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A new type of solutions for a singularly perturbed elliptic Neumann problem

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Abstract

We prove the existence of positive solutions concentrating simultaneously on some higher dimensional manifolds near and on the boundary of the domain for a nonlinear singularly perturbed elliptic Neumann problem.

1. Introduction

The aim of this paper is to construct solutions concentrating on some higher dimensional manifolds for the following singularly perturbed elliptic problem:

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^{p-1}, \ u > 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small number, Ω is an open domain and n is the outward unit normal of $\partial\Omega$ at $y \in \partial\Omega$.

We assume that Ω is a domain in $\mathbb{R}^N$, whose boundary is Lipschitz continuous, and satisfies the following condition:

(Ω_1): there is an integer m, $1 < m < N$, such that $y \in \Omega$, if and only if $(|y'|, y'') \in D$, where $y = (y', y'')$, $y' \in \mathbb{R}^m$, $y'' \in \mathbb{R}^{N-m}$, D is a relatively open domain in $\mathbb{R}^{N-m+1}_+$, and

$$\mathbb{R}^{N-m+1}_+ = \big\{ z = (z_1, \ldots, z_{N-m+1}) : z_1 \geq 0 \big\}.$$

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In this paper, we do not assume that Ω is bounded. The domain Ω can be a bounded domain, or an exterior domain in $\mathbb{R}^N$, or many other unbounded domains.

We assume that p satisfies

$$p \in (2, 2(N-m+1)/(N-m-1)) \quad \text{if } m < N-1,$$
$$p \in (0, +\infty) \quad \text{if } m \geq N-1.$$

In view of the assumption on Ω, we will work on the following subspace of $H^1(\Omega)$:

$$H_s = \big\{ u : u \in H^1(\Omega), u(y) = u(|y'|, y'')\big\}.$$

Let U be the unique solution of the following problem:

$$\begin{cases} -\Delta v + v = v^{p-1}, \ v > 0, & \text{in } \mathbb{R}^{N-m+1}, \\ v(0) = \max_{z\in\mathbb{R}^{N-m+1}} v(z), & v \in H^1(\mathbb{R}^{N-m+1}). \end{cases}$$

Then $U(z) = U(|z|)$, $U' < 0$,

$$|z|^{(N-m)/2} e^{|z|} U(|z|) \to c > 0, \ |z| \to +\infty,$$

and $U(z)$ also satisfies

$$\begin{cases} -\Delta v + v = v^{p-1}, \ v > 0, & \text{in } \mathbb{R}^{N-m+1}_+, \\ v(0) = \max_{z\in\mathbb{R}^{N-m+1}_+} v(z), & v \in H^1(\mathbb{R}^{N-m+1}_+), \\ \frac{\partial v}{\partial x_{N-m+1}} = 0, & x \in \partial\mathbb{R}^{N-m+1}_+. \end{cases}$$

Furthermore, $U(z)$ is nondegenerate. That is, the kernel of the operator $-\Delta w + w - (p-1)U^{p-2}w$ in $H^1(\mathbb{R}^{N-m+1})$ is spanned by

$$\left\{ \frac{\partial U(z)}{\partial z_i}, i = 1, \cdots, N-m+1 \right\},$$

and the kernel of the operator $-\Delta w + w - (p-1)U^{p-2}w$ in $H^1(\mathbb{R}^{N-m+1}_+)$ with Neumann boundary condition $\frac{\partial w}{\partial x_{N-m+1}} = 0$ is spanned by

$$\left\{ \frac{\partial U(z)}{\partial x_i}, i = 1, \cdots, N-m \right\}.$$

See [15, 23].

For any $y = (y', y'') \in \mathbb{R}^N$, $y' \in \mathbb{R}^m$, $y'' \in \mathbb{R}^{N-m}$, we denote $\tilde{y} = (|y'|, y'') \in \mathbb{R}^{N-m+1}$. Let $\bar{W}(y) = U(\tilde{y})$. For any $\bar{x} \in D$, let $\bar{W}_{\varepsilon,\bar{x}}(y) = U(|\tilde{y} - \bar{x}|/\varepsilon)$. Then, $\bar{W}_{\varepsilon,\bar{x}}$ satisfies

$$(1.2) \quad -\varepsilon^2 \Delta \bar{W}_{\varepsilon,\bar{x}} + \bar{W}_{\varepsilon,\bar{x}} = \bar{W}^{p-1}_{\varepsilon,\bar{x}} - \varepsilon \frac{m-1}{|y'|} \frac{|y'| - \bar{x}_1}{|\tilde{y} - \bar{x}|} U'\Big(\frac{|\tilde{y} - \bar{x}|}{\varepsilon}\Big), \quad \text{in } \Omega.$$

