(C_j).$$

If $C_j \subset \overline{K_n \backslash K_{n-1}}$, then $\sup\big\{\nabla u(x) : x \in C_j\big\} \leq M_n$. In that case, using $(c)$, we obtain:

$$\sup\big\{u(x) : x \in C_j\big\} \leq \frac{1}{2^n}\, d_H(K_n, \partial\Omega).$$

So whenever $x \in K_n \backslash K_{n-1}$, we have that $0 \leq u(x) \leq dist(x, \partial\Omega)/2^n$. This implies that for each point $x \in \partial\Omega$, $u$ is differentiable at $x$ and $\nabla u(x) = 0$. ■

**Proof of Lemma 4.1.** Observe that if $F(0, x, 0) = 0$ for almost every $x \in C$, we can take $u_C = 0$ and the above assertions are satisfied. From now on, we assume that

$$\lambda_d\big(\{x \in C : F(0, x, 0) < 0\}\big) > 0$$

By assumption $(B)$, there exists $\alpha > 0$ such that

$$r := \sup\big\{\|p\| : F(u, x, p) \leq 0 \text{ for some } x \in \overline{C} \text{ and } u \in [0, \alpha]\big\}$$

is finite. We fix a map $\mathbf{t} : B(0, 1 + r) \to \mathbb{R}^d$ satisfying the conditions of Lemma 2.3.

The function $u_C$ will be given by a series

$$u_C = \sum_{n=1}^{\infty} u_n,$$

where each $u_n$ is a $\mathcal{C}^\infty$-smooth function on $\mathbb{R}^d$. For each $n$, we will write $\nabla u_n = v_n + w_n$, and we will denote

$$U_n = \sum_{k=1}^{n} u_k \qquad \text{and} \qquad \sigma_n = \sum_{k=1}^{n} v_k.$$

***Construction of the functions $u_n$***: The functions $u_n$ will be constructed together with a sequence $(\mathcal{Q}_n)_{n\geq 0}$ of partitions of $C$ into cubes, where each $\mathcal{Q}_{n+1}$ is a refinement of $\mathcal{Q}_n$. We also fix a sequence $(\varepsilon_k)_{k\geq 1}$ of positive numbers, with $(\varepsilon_k) \downarrow 0$ and such that $\inf\{F(0,x,0) : x \in C\} < -\varepsilon_1$ and $\varepsilon_1 < 1$, and we construct an increasing sequence of integers $(N_k)_{k\geq 0}$ with $N_0 = 0$. The following conditions will be proved by induction:

(0) There exists $x_0 \in C$ such that, for each $n \geq 1$, $u_n(x_0) = 0$ and $\nabla u_n(x_0) = 0$.

(i) For each $n \geq 1$, $u_n$ and $v_n$ have their support included in the interior of $C$, $v_n$ is constant on each cube of $\mathcal{Q}_n$, and $\|w_n\|_\infty \leq 2^{-n}$.

(ii) For each $n \geq 1$ and $x \in C$, $F(U_n(x), x, \nabla U_n(x)) \leq 0$.

(iii) For each $n \geq 1$ and $x \in C$, we have

$$\|\sigma_n(x)\| \leq 1 + r \qquad \text{and} \qquad \langle \mathbf{t}(\sigma_n(x)) \,,\, \sigma_{n+1}(x) - \sigma_n(x)\rangle = 0,$$

(iv) $\|u_1\|_\infty \leq \alpha/2$, and, for each $n \geq 1$, we have

$$0 < \|u_{n+1}\|_\infty \leq 2^{-n}\|u_n\|_\infty \qquad \text{and} \qquad osc(\nabla U_n, \|u_{n+1}\|_\infty) \leq 1/2^n$$

(v) For each $k \geq 1$ and each $N_{k-1} < n \leq N_k$, we have $\|v_n\|_\infty \leq \varepsilon_k$.

(vi) For each $k \geq 1$,

$$\lambda_d\big\{x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) \leq -\varepsilon_k\big\} \leq 2^{-k}\lambda_d(C).$$

*Construction of* $u_1$: Fix a cube $Q_0 \subset C$ with

$$d_H(Q_0, \partial C) > 0 \qquad \text{and} \qquad \sup\big\{F(0,x,0) : x \in Q_0\big\} \leq -\varepsilon_1$$

This implies, using the uniform continuity of $F$ on compact sets, that there exists $0 < \delta_1 \leq \varepsilon_1$ such that, whenever $x \in Q_0$, $0 \leq h \leq \delta_1$ and $\|q\| \leq 2\delta_1$, then:

$$F(h, x, q) \leq 0. \tag{4.1}$$

Choose $a = a(Q_0) \in \mathbb{R}^d$ such that $\|a(Q_0)\| = \delta_1$ and $\langle \mathbf{t}(0)\,,\, a\rangle = 0$ (this is possible since $d \geq 2$). Now applying Lemma 2.2 to the cube $Q_0$, we obtain a $C^\infty$ function $u_1$ on $\mathbb{R}^d$, and a cube partition $\mathcal{Q}_1$ of $C$ such that $Q_0$ is a union of some elements of $\mathcal{Q}_1$, such that:

