

# A note on boundaries of open polynomial images of $\mathbb{R}^2$

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## Abstract

We construct a family of polynomial maps  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that their images are open semialgebraic sets whose topological exteriors have arbitrarily many connected components, which are parametric semilines.

## 1. Introduction

The problem of deciding whether or not a given subset of  $\mathbb{R}^n$  is the image of an euclidean space  $\mathbb{R}^m$  under a polynomial map  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  has revealed as a difficult task. Such problem was proposed by J. M. Gamboa in the 90's in an *Oberwolfach* week (see [3]) and has been mainly studied in [1] and [2]. In [1] the authors found some “initial” obstructions to be a polynomial image, which allowed them to discard many sets as polynomial images.

Such obstructions appear by using some relevant results about the set of points where a real polynomial map is not proper (see [4]). Moreover, they also gave some positive constructive answers for certain relevant sets. It is worthwhile mentioning the construction of a polynomial map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  whose image is the open quadrant

$$\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

That article has a continuation in [2] where the authors develop a more systematic approach and, among other things, find new properties that must be satisfied by those sets that are polynomial images of some  $\mathbb{R}^n$  [Thm. 3.1 and Cor. 3.4]. From these results it follows, in particular, that each open convex unbounded polygon with  $e \geq 3$  nonparallel linear sides  $S \subseteq \mathbb{R}^2$  is

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a polynomial image of  $\mathbb{R}^{2e-3}$  but it is not a polynomial image of  $\mathbb{R}^2$ . Furthermore, some open questions are proposed there. In particular Question in [2, 7.3.2] asks:

- (\*) *Is there a bound for the number of connected components of the topological exterior  $\mathbb{R}^2 \setminus S$  of a polynomial image  $S \subset \mathbb{R}^2$  of  $\mathbb{R}^2$ ?*

The main aim of this work is to give a negative answer to this question, and in fact we construct a family of polynomial maps whose images are open sets whose topological exterior has an arbitrarily large amount of connected components. More precisely, we will prove the following result in Section 2:

**Theorem 1.1** *For every positive integer  $n$  there exists a polynomial map  $p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  whose image  $S$  is open and has  $n + 1$  boundary connected components, which are parametric semilines. Moreover, the exterior of  $S$  has also  $n + 1$  connected components.*

Surprisingly, all the known examples of open polynomial images of  $\mathbb{R}^2$  presented in [1] and [2] had connected exterior. This is why the authors of [2] wondered if the semialgebraic set

$$R = \{(x, y) \in \mathbb{R}^2 : 0 < y < x^2 + 1\},$$

whose exterior has two connected components, is a polynomial image of  $\mathbb{R}^2$ . We answer this in the affirmative in Proposition 1. Indeed, the construction of a polynomial map whose image is  $R$  will be the first step to prove Theorem 1.1, which will be done in Section 2. The paper ends with some natural questions which appear when one tries to generalize the construction developed to prove Theorem 1.1.

## 2. Proof of the main result

Let us consider the following polynomial maps from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ :

$$f(x, y) = (y + (y^2 + 1)x, (xy + 1)^2 + x^2) \quad ; \quad \phi(x, y) = (x + y, y).$$

Note that  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a homeomorphism keeping invariant the upper half-plane

$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Moreover, let us consider the open region in  $\mathbb{R}^2$  defined as

$$R = \{(x, y) \in \mathbb{R}^2 : 0 < y < x^2 + 1\}.$$

Then we have the following result:

**Lemma 2.1** *The restriction map  $f|_H : H \rightarrow R \subset H$  is a homeomorphism between open subsets in  $\mathbb{R}^2$ .*

**Proof.** By the domain invariance theorem it is enough to show that the continuous map  $f|_H$  is bijective. Pick a point  $(a, b) \in \mathbb{R}^2$  and write

$$a = y + (y^2 + 1)x \quad \text{and} \quad b = (1 + xy)^2 + x^2.$$

It is clear that  $b$  must be  $> 0$ . Moreover, we have

$$a^2 + 1 - b = y^2((1 + xy)^2 + x^2) > 0$$

whenever  $y > 0$  is satisfied. This proves that  $f(H) \subset R$ . Conversely, given  $(a, b) \in R$ , it can be checked that there is a unique point  $(x, y) \in H$  such that  $f(x, y) = (a, b)$ , namely

$$y = +\sqrt{\frac{a^2 + 1 - b}{b}}, \quad x = \frac{a - y}{y^2 + 1}.$$

From these facts we conclude that  $f$  is a bijection between  $H$  and  $R$ , and the lemma follows. ■

As a consequence we answer in the next Proposition the second part of Question [2, 7.3.2].