In this paper, we assume that Ω also satisfies the following condition: (Ω_2): there exists $\bar{x} = (\bar{x}_1, \bar{x}'') \in \partial D$, such that

(i) there is a C^2 function $\psi(z'')$ in $\mathbb{R}^{N-m}$, such that

$$D \cap B_\delta(\bar{x}) = \{z = (z_1, z'') : z_1 < \psi(z'')\} \cap B_\delta(\bar{x}),$$

and

$$\partial D \cap B_\delta(\bar{x}) = \{z = (z_1, z'') : z_1 = \psi(z'')\} \cap B_\delta(\bar{x}),$$

where $\delta > 0$ is a constant;

(ii) $\bar{x}_1 = \psi(\bar{x}'') = \max_{z'' \in B_\delta(\bar{x}'')} \psi(z'') > 0$, and $\bar{x}_1 > \max_{z'' \in \partial B_\delta(\bar{x}'')} \psi(z'')$.

We will prove that for any positive integer pair (k_1, k_2), (1.1) has a solution u_ε, which is close to $\sum_{j=1}^{k_1+k_2} \bar{W}_{\varepsilon,x_j}$ in a small neighbourhood of $|y'| = \bar{x}_1$ with $x_j \in D$ for $j = 1, \dots, k_1$ and $x_j \in \partial D$ for $j = k_1, \dots, k_2$, and is close to zero elsewhere. Since the right hand side of (1.2) has a singularity at $y' = 0$, we truncate $\bar{W}_{\varepsilon,\bar{x}}$ as follows.

Let $\xi \in C_0^\infty(\mathbb{R}^{N-m+1})$ be a function such that $\xi = 0$ if $z_1 \le \kappa$, $\xi = 1$ if $z_1 \ge 2\kappa$, for some small $\kappa > 0$. For any $x_j \in D$ with $x_{j,1} \ge 6\kappa$, define

$$W_{\varepsilon,x_j}(y) = \xi(|y'|, y'')\bar{W}_{\varepsilon,x_j}(y).$$

Then W_{ε,x_j} satisfies

$$-\varepsilon^2 \Delta W_{\varepsilon,x_j} + W_{\varepsilon,x_j} = \xi \bar{W}_{\varepsilon,x_j}^{p-1} + \tilde{f}_{\varepsilon,x_j}(y) \quad \text{in } \Omega, \tag{1.3}$$

where

$$\begin{aligned}\tilde{f}_{\varepsilon,x_j}(y) = &-\xi\varepsilon \frac{m-1}{|y'|} \frac{|y'| - x_{j,1}}{|\tilde{y} - x_j|} U'\Big(\frac{|\tilde{y} - x_j|}{\varepsilon}\Big) - 2\varepsilon D\xi DU\big(\frac{|\tilde{y} - x_j|}{\varepsilon}\big) \\ &- \varepsilon^2 U\Big(\frac{|\tilde{y} - x_j|}{\varepsilon}\Big)\Delta\xi.\end{aligned}$$

Since

$$\frac{z_1 - x_{j,1}}{|z - x_j|}\, U'\Big(\frac{|z - x_j|}{\varepsilon}\Big) = \varepsilon \frac{\partial}{\partial z_1} U\Big(\frac{|z - x_j|}{\varepsilon}\Big),$$

and $\xi = 0$ in a neighbourhood of $|y'| = 0$, it is easy to see that $\tilde{f}_{\varepsilon,x_j}$ is a smooth function in both y and x_j, and satisfies

$$|\tilde{f}_{\varepsilon,x_j}| \le C\varepsilon U\Big(\frac{|\tilde{y} - x_j|}{\varepsilon}\Big).$$

Let $P_{\varepsilon,\Omega}W_{\varepsilon,x_j}$ be the solution of

$$(1.4)\qquad \begin{cases} -\varepsilon^2\Delta v + v = \xi \bar{W}^{p-1}_{\varepsilon,x_j} + \tilde{f}_{\varepsilon,x_j}(y), & \text{in } \Omega \\ \frac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

By the uniqueness, we know that $P_{\varepsilon,\Omega}W_{\varepsilon,x_j} \in H_s$.

Let

$$\langle u, v\rangle_\varepsilon = \int_\Omega (\varepsilon^2 DuDv + uv), \quad \|v\|_\varepsilon = \langle u, v\rangle_\varepsilon^{1/2}.$$

The main result of this paper is the following.