- $u_1$ vanishes on a neighborhood of $\partial Q_0$ and outside of $Q_0$.
- $0 < \|u_1\|_\infty \leq \min\{\delta_1, a/2\}$.
- $\nabla u_1 = v_1 + w_1$, where $\|w_1\|_\infty \leq \min\{\delta_1, 1/2\}$, $v_1$ is constant on each cube of $\mathcal{Q}_1$, and $v_1(Q_0) \subset [-a(Q_0), a(Q_0)]$.

Fix $x_0 \in \partial Q_0$: we have $u_1(x_0) = 0$ and $\nabla u_1(x_0) = 0$, so condition (0) is satisfied. Conditions (i),(iii) and (v) are clearly satisfied, and (ii) follows from (4.1). So we can start the induction.

*Inductive step*: Fix $k \geq 1$, assume that $N_{k-1}$ has been defined, and for some $n \geq N_{k-1}$ the partition $\mathcal{Q}_n$ and the function $u_n$ have been constructed.

First, there exists $0 < \delta_k \leq \varepsilon_k$ such that whenever $x \in \overline{C}$, $u \in [0, \alpha]$, $p \in B(0, 1+r)$, $0 \leq h \leq \delta_k$ and $\|q\| \leq 2\delta_k$, then

$$F(u, x, p) \leq -\varepsilon_k/2 \quad \Longrightarrow \quad F(u+h, x, p+q) \leq 0 \tag{4.2}$$

Next, choose a cube partition $\widehat{\mathcal{Q}}_n$ of $C$ refining $\mathcal{Q}_n$ such that:

∘ If we denote $\widehat{\mathcal{R}}_n$ the family of all cubes $Q \in \widehat{\mathcal{Q}}_n$ such that $d_H(Q, \partial C) > 0$, and $K_n = \cup\{Q : Q \in \widehat{\mathcal{R}}_n\}$, we have that

$$\lambda_d(C \setminus K_n) < 2^{-(k+1)}\lambda_d(C). \tag{4.3}$$

∘ For all $Q \in \widehat{\mathcal{R}}_n$ and every $x, y \in Q$, we have

$$\big|F\big(U_n(x), x, \nabla U_n(x)\big) - F\big(U_n(y), y, \nabla U_n(y)\big)\big| < \varepsilon_k/2. \tag{4.4}$$

The second condition above can be obtained using the uniform continuity of the mapping $x \mapsto F\big(U_n(x), x, \nabla U_n(x)\big)$ on the compact set $\overline{C}$.

Now, each cube $Q$ of $\widehat{\mathcal{Q}}_n$ is contained in a cube $Q'$ of $\mathcal{Q}_n$ and by (i), $\sigma_n$ is constant on $Q'$. We denote by $\sigma_n(Q)$ the constant value of $\sigma_n|_{Q'}$. Choose $a = a(Q) \in \mathbb{R}^d$ such that $\|a(Q)\| = \delta_k$ and $\langle \mathbf{t}(\sigma_n(Q))\,,\, a\rangle = 0$. Now applying Lemma 2.2, for each cube $Q \in \widehat{\mathcal{R}}_n$ we obtain a $C^\infty$ function $u_Q$ on $\mathbb{R}^d$, and a cube partition $\mathcal{Q}_{n+1}$ of $Q_0$ which is a refinement of $\widehat{\mathcal{Q}}_n$ (and therefore of $\mathcal{Q}_n$), such that:

(a) $u_Q$ vanishes on a neighborhood of $\partial Q$.

(b) $0 < \|u_Q\|_\infty \leq \min\left\{2^{-n}\|u_n\|_\infty, \delta_k\right\}$ and $osc(\nabla U_n, \|u_Q\|_\infty) < 1/2^n$.

(c) $\lambda_d\big\{x\in Q : \nabla u_Q(x)=\pm a(Q)\big\}\geq (1-2^{-k})\lambda_d(Q)$.

(d) $\nabla u_Q = v_Q + w_Q$, where $\|w_Q\|_\infty \leq \min\{\delta_k, 1/2^{n+2}\}$, $v_Q$ is constant on each cube of $\mathcal{Q}_{n+1}$, and $v_Q(Q)\subset[-a(Q),a(Q)]$. In particular, we have $\|v_Q\|_\infty\leq\|a(Q)\|=\delta_k\leq\varepsilon_k$ and $\|\nabla u_Q\|_\infty\leq 2\delta_k$.