**Proposition 2.2** *The open set  $R \subset \mathbb{R}^2$  is a polynomial image of  $\mathbb{R}^2$ .*

**Proof.** As it is claimed in [1] the half-plane  $H$  is the image of the polynomial map

$$g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto (y(xy - 1), (xy - 1)^2 + x^2)$$

Thus,  $R$  is the image of  $\mathbb{R}^2$  under the composite map  $f \circ g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . ■

In what follows, and in order to simplify the notation, whenever we write  $f$  we will be referring to the polynomial homeomorphism  $f|_H : H \rightarrow R$ . Moreover, we will use in the sequel the polynomial homeomorphism

$$F = \phi \circ f : H \rightarrow \phi(R) \subset H,$$

between  $H$  and the affine deformation  $\phi(R)$  of  $R$ .

**Lemma 2.3** *Let us consider the open semialgebraic subsets of  $\mathbb{R}^2$  defined as*

$$M = \{(x, y) \in \mathbb{R}^2 : -2y < x(y^2 + 1) < 0\}, \quad N = \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$$

$$P = \{(x, y) \in \mathbb{R}^2 : x^2 + 1 < 2y < 2(x^2 + 1)\}, \quad Q = \{(x, y) \in \mathbb{R}^2 : x < 0 < y\}$$

(see figure 1). The polynomial map  $F$  satisfies the following properties:

- (1)  $F(M) = N$ ;
- (2)  $F(N) = \phi(P)$ ;
- (3)  $M \subset Q \subset F(Q)$ .

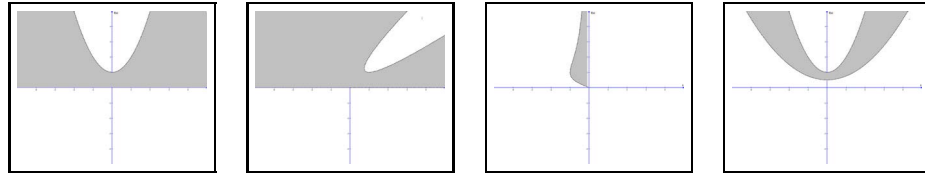


Figure 1. The open regions  $R$ ,  $\phi(R)$ ,  $M$  and  $P$ .

**Proof.** (1) To prove the first assertion, we observe that

$$\begin{aligned} f^{-1}(N) &= \{(x, y) \in H : (xy + 1)^2 + x^2 < 1\} \\ &= \{(x, y) \in H : x(xy^2 + x + 2y) < 0\}. \end{aligned}$$

Note that if  $(x, y) \in f^{-1}(N)$ , then  $y > 0$  and so  $x < 0$ . Thus, we deduce that

$$f^{-1}(N) = \{(x, y) \in \mathbb{R}^2 : -2y < x(y^2 + 1), x < 0\} = M,$$

and so  $f(M) = ff^{-1}(N) = N$ , since  $f : H \rightarrow R$  is surjective. Now, since  $\phi(N) = N$ , we conclude that  $F(M) = N$ .

(2) For the second assertion, it is enough to show that  $f(N) = P$ . Denote  $\ell = \{(x, y) \in \mathbb{R}^2 : y = 1\}$ , and observe that its complementary set  $H \setminus \ell$  has two connected components  $N$  and  $H \setminus \overline{N}$ . This line is mapped by  $f$  onto the parabola

$$f(\ell) = \{(1 + 2t, (t + 1)^2 + t^2) : t \in \mathbb{R}\} = \{(x, y) \in \mathbb{R}^2 : x^2 + 1 = 2y\}$$

Since  $f : H \rightarrow R$  is a homeomorphism (see 2.1), the set  $R \setminus f(\ell)$  has two connected components, namely  $P$  and  $R \setminus \overline{P}$ , and necessarily we have  $f(N) = P$ , since for instance  $f(0, 1/2) = (1/2, 1) \in P$ .