Theorem 1.1. *Assume that $1 < m < N$. Suppose that Ω satisfies (Ω_1) and (Ω_2). Then, for any positive integers k_1 and k_2, there is an $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$, (1.1) has a solution of the form*

$$(1.5)\qquad u_\varepsilon = \sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}} + \omega_\varepsilon,$$

with $\omega_\varepsilon \in H_s$, where for $j = 1, \dots, k_1$, $x_{\varepsilon,j} = (x_{\varepsilon,j,1}, x''_{\varepsilon,j}) \in D$, and as $\varepsilon \to 0$,

$$\frac{d(x_{\varepsilon,j}, \partial D)}{\varepsilon} \to +\infty.$$

For $j = k_1 + 1, \dots, k_1 + k_2$, $x_{\varepsilon,j} = (x_{\varepsilon,j,1}, x''_{\varepsilon,j}) \in \partial D$.

Moreover, for $i, j = 1, \dots, k_1 + k_2$,

$$\frac{|x_{\varepsilon,j} - x_{\varepsilon,i}|}{\varepsilon} \to +\infty, \ \forall\ j \neq i,$$

$$x_{\varepsilon,j} \to \hat{x}_j = (\hat{x}_{j,1}, \hat{x}''_j) \in \partial D \cap B_\delta(\bar{x}), \quad \text{with } \hat{x}_{j,1} = \psi(\hat{x}''_j) = \max_{z'' \in B_\delta(\bar{x}'')} \psi(z''),$$

and

$$\|\omega_\varepsilon\|_\varepsilon^2 = o(\varepsilon^{N-m+1}).$$

In [18, 19], Malchiodi and Montenegro obtained solutions concentrating on higher dimensional subsets of the boundary, which seems to be the first results concerning solutions concentrating on higher-dimensional sets.

For (1.1) with the Dirichlet boundary condition, the results in [7, 10] show that it has a solution concentrating on a manifold near $|y'| = x_1$, where $x_1 > 0$ is a local minimum of the distance of x to $\{z_1 = 0\}$ for $x \in D$.

For the Neumann problem, in [10], Dance and Yan constructed solutions concentrating on higher dimensional subsets inside the domain and on the boundary of the domain separately, and all the manifolds are close to $|y'| = \hat{x}_1$, where $\hat{x}_1 > 0$ is a local maximum of the distance of x to $\{z_1 = 0\}$ for $x \in D$.

For problem (1.1) with potential functions $V(x)$ and $K(x)$ multiplying the linear term u and nonlinear term u^{p-1} respectively, Ambrosetti, Malchiodi and Ni [1, 2] extended [18] to higher-dimensional spike-layers for problem (1.1) with $K(|x|) \equiv 1$, $\Omega = \mathbb{R}^N$ and $V(x)$ being radially symmetric.

Also, in [3, 4], Bartsch and Peng show that (1.1) has a solution concentrating on multi-dimensional subsets inside the domain, moreover, these results are true for both Dirichlet and Neumann boundary conditions.

For a survey of this kind of results we can also refer to [17].

Our result here shows that (1.1) has solutions concentrating simultaneously on several higher dimensional interior and boundary manifolds.

There are many works in the case $m = 1$ since the pioneering works [22, 23]. See for example [5, 8, 9, 11, 12, 13, 14, 16, 24, 25]. To obtain the results mentioned above for the case $m = 1$, no symmetry condition is imposed on the domain Ω.

In the case $m > 1$, we use the solution U of a lower dimensional problem as an approximate solution for problem (1.1). So, there is no control in some directions for the corresponding linear operator

$$L_\varepsilon v =: -\varepsilon^2 \Delta v + v - (p-1)\bar{W}_{\varepsilon,\bar{x}}^{p-2} v \quad \text{in} \quad H^1(\Omega).$$

As a consequence, $L_\varepsilon v = \lambda v$, $v \in H^1(\Omega)$, will have many small eigenvalues. This is the main reason that Malchiodi and Montenegro [18] could only prove the existence of solutions concentrating on a whole connected component of $\partial\Omega$ for a sequence of $\varepsilon_j \to 0$. By imposing some partial symmetry conditions on Ω, we can get rid of the small eigenvalues if we work on the subspace H_s.

The functional corresponding to (1.1) may not be well defined in H_s, because the exponent p may be supercritical.