Next we define the function $u_{n+1}$ on $\mathbb{R}^d$. We first choose for each $Q\in\widehat{\mathcal{R}}_n$ a point $x_Q$ in the closure of $Q$ such that

$$F\big(U_n(x_Q),x_Q,\nabla U_n(x_Q)\big)=\inf\big\{F\big(U_n(x),x,\nabla U_n(x)\big) : x\in Q\big\}$$

We define $u_{n+1}$ on each cube of $\widehat{\mathcal{R}}_n$ in the following way:

1. If $F\big(U_n(x_Q),x_Q,\nabla U_n(x_Q)\big) > -\varepsilon_k$, we set $u_{n+1}=0$ and $v_{n+1}=w_{n+1}=0$ on $Q$.
2. If $F\big(U_n(x_Q),x_Q,\nabla U_n(x_Q)\big)\leq-\varepsilon_k$, we set $u_{n+1}=u_Q$ on $Q$. In this case, $v_{n+1}=v_Q, w_{n+1}=w_Q$, and we have
$$\lambda_d\big\{x\in Q : \|\nabla u_{n+1}(x)\|=\delta_k\big\}\geq(1-2^{-k})\cdot\lambda_d(Q).$$
3. Finally, on $(K_n)^c$, we set $u_{n+1}=0$, and $v_{n+1}=w_{n+1}=0$.

In this way we obtain that $u_{n+1}$ is a $C^\infty$ function on $\mathbb{R}^d$, which vanish on a neighborhood of $\partial Q$ for every $Q\in\widehat{\mathcal{R}}_n$.

Next we are going to check conditions (0) to (vi) for $n+1$. Since $\widehat{\mathcal{Q}}_n$ is a refinement of $\mathcal{Q}_1$ and $x_0\in\partial Q_0\in\mathcal{Q}_1$, there exists $Q\in\widehat{\mathcal{Q}}_n$ such that $x_0\in\partial Q$, so $u_{n+1}(x_0)=0$ and $\nabla u_{n+1}(x_0)=0$. This proves condition (0). Condition (i) is clearly satisfied. In order to prove condition (ii), fix $x\in\Omega$. By induction hypothesis, $F\big(U_n(x),x,\nabla U_n(x)\big)\leq0$. Let us prove that $F\big(U_{n+1}(x),x,\nabla U_{n+1}(x)\big)\leq0$:

- If $x\in(K_n)^c$, then $u_{n+1}=0$ on a neighbourhood of $x$ and $\nabla U_{n+1}(x)=\nabla U_n(x)$, so $F\big(U_{n+1}(x),x,\nabla U_{n+1}(x)\big)=F\big(U_n(x),x,\nabla U_n(x)\big)\leq0$.

- If $x\in Q\in\widehat{\mathcal{R}}_n$ with $F\big(U_n(x_Q),x_Q,\nabla U_n(x_Q)\big)>-\varepsilon_k$, then $u_{n+1}=0$ on a neighborhood of $Q$ and

$$F\big(U_{n+1}(x_Q),x_Q,\nabla U_{n+1}(x_Q)\big)=F\big(U_n(x_Q),x_Q,\nabla U_n(x_Q)\big)\leq0.$$

- Finally, if $x\in Q\in\widehat{\mathcal{R}}_n$ with $F\big(U_n(x_Q),x_Q,\nabla U_n(x_Q)\big)\leq-\varepsilon_k$, by 4.4, we have for all $x\in Q$, $F\big(U_n(x),x,\nabla U_n(x)\big)\leq-\varepsilon_k/2$. Since

$$|U_n(x)|\leq\sum_{k=1}^{\infty}|u_k(x)|\leq\alpha,$$

from the definition of $r$ and the fact that $F\big(U_n(x), x, \nabla U_n(x)\big) \leq 0$, it follows that $\|\nabla U_n(x)\| \leq r$. Since we have also $\|u_{n+1}\|_\infty \leq \delta_k$ and $\|\nabla u_{n+1}\|_\infty \leq 2\delta_k$, we deduce from 4.2 that

$$F\big(U_{n+1}(x), x, \nabla U_{n+1}(x)\big) \leq 0$$

In order to prove (iii), fix $x \in C$. First, $F\big(U_{n+1}(x), x, \nabla U_{n+1}(x)\big) \leq 0$, so $\|\nabla U_{n+1}(x)\| \leq r$, and

$$\|\nabla U_{n+1}(x) - \sigma_{n+1}(x)\| \leq \sum_{k=1}^{n+1} \|w_k(x)\| \leq 1.$$

Therefore $\|\sigma_n(x)\| \leq 1+r$. Then, if $v_{n+1}(x) = 0$, we have $\sigma_{n+1}(x) - \sigma_n(x) = 0$, so $\langle \mathbf{t}(\sigma_n(x)), \sigma_{n+1}(x) - \sigma_n(x)\rangle = 0$. On the other hand, if $v_{n+1}(x) \neq 0$, then$x \in Q$ for some $Q \in \widehat{\mathcal{R}}_n$, and $v_{n+1} = v_Q$. In this case $\sigma_n(x) = \sigma_n(Q)$, and $v_{n+1}(x) \in [-a(Q), a(Q)]$. Thus $v_{n+1}(x) = \sigma_{n+1}(x) - \sigma_n(x)$ is proportional to $a(Q)$, and therefore orthogonal to $\mathbf{t}(\sigma_n(Q))$, and the condition $\langle \mathbf{t}(\sigma_n(x)), \sigma_{n+1}(x) - \sigma_n(x)\rangle = 0$ is again satisfied.