(3) Finally, to check the third assertion we observe that  $Q \setminus M$  and  $H \setminus Q$  are the connected components of  $H \setminus M$  and,  $f : H \rightarrow R$  being a homeomorphism, they are respectively mapped by  $f$  to the two connected components

$$R_1 = \{(x, y) \in R : y \geq 1, x < 0\} \quad \text{and} \quad R_2 = \{(x, y) \in R : y \geq 1, x > 0\}$$

of  $f(H \setminus M) = f(H) \setminus f(M) = R \setminus N$ . From this we get

$$f(Q) = f((Q \setminus M) \cup M) = f(Q \setminus M) \cup f(M) = R_1 \cup N.$$

Moreover, since

$$\{(x, y) \in \mathbb{R}^2 : 0 < y < -x\} \subset \{(x, y) \in R : x < 0\} \subset R_1 \cup N$$

and  $\phi(\{(x, y) \in \mathbb{R}^2 : 0 < y < -x\}) = Q$ , we conclude that

$$Q \subset \phi(\{(x, y) \in R : x < 0\}) \subset (\phi \circ f)(Q) = F(Q),$$

and we are done. ■

Let us recall now some properties about the behaviour of the boundary operator. Given a topological space  $X$  and two subsets  $C \subset A \subset X$ , we will denote by  $\partial C$  the boundary of  $C$  as a subset of  $X$  and by  $\partial^A C$  its boundary as a subset of  $A$ . As one can check straightforwardly we have the following:

**Lemma 2.4** *Let  $X, Y$  be two topological spaces and let  $A \subset X$  and  $B \subset Y$  be two subsets. Let  $G : A \rightarrow B$  be a homeomorphism between  $A$  and  $B$  and consider a subset  $C \subset A$ . Then:*

$$(1) \partial^A C \subset \partial C \subset \partial^A C \cup \partial A.$$

$$(2) G(\partial^A C) = \partial^B G(C).$$

**(2.5) Auxiliary curves.** For later purposes we introduce now the following sequence of semialgebraic curves in the real plane:

$$\begin{aligned} \Gamma_0 &= \{(x, y) \in \mathbb{R}^2 : x = t, y = 0, t \in \mathbb{R}\}, \\ \Gamma_1 &= \phi(\{(x, y) \in \mathbb{R}^2 : x = t, y = t^2 + 1, t \in \mathbb{R}\}), \\ &\vdots \\ \Gamma_n &= F(\Gamma_{n-1}), \quad n > 1. \end{aligned}$$

We claim that: *These curves are connected, pairwise disjoint and closed in  $\mathbb{R}^2$ .*

Note that all these curves are parametric semilines. Recall that a *parametric semiline* is the image of the real line under a nonconstant polynomial map, that is, a one dimensional semialgebraic set which admits a polynomial parametrization (see [4]).

**Proof of 2.5.** We will use an inductive argument on  $n$  to show that they are pairwise disjoint. For  $n = 1$  the statement is obvious. Assume that  $\Gamma_0, \Gamma_1, \dots, \Gamma_n$  are pairwise disjoint and let us see that they are also disjoint to  $\Gamma_{n+1}$ .

First note that  $F : H \rightarrow \phi(R)$  is a homeomorphism (see 2.1) and that for  $2 \leq k \leq n$  we have by the induction hypothesis that  $\Gamma_{k-1} \cap \Gamma_n = \emptyset$ . Therefore

$$\Gamma_k \cap \Gamma_{n+1} = F(\Gamma_{k-1}) \cap F(\Gamma_n) = F(\Gamma_{k-1} \cap \Gamma_n) = F(\emptyset) = \emptyset.$$

On the other hand notice that  $\Gamma_{n+1} \subset \phi(R) \subset H$  does not meet neither  $\Gamma_0 \subset \mathbb{R}^2 \setminus H$  nor  $\Gamma_1 \subset H \setminus \phi(R)$ .