Our objective is to construct solutions concentrating near the $m-1$ dimensional manifolds $|y'| = \bar{x}_1$. So we can modify the nonlinear term u^{p-1} in such a way that corresponding to the modified problem, the functional is well defined in H_s, and the modified problem has a solution concentrating near $|y'| = \bar{x}_1$, which is also a solution of the original problem. To this aim, we define

$$f(y,t) = 1_B t_+^{p-1} + \big(1 - 1_B\big)\bar{f}(t), \tag{1.6}$$

where $B = \{y : y \in \Omega, (|y'|, y'') \in D \cap B_\delta(\bar{x}) \subset \{(|y'|, y'') : |y'| \geq \kappa\}\}$, $1_B = 1$ in B, and is zero otherwise, and

$$\bar{f}(t) = \begin{cases} t_+^{p-1}, & t \leq 1; \\ 1 + (p-1)(t-1), & t > 1. \end{cases}$$

Now we consider the following problem:

$$\begin{cases} -\varepsilon^2\Delta u + u = f(y,u),\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The functional corresponding to (1.7) is

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \big(\varepsilon^2|Du|^2 + u^2\big) - \int_\Omega F(y,u),$$

where $F(y,t) = \int_0^t f(y,\tau)\,d\tau$. For any $y \in B$, we have $|y'| \geq \kappa > 0$. We see that $I_\varepsilon(u)$ is well defined in H_s if $p \in (2, 2(N-m+1)/(N-m-1))$.

Remark 1.1. After this work was completed, the paper [20] was published. In [20], Ω is supposed to be a unit ball $B_1(0)$ in $\mathbb{R}^N$ and a solution concentrating on several spheres $S^{N-1}_{r_{j,\varepsilon}}$ was constructed, where $S^{N-1}_{r_{j,\varepsilon}} \subset B_1(0)$ is an $(N-1)$-dimensional sphere with radius $r_{j,\varepsilon}$ and $r_{j,\varepsilon} \to 1$ as $\varepsilon \to 0$. We point out here that if $\Omega = B_1(0)$, our result shows that for any integer $1 \leq k < N-1$, there exists a solution concentrating on several k-dimensional spheres which can be simultaneously in $B_1(0)$ and on $\partial B_1(0)$, and all the radii of these spheres tent to 1 as $\varepsilon \to 0$. Hence our result gives a positive answer to the conjecture proposed by Ni in [21].

2. Basic Estimates

In this section, we give some basic estimates needed in the proof of the main result, under the assumption that $x_j \in D$, $x_{j,1} \geq 6\kappa$, $d(x_j,\partial D)/\varepsilon$ is large and $d(x_j,\partial D)$ is small. We assume that $d(x_j,\partial D)$ is small enough such that for $x_j \in D$ with $x_{j,1} \geq 6\kappa$,

$$d(x_j,\partial B) = d(x_j,\partial D).$$

Let $\varphi_{\varepsilon,x_j} = W_{\varepsilon,x_j} - P_{\varepsilon,\Omega}U_{\varepsilon,x_j}$. Then $\varphi_{\varepsilon,x_j}$ satisfies

$$\begin{cases} -\varepsilon^2\Delta\varphi_{\varepsilon,x_j} + \varphi_{\varepsilon,x_j} = 0, & \text{in } \Omega, \\ \frac{\partial\varphi_{\varepsilon,x_j}}{\partial n} = \frac{\partial W_{\varepsilon,x_j}}{\partial n}, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.1. *For any small $\theta > 0$, there is a constant $C > 0$, such that*

$$|\varphi_{\varepsilon,x}(y)| \leq \begin{cases} C\min\Big\{e^{-d(x,\partial D)/\varepsilon}e^{-(1-\theta)d(\tilde y,\partial D)/\varepsilon},\ e^{-(1-\theta)|\tilde y - x|/\varepsilon}\Big\}, & x \in D, \\ C\varepsilon e^{-(1-\theta)|\tilde y - x|/\varepsilon}, & x \in \partial D. \end{cases}$$

Proof. Let $G_\varepsilon(z,y)$ and $G(z,y)$ be the corresponding Green's functions of $-\varepsilon\Delta + I$ in Ω and $-\Delta + I$ in $\Omega_{\varepsilon,y} = \{z \in \mathbb{R}^N :\ \varepsilon z + y \in \Omega\}$ subject to the Neumann boundary condition respectively. Then

$$G_\varepsilon(z,y) = \frac{1}{\varepsilon^N}\, G\Big(\frac{z-y}{\varepsilon}, 0\Big).$$