Now we are going to see that $u_{n+1} \neq 0$. Indeed, if $Q \in \widehat{\mathcal{Q}}_n$ is such that $Q \subset Q_0$ and $x_0 \in \partial Q$, then $d_H(Q, \partial C) \geq d_H(Q_0, \partial C) > 0$, so $Q \in \widehat{\mathcal{R}}_n$. Since $U_n(x_0) = 0$ and $\nabla U_n(x_0) = 0$, we have

$$F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) \leq F\big(U_n(x_0), x_0, \nabla U_n(x_0)\big) \leq -\varepsilon_1 \leq -\varepsilon_k,$$

and therefore $u_{n+1} = u_Q \neq 0$ on $Q$. Condition (iv) follows now from (b). Next, condition (v) also holds, although we still have to define the integer $N_k$.

Finally, let us prove that (vi) is satisfied. Suppose, to the contrary, that for every $n > N_{k-1}$ we have:

$$\lambda_d\big\{x \in C : F(U_n(x), x, \nabla U_n(x)) \leq -\varepsilon_k\big\} > 2^{-k} \cdot \lambda_d(C).$$

By (4.3), we obtain that

$$\lambda_d\big\{x \in K_n : F(U_n(x), x, \nabla U_n(x)) \leq -\varepsilon_k\big\} > 2^{-(k+1)} \cdot \lambda_d(K_n).$$

Suppose now that

$$F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) \leq -\varepsilon_k.$$

As we have noticed, in this case

$$\begin{aligned}\lambda_d\big\{x \in Q : \|\nabla U_{n+1}(x) - \nabla U_n(x)\| \geq \delta_k\big\} &\geq \\ \geq \lambda_d\big\{x \in Q : \|\nabla u_{n+1}(x)\| = \delta_k\big\} &\geq (1 - 2^{-k}) \cdot \lambda_d(Q).\end{aligned}$$

Now the proportion of cubes $Q$ in $\widehat{\mathcal{R}}_n$ satisfying this has to be at least $2^{-(k+1)}$. Therefore

$$\lambda_d\big\{x \in K_n : \|\nabla U_{n+1}(x) - \nabla U_n(x)\| \geq \delta_k\big\} \geq (1-2^{-k})2^{-(k+1)} > 0.$$

This will be a contradiction with Lemma 2.3, since we are going to prove that the sequence $(\nabla U_n)_{n\geq 1}$ is pointwise convergent. Indeed, for each $x \in \mathbb{R}^d$, it follows from (i) that the sequence $\big(\sum_{k=1}^n w_k(x)\big)_{n\geq 1}$ converges, and from (iii) and Lemma 2.3 that the sequence $\big(\sigma_n(x)\big)_{n\geq 1}$ converges. Since

$$\nabla U_n(x) = \sigma_n(x) + \sum_{k=1}^{n} w_k(x)$$

we have that the sequence $\big(\nabla U_n(x)\big)_{n\geq 1}$ is convergent. This contradiction shows that there exists an integer $N_k > N_{k+1}$ satisfying (vi). This concludes the inductive step.

***The function $u_C$***: We now define

$$u_C = \sum_{n=1}^{\infty} u_n.$$

By (vi) the series is uniformly convergent on $\mathbb{R}^d$, so that $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, and it is clear that $u_C$ vanishes outside $C$. In order to see that $u_C$ is differentiable on $\mathbb{R}^d$, we check the conditions of Lemma 2.1. For each $n \geq 1$ let $k_n$ be an integer with $N_{k_n-1} < n \leq N_{k_n}$. From (i) and (v), we have that

- $\|\nabla u_n\|_\infty \leq \|v_n\|_\infty + \|w_n\|_\infty \leq \varepsilon_{k_n} + 2^{-n} \to 0,$

and from (iv), we obtain:

- $\|\nabla u_{n+1}\|_\infty = o\big(\|\nabla u_n\|_\infty\big),$
- $osc\big(\nabla U_n\,, \|\nabla u_n\|_\infty\big) \leq 2^{-n} \to 0.$