Finally, observe that each  $\Gamma_j$  is a parametric semiline, and so it is a closed subset of  $\mathbb{R}^2$  (see [4]). ■

**Proposition 2.6** *The map  $F^n : H \rightarrow F^n(H) \subset \mathbb{R}^2$  is a homeomorphism and  $\partial F^n(H)$  has  $n+1$  connected components, which are the curves  $\Gamma_0, \Gamma_1, \dots, \Gamma_n$ .*

**Proof.** We will prove by induction on  $n$  that

$$\partial(F^n(H)) = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_n.$$

For  $n = 1$  the result follows from Lemma 2.1, since

$$\partial F(H) = \partial\phi(R) = \phi(\partial R) = \Gamma_0 \cup \Gamma_1.$$

Now, let us suppose the proposition is true for  $n > 1$ , so that  $\partial F^n(H) = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_n$ , and let us see that the result is also true for  $n + 1$ . On the one hand, by 2.3, we have

$$H \supset F^{n+1}(H) \supset F(F^n(Q)) \supset F(Q) \supset F(M) = N,$$

and this implies that  $\partial(F^{n+1}(H)) \supset \Gamma_0$ . On the other hand,

$$\begin{aligned} \phi(R) &= F(H) \supset F^{n+1}(H) = F^2(F^{n-1}(H)) \supset F^2(F^{n-1}(Q)) \\ &\supset F^2(Q) = F(F(Q)) \supset F(F(M)) = F(N) = \phi(P), \end{aligned}$$

so that  $\partial(F^{n+1}(H)) \supset \Gamma_1$ . Since  $F$  is a homeomorphism between  $H$  and  $\phi(R)$ , by 2.4 applied to  $F^n(H)$  we obtain

$$\partial^{\phi(R)}(F^{n+1}(H)) \subset \partial(F^{n+1}(H)) \subset \partial^{\phi(R)}(F^{n+1}(H)) \cup \partial(\phi(R))$$

and, taking into account that  $\partial(\phi(R)) = \Gamma_0 \cup \Gamma_1 \subset \partial F^{n-1}(H)$ , we finally get

$$\begin{aligned} \partial(F^{n+1}(H)) &= \partial^{\phi(R)}(F^{n+1}(H)) \cup \Gamma_0 \cup \Gamma_1 = F(\partial^H(F^n(H))) \cup \Gamma_0 \cup \Gamma_1 \\ &= F(\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n) \cup \Gamma_0 \cup \Gamma_1 = (\Gamma_2 \cup \Gamma_3 \cup \cdots \cup \Gamma_{n+1}) \cup \Gamma_0 \cup \Gamma_1, \end{aligned}$$

as claimed. ■

Now we are ready to show open polynomial images of the real plane whose exteriors have as many connected components as desired.

**(2.7) Auxiliary regions.** To that end we define, with the notations above, the regions in  $\mathbb{R}^2$

$$\begin{aligned} \Delta_0 &= \{(x, y) \in \mathbb{R}^2 : y < 0\}, \\ \Delta_1 &= \phi(\{(x, y) \in \mathbb{R}^2 : y > x^2 + 1\}), \\ &\vdots \\ \Delta_n &= F(\Delta_{n-1}), \quad n > 1. \end{aligned}$$

All these regions are connected open subspaces of  $\mathbb{R}^2$  and pairwise disjoint. This last fact can be proved, by induction on  $n$ , using a similar argument to the one developed in 2.5 to prove that the curves  $\Gamma_0, \Gamma_1, \dots, \Gamma_n$  are pairwise disjoint. Namely, for  $n = 1$  we have that  $\Delta_0$  and  $\Delta_1$  do not intersect, and

assuming that the sets  $\{\Delta_i\}_{0 \leq i \leq n}$  are pairwise disjoint, we have for  $2 \leq k \leq n$ , by the induction hypothesis, that

$$\Delta_k \cap \Delta_{n+1} = F(\Delta_{k-1}) \cap F(\Delta_n) = F(\Delta_{k-1} \cap \Delta_n) = F(\emptyset) = \emptyset.$$

Since also  $\Delta_{n+1} \subset \phi(R) \subset H$  does not meet  $\Delta_0 \subset \mathbb{R}^2 \setminus H$  nor  $\Delta_1 \subset H \setminus \phi(R)$ , we get that  $\Delta_{n+1}$  does not intersect any  $\Delta_i$ ,  $0 \leq i \leq n$ .