Moreover, applying as before Lemma 2.3, the sequence $(\nabla U_n)_{n\geq 1}$ is pointwise convergent, that is,

$$\sum_{n=1}^{\infty} \nabla u(x)$$

is convergent for every $x \in \mathbb{R}^d$. Applying Lemma 2.1, we obtain that $u_C$ is everywhere differentiable and $\nabla u_C$ is the pointwise limit of

$$\sum_{k=1}^{n} u_k.$$

Now (ii) implies that $F\big(u_C(x), x, \nabla u_C(x)\big) \leq 0$ for every $x \in \Omega$. Finally, let us prove that

$$F\big(u_C(x), x, \nabla u_C(x)\big) = 0$$

almost everywhere on $C$. Consider $x \in C$ such that $F\big(u_C(x), x, \nabla u_C(x)\big) < 0$. Taking into account that $\nabla u_C(x) = \lim_k \nabla U_{N_k}(x)$, we can find some integer $k_0$ such that

$$F\big(U_{N_k}(x), x, \nabla U_{N_k}(x)\big) < -\varepsilon_{k_0} \leq -\varepsilon_k,$$

for every $k \geq k_0$. Therefore the set $\big\{x \in C : F(u_C(x), x, \nabla u_C(x)) < 0\big\}$ is contained in the set

$$\limsup_k \big\{x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) < -\varepsilon_k\big\}.$$

Since

$$\lambda_d\big\{x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) < -\varepsilon_k\big\} \leq 2^{-k}\lambda_d(C) \to 0,$$

by Borel-Cantelli lemma $\lambda_d\Big(\big\{x \in C : F\big(u_C(x), x, \nabla u_C(x)\big) < 0\big\}\Big) = 0$. That is, $F\big(u_C(x), x, \nabla u_C(x)\big) = 0$ for almost every $x \in C$. ■

## 5. Almost classical solutions on Riemannian manifolds

In order to obtain our results for smooth manifolds, we will use the concept of triangulation, as it is given by Whitney in [11] (see also [3]). In what follows we assume that every smooth manifold is Hausdorff and second countable. If $M$ is a smooth $d$-dimensional manifold, a *triangulation* of $M$ is a pair $(K, \pi)$, where $K$ is a simplicial complex and $\pi : K \to M$ is a homeomorphism, such that for each $d$-dimensional simplex $S$ of $K$ there exists a local chart $(W, \varphi)$ of $M$, where $W$ is a neighborhood of $\pi(S)$ and $\varphi \circ \pi$ is affine on $S$. According to Whitney [11], every smooth manifold admits a triangulation.

Our first result is an extension of Theorem 3.1 to the setting of smooth manifolds. The Riemannian structure is not needed here. We consider an open subset $\Omega$ of the manifold $M$ and we denote $T^*\Omega$ the corresponding cotangent bundle. Then we consider equations of the form $F\big(u(x), x, du(x)\big) = 0$, where $F : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ is a continuous function. We obtain the following:

**Theorem 5.1.** *Let $M$ be a smooth manifold of dimension $d \geq 2$, consider an open subset $\Omega$ of $M$, and let $F : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

- $(A)$ *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$ such that $F\big(u_0(x), x, du_0(x)\big) \leq 0$, for every $x \in \Omega$.*
- $(B)$ *For each $x_0 \in \Omega$, there exist a compact neighborhood $V^{x_0}$ in $\Omega$ and $\alpha > 0$, such that the set $B(V^{x_0}; \alpha)$ is compact in $\mathbb{R} \times T^*M$, where*

$$B(V^{x_0}; \alpha) = \big\{(u, x, \xi) \in \mathbb{R} \times T^*M : u \in [0, \alpha];\ x \in V^{x_0};\\ F\big(u + u_0(x), x, \xi\big) \leq 0\big\}.$$

*Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:*

1. $u \geq u_0$ *on* $M$, $u = u_0$ *on* $\Omega^c$ *and* $du = du_0$ *on* $\Omega^c$.
2. $F\big(u(x), x, du(x)\big) \leq 0$ *for every* $x \in \Omega$.
3. $F\big(u(x), x, du(x)\big) = 0$ *for almost every* $x \in \Omega$.

**Proof.** We will consider two cases.

**First Case:** Suppose first that $u_0 \equiv 0$ on $M$. Let $(K, \pi)$ be a triangulation of $M$, where $K$ is a simplicial complex and $\pi : K \to M$ is a homeomorphism, and consider the family $\{S_i\}_{i \in I}$ of all $d$-dimensional simplices of $K$. For each $i \in I$, denote $T_i = \pi(S_i)$. Then

$$M = \bigcup_{i \in I} T_i,$$

each $\partial T_i$ has measure zero in $M$, and $int(T_i) \cap int(T_j) = \emptyset$ if $i \neq j$. Since $M$ is locally compact and $\pi$ is a homeomorphism, we have that the simplicial complex $K$ is locally compact, and therefore locally finite. Thus the family $\{T_i\}_{i \in I}$ is locally finite. Since $M$ is also $\sigma$-compact, we obtain that the index set $I$ is countable. For each $i \in I$, denote $\Omega_i = \Omega \cap int(T_i)$. Then the set $\Omega \setminus (\cup_{i \in I} \Omega_i)$ has measure zero in $M$.

For each $i \in I$ there is a chart $(W_i, \varphi_i)$ in $M$ with $T_i \subset W_i$. Associated to this chart there is a natural diffeomorphism

$$\Phi_i : \mathbb{R} \times T^*W_i \to \mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d$$

of the form $\Phi_i(u, x, \xi) = \big(u, \varphi_i(x), h_i(x, \xi)\big)$, where $h_i(x, \xi) \in \mathbb{R}^d$ satisfies that, for every $p \in \mathbb{R}^d$:

$$\langle h_i(x, \xi), p \rangle = \xi \circ d\varphi_i(x)^{-1}(p).$$

If $\varphi_i(\Omega_i) \neq \emptyset$, consider the function $G_i = F \circ \Phi_i^{-1} : \mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d \to \mathbb{R}$. In order to apply Theorem 3.1 to the function $G_i$ note that the following conditions hold:

(A) $G_i(0, z, 0) = F(0, \varphi_i^{-1}(z), 0) \leq 0$, for each $z \in \varphi_i(\Omega_i)$.

(B) For each compact subset $H$ of $\varphi_i(\Omega_i)$, there exists $\alpha_H > 0$ such that the set

$$B(H; \alpha_H) = \big\{(u, z, p) \in [0, \alpha_H] \times H \times \mathbb{R}^d : G_i(u, z, p) \leq 0\big\}$$

is compact in $\mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d$.

Therefore, taking into account Remark 3.2, we obtain that there exists a differentiable function $v_i : \varphi_i(W_i) \to \mathbb{R}$ such that:

1. $v_i \mid_{\varphi_i(\Omega_i)^c} \equiv 0$ and $\nabla v_i \mid_{\varphi_i(\Omega_i)^c} \equiv 0$;
2. $G_i\big(v_i(z), z, \nabla v_i(z)\big) \leq 0$ for every $z \in \varphi_i(\Omega_i)$.
3. $G_i\big(v_i(z), z, \nabla v_i(z)\big) = 0$ for almost every $z \in \varphi_i(\Omega_i)$.

Then the function $u_i = v_i \circ \varphi_i : W_i \to \mathbb{R}$ is differentiable on $W_i$, and for each $x \in W_i$ we have that

$$\begin{aligned} F\big(u_i(x), x, du_i(x)\big) &= F\big(u_i(x), x, dv_i(\varphi_i(x)) \circ d\varphi_i(x)\big) \\ &= F\big(\Phi_i^{-1}\big(v_i(\varphi_i(x)), \varphi_i(x), \nabla v_i(\varphi_i(x))\big)\big) \\ &= G_i\big(v_i(\varphi_i(x)), \varphi_i(x), \nabla v_i(\varphi_i(x))\big). \end{aligned}$$

As a consequence, we obtain that

1. $u_i \mid_{\Omega_i^c} \equiv 0$ and $\nabla u_i \mid_{\Omega_i^c} \equiv 0$.
2. $F\big(u_i(x), x, du_i(x)\big) \leq 0$ for every $x \in \Omega_i$.
3. $F\big(u_i(x), x, du_i(x)\big) = 0$ for almost every $x \in \Omega_i$.

On the other hand, if If $\varphi_i(\Omega_i) = \emptyset$, we set $u_i = 0$. Now we define $u : M \to \mathbb{R}$ by setting $u = u_i$ on each $T_i$. Then $u$ is well-defined, since $\partial T_i \subset \Omega_i^c$ for each $i \in I$. Taking into account that the family $\{T_i\}_{i \in I}$ is locally finite, we see that $u$ is differentiable on $M$, and it satisfies the required conditions.

**General Case:** In general, consider the continuous function $G : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ defined by:

$$G(u, x, \eta) = F\big(u + u_0(x), x, \eta + du_0(x)\big).$$

We know that, or each $x_0 \in \Omega$, there exist a compact neighborhood $V^{x_0}$ in $\Omega$ and $\alpha > 0$, such that

$$B_F(V^{x_0};\alpha) = \big\{(u,x,\xi) \in [0,\alpha] \times T^*\Omega : \, x \in V^{x_0};\, F(u+u_0(x),x,\xi) \leq 0\big\}$$

is compact in $\mathbb{R} \times T^*\Omega$. Since the mapping $\tau : \mathbb{R} \times T^*\Omega \to \mathbb{R} \times T^*\Omega$ given by

$$\tau(u,x,\xi) = (u,x,\xi - du_0(x))$$

is continuous, we have that the set

$$\begin{aligned} B_G(V^{x_0};\alpha) &= \big\{(u,x,\eta) \in [0,\alpha] \times T^*\Omega : \, x \in V^{x_0};\, G(u+u_0(x),x,\eta) \leq 0\big\} \\ &= \tau\big(B_F(V^{x_0};\alpha)\big) \end{aligned}$$

is compact in $\mathbb{R} \times T^*\Omega$. Thus by the first case we obtain that there exists a differentiable function $v : M \to \mathbb{R}$ such that:

1. $v \,|_{\Omega_0^c} \equiv 0$ and $dv \,|_{\Omega_0^c} \equiv 0$.
2. $F(v(x),x,dv(x)) \leq 0$ for every $x \in \Omega$.
3. $F(v(x),x,dv(x)) = 0$ for almost every $x \in \Omega$.