**Proposition 2.8** *The exterior of the image of the polynomial map  $F^n : H \rightarrow F^n(H) \subset \mathbb{R}^2$  is*

$$\mathbb{R}^2 \setminus \overline{F^n(H)} = \Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_n.$$

**Proof.** We use again induction on  $n$ . For  $n = 1$  the statement is clear. Let us suppose now that it is also true for  $n$ . Then, for  $n + 1$  we have

$$\begin{aligned} \mathbb{R}^2 \setminus \overline{F^{n+1}(H)} &= (\overline{F(H)} \cup \Delta_0 \cup \Delta_1) \setminus \overline{F^{n+1}(H)} \\ &= \Delta_0 \cup \Delta_1 \cup (\overline{F(H)} \setminus \overline{F^{n+1}(H)}) = \Delta_0 \cup \Delta_1 \cup F(H \setminus \overline{F^n(H)}) \\ &= \Delta_0 \cup \Delta_1 \cup F(\Delta_1 \cup \dots \cup \Delta_n) = \Delta_0 \cup \Delta_1 \cup (\Delta_2 \cup \dots \cup \Delta_{n+1}), \end{aligned}$$

as needed. The identity  $\overline{F(H)} \setminus \overline{F^{n+1}(H)} = F(H \setminus \overline{F^n(H)})$  is straightforwardly deduced from 2.6. Indeed,

$$\begin{aligned} \overline{F(H)} \setminus \overline{F^{n+1}(H)} &= \left( F(H) \cup \Gamma_0 \cup \Gamma_1 \right) \setminus \left( F^{n+1}(H) \cup \Gamma_0 \cup \Gamma_1 \cup \bigcup_{k=2}^{n+1} \Gamma_k \right) \\ &= F(H) \setminus \left( F^{n+1}(H) \cup \bigcup_{k=1}^n F(\Gamma_k) \right) = F\left( H \setminus \left( F^n(H) \cup \bigcup_{k=1}^n \Gamma_k \right) \right) \\ &= F\left( H \setminus \left( F^n(H) \cup \bigcup_{k=0}^n \Gamma_k \right) \right) = F(H \setminus \overline{F^n(H)}), \end{aligned}$$

and we are done. ■

Finally from the previous result we conclude Theorem 1.1 which was stated in the Introduction and provides a negative answer to Question [2, 7.3.2].

**Proof of Theorem 1.1** Recall that  $g$  denotes the polynomial map

$$g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto (y(xy - 1), (xy - 1)^2 + x^2)$$

which maps  $\mathbb{R}^2$  onto the open half-plane  $H : y > 0$ . Now, by 2.6 and 2.8, we have that the image of the polynomial map  $p = F^n \circ g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is open and has as boundary the set  $\Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_n$  and as exterior the set  $\Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_n$ . ■

**(2.9) Open questions.** As we have already pointed out it is well-known that determining the open subsets of  $\mathbb{R}^2$  which are images of polynomial mappings  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a challenging problem. We propose here two questions that are particular cases of this problem and which naturally appear when one tries to generalize the construction developed above.

**Question 1:** Let  $A$  be an open polynomial image of  $\mathbb{R}^2$  and let  $\phi_1, \phi_2, \dots, \phi_n$  be bijective affine transformations of the real plane so that the open set

$$G = \phi_1(A) \cup \phi_2(A) \cup \dots \cup \phi_n(A)$$

is connected. Is there a polynomial map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  whose image is  $G$ ? Note that [1, Corollary 4.1] answers this question in the affirmative for  $n = 2$  and  $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ .

**Question 2:** Let  $B$  be a closed polynomial image of  $\mathbb{R}^2$  and let  $\phi_1, \phi_2, \dots, \phi_n$  be bijective affine transformations of the real plane such that the closed semialgebraic sets  $\phi_1(B), \phi_2(B), \dots, \phi_n(B)$  are pairwise disjoint. Is there a polynomial map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  whose image is the open set

$$\mathbb{R}^2 \setminus (\phi_1(B) \cup \phi_2(B) \cup \dots \cup \phi_n(B)) ?$$

In particular, it seems interesting to consider the case of the set

$$B = \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}.$$

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