Now it is easy to see that the function $u = u_0 + v$ satisfies the required properties. ■

Our next Corollary, which is analogous to Corollary 3.4, is an easy consequence of Theorem 5.1.

**Corollary 5.2.** *Let $M$ be a smooth manifold of dimension $d \geq 2$, consider an open subset $\Omega$ of $M$, and let $F : T^*\Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(*A*) *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$ such that $u_0(x) + F\big(x, du_0(x)\big) \leq 0$, for every $x \in \Omega$.*

(*B*) *For each $x_0 \in \Omega$, there exists a compact neighborhood $V^{x_0}$ in $\Omega$ such that the set $B(V^{x_0}) = \big\{(x,\xi) \in T^*M : x \in V^{x_0};\, u_0(x) + F(x,\xi) \leq 0\big\}$ is compact in $T^*M$.*

*Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:*

1. *$u \geq u_0$ on $M$, $u = u_0$ on $\Omega^c$ and $du = du_0$ on $\Omega^c$.*
2. *$u(x) + F\big(x, du(x)\big) \leq 0$ for every $x \in \Omega$.*
3. *$u(x) + F\big(x, du(x)\big) = 0$ for almost every $x \in \Omega$.*

Also as a consequence of Theorem 5.1 we obtain the following result, which extends Theorem 1.3:

**Theorem 5.3.** *Let $M$ be a smooth manifold of dimension $d \geq 2$, consider an open subset $\Omega$ of $M$, and let $F : T^*\Omega \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function. Suppose that the following conditions hold:*

- $(A)$ *There exists a $\mathcal{C}^1$-smooth function $u_0 : M \to \mathbb{R}$ such that $F\big(x, du_0(x)\big) \leq 0$, for every $x \in \Omega$.*
- $(B)$ *For each $x \in \Omega$, the set $B(x) = \big\{\xi \in T^*_x M : F(x,\xi) \leq 0\big\}$ is compact, the set $S(x) = \big\{\xi \in T^*_x M : F(x,\xi) = 0\big\}$ is connected, and the function $F(x,\cdot)$ has maximal rank on the set $S(x)$.*

*Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:*

1. *$u \geq u_0$ on $M$, $u = u_0$ on $\Omega^c$ and $du = du_0$ on $\Omega^c$.*
2. *$F\big(x, du(x)\big) \leq 0$ for every $x \in \Omega$.*
3. *$F\big(x, du(x)\big) = 0$ for almost every $x \in \Omega$.*

**Proof.** We are going to see that the conditions of Theorem 5.1 are satisfied. Fix $x_0 \in \Omega$, and consider a chart $(W, \varphi)$ in $M$ with $x_0 \in W$. Associated to this chart, consider as before the natural diffeomorphism

$$\Phi : T^*W \to \varphi(W) \times \mathbb{R}^d$$

of the form $\Phi(x,\xi) = \big(\varphi(x), h(x,\xi)\big)$, where $h(x,\xi) \in \mathbb{R}^d$ satisfies that, for every $p \in \mathbb{R}^d$:

$$\langle h(x,\xi), p\rangle = \xi \circ d\varphi(x)^{-1}(p).$$

Denote $z_0 = \varphi(x_0)$. We take into account that $\Phi(S(x_0))$ is compact, and that $F \circ \Phi^{-1}$ has maximal rank on $\{z_0\} \times \Phi(S(x_0))$, and we apply the Implicit Function Theorem. Then we can find a neighborhood $U^{z_0}$ contained in $\varphi(W)$ and a finite family $V_1, \dots V_m$ of open subsets of $\mathbb{R}^d$ with compact closure such that $\Phi(S(x_0)) \subset V_1 \cup \dots \cup V_m$ and, for each $j = 1, \dots, m$, the set of points $(z,p) \in U^{z_0} \times V_j$ satisfying $F \circ \Phi^{-1}(z,p) = 0$ coincides, up to a permutation in the coordinates of $p$, with the graph of a $\mathcal{C}^1$-smooth mapping $g_j : U^{z_0} \times W_j \to \mathbb{R}$, where $W_j$ is an open subset of $\mathbb{R}^{d-1}$.

We claim that there exists a compact neighborhood $V^{x_0}$ such that, for every $x \in V^{x_0}$, we have that $\Phi(S(x)) \subset V_1 \cup \dots \cup V_m$. Indeed, if this is not the case, there exist a sequence $(z_n)_n \subset U^{z_0}$ converging to $z_0$ and a sequence $(p_n)_n \subset (V_1 \cup \dots \cup V_m)^c$ such that $F \circ \Phi^{-1}(z_n, p_n) = 0$ for every $n$. Since each

$S(x_n)$ is connected and $\Phi(S(x_n)) \cap (V_1 \cup \cdots \cup V_m) \neq \emptyset$ for every $n$, we can assume that, in fact, $(p_n)_n \subset \partial(V_1 \cup \cdots \cup V_m)$, which is a compact set. Then, taking a subsequence, we can assume that $(p_n)_n$ is convergent to some point $p_0 \in \partial(V_1 \cup \cdots \cup V_m)$. Now $F \circ \Phi^{-1}(z_0, p_0) = \lim_n F \circ \Phi^{-1}(z_n, p_n) = 0$, that is, $p_0 \in \Phi(S(x_0))$, and this contradicts the fact that $\Phi(S(x_0)) \subset V_1 \cup \cdots \cup V_m$. Then there exists $R > 0$ such that $\Phi(S(x)) \subset B(0, R)$ for every $x \in V^{x_0}$. Since $\Phi(S(x))$ is the boundary of $\Phi(B(x))$ we have that in fact $\Phi(B(x)) \subset B(0, R)$ for every $x \in V^{x_0}$. Thus the set

$$B(V^{x_0}) = \big\{(x, \xi) \in T^*\Omega : \, x \in V^{x_0}; \, F(x, \xi) \leq 0\big\}$$

is compact in $T^*\Omega$, and the requirements of Theorem 5.1 are satisfied. ■

In our next result we consider a Riemannian manifold $(M, g)$. As we mentioned before, if $u : M \to \mathbb{R}$ is differentiable, for every $x \in M$ we identify in the usual way the differential $du(x)$ with the gradient $\nabla u(x)$ by means of the scalar product $g_x(\cdot, \cdot)$ on the tangent space $T_xM$. In this case we obtain the following extension of Theorem 1.4:

**Theorem 5.4.** *Let $(M, g)$ be a Riemannian manifold of dimension $d \geq 2$, consider an open subset $\Omega$ of $M$, and let $F : T\Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

- ($A$) *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$, such that $F\big(x, \nabla u_0(x)\big) \leq 0$, for every $x \in \Omega$.*
- ($B$) *There exists a locally bounded function $\rho : \Omega \to (0, \infty)$ such that, for every $x \in \Omega$, the set $B(x) = \big\{v \in T_xM : F(x, v) \leq 0\big\}$ is contained in the ball of center 0 and radius $\rho(x)$ in $T_xM$.*

*Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:*

1. *$u \geq u_0$ on $M$, $u = u_0$ on $\Omega^c$ and $\nabla u = \nabla u_0$ on $\Omega^c$.*
2. *$F\big(x, \nabla u(x)\big) \leq 0$ for every $x \in \Omega$.*
3. *$F\big(x, \nabla u(x)\big) = 0$ for almost every $x \in \Omega$.*

**Proof.** We are going to see that the conditions of Theorem 5.1 are satisfied. Fix $x_0 \in \Omega$ and consider a chart $(W, \varphi)$ in $M$ with $x_0 \in W$. Associated to this chart, consider the natural diffeomorphism

$$\Phi : TW \to \varphi(W) \times \mathbb{R}^d$$

given by $\Phi(x, v) = (\varphi(x), d\varphi(x)(v))$. Choose a compact neighborhood $V^{x_0}$ of $x_0$ contained in $W$ and $R > 0$ such that for every $x \in V^{x_0}$ the set

$B(x) = \big\{v \in T_xM : F(x,v) \leq 0\big\}$ is contained in the closed ball of center 0 and radius $R$ in $T_xM$. Now set

$$r = \inf \big\{\|v\|_x : x \in V^{x_0};\ \|d\varphi(x)(v)\|_{\mathbb{R}^d} = 1\big\}.$$

By compactness, it is clear that $r > 0$. For every $x \in V^{x_0}$ and every $v \in T_xM$, we have

$$\|d\varphi(x)(v)\|_{\mathbb{R}^d} \leq \frac{1}{r}\,\|v\|_x.$$

Therefore

$$B(V^{x_0}) = \{(x,v) \in TM : x \in V^{x_0};\ F(x,v) \leq 0\}$$

$$\subset \{(x,v) \in TM : x \in V^{x_0};\ \|v\|_x \leq R\} \subset \Phi^{-1}\Big(\varphi(V^{x_0}) \times \overline{B}\Big(0;\frac{R}{r}\Big)\Big).$$

It follows that $B(V^{x_0})$ is a compact subset of $TM$. ■

**Corollary 5.5.** *Let $(M,g)$ be a Riemannian manifold of dimension $\geq 2$ and let $\Omega$ be an open subset of $M$. Then there exists a differentiable function $u : M \to \mathbb{R}$ such that $u\,|_{\Omega^c} \equiv 0$ and $\|\nabla u(x)\|_x = 1$ for almost every $x \in \Omega$.*

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