Suppose $x \in D$, then we have

$$\begin{aligned}|\varphi_{\varepsilon,x}(y)| &= \varepsilon^2 \Big| \int_{\partial\Omega} G_\varepsilon(z,y) \frac{\partial W_{\varepsilon,x}}{\partial n} dz \Big| \le C\varepsilon e^{-d(x,\partial D)/\varepsilon} \int_{\partial\Omega} |G_\varepsilon(z,y)| dy \\ &= Ce^{-d(x,\partial D)/\varepsilon} \int_{\partial\Omega_{\varepsilon,y}} |G(z,0)| dz \le Ce^{-d(x,\partial D)/\varepsilon} e^{-(1-\theta)d(\tilde{y},\partial D)/\varepsilon},\end{aligned}$$

since $G(z,0) \sim 1/|z|^{N-2}$ as $|z| \to 0$ and $|G(z,0)| \le Ce^{-|z|}$ as $|z| \to \infty$.

Since the solution of (2.1) is unique, we know $\varphi_{\varepsilon,x} \in H_s$. For any $y \in \Omega$, let $y^* = (|y'|, 0, \dots, 0, y'')$. Then

$$\begin{aligned}|z - y^*| &= ((z_1 - |y'|)^2 + \sum_{l=2}^{m} z_l^2 + |z'' - y''|^2)^{1/2} \\ &\ge ((|z'| - |y'|)^2 + |z'' - y''|^2)^{1/2} = |\tilde{z} - \tilde{y}|.\end{aligned}$$

As a consequence,

$$\begin{aligned}|\varphi_{\varepsilon,x}(y)| = |\varphi_{\varepsilon,x}(y^*)| &\le C\varepsilon \int_{\partial\Omega} |G_\varepsilon(z,y)| e^{-|\tilde{z}-x|/\varepsilon} dz \\ &= C\varepsilon^{1-N} \int_{\partial\Omega} \Big|G\Big(\frac{z-y^*}{\varepsilon}, 0\Big)\Big| e^{-|\tilde{z}-x|/\varepsilon} dz \\ &\le C\varepsilon^{1-N} \int_{\partial\Omega} \frac{1}{\big(\frac{|z-y^*|}{\varepsilon}\big)^{N-2}} e^{-(1-\theta)|z-y^*|/\varepsilon} e^{-|\tilde{z}-x|/\varepsilon} dz \\ &\le C\varepsilon^{1-N} e^{-(1-2\theta)|\tilde{y}-x|/\varepsilon} \int_{\partial\Omega} \frac{1}{\big(\frac{|z-y^*|}{\varepsilon}\big)^{N-2}} e^{-2\theta|z-y^*|/\varepsilon} dz \\ &\le Ce^{-(1-2\theta)|\tilde{y}-x|/\varepsilon}.\end{aligned}$$

Suppose $x \in \partial D$, then

$$\Big|\frac{\partial W_{\varepsilon,x}}{\partial n}\Big| \le C\varepsilon e^{-(1-\theta)|\tilde{y}-x|/\varepsilon}.$$

Using the same arguments as the case $x \in D$, we deduce

$$|\varphi_{\varepsilon,x}(y)| \le C\varepsilon e^{-(1-\theta)|\tilde{y}-x|/\varepsilon}.$$

Hence we complete the proof. ■

Lemma 2.2. *Let $q \in [1, p-1)$ be a constant and suppose $x_j \in D$, $x_i \in \partial D$. Then there is a constant $\sigma > 0$, such that*

$$\int_\Omega W_{\varepsilon,x_j}^{p-1-q} |\varphi_{\varepsilon,x_j}|^q W_{\varepsilon,x_i} = \varepsilon^{N-m+1} O\Big(e^{-(2+\sigma)d_j/\varepsilon} + e^{-(1+\sigma)|x_i - x_j|/\varepsilon} \Big).$$

Proof. We have

$$\int_\Omega W_{\varepsilon,x_j}^{p-1-q} |\varphi_{\varepsilon,x_j}|^q W_{\varepsilon,x_i} = c_{m-1} \int_D |z_1|^{m-1} W_{\varepsilon,x_j}^{p-1-q} |\varphi_{\varepsilon,x_j}|^q W_{\varepsilon,x_i}.$$

Let $d_{ij} = |x_i - x_j|$. Let $\sigma' > 0$ be a fixed small constant. Then using Lemma 2.1, we obtain

$$\begin{aligned}
&\int_{D \setminus B_{\sigma' d_{ij}}(x_j)} |z_1|^{m-1} W_{\varepsilon,x_j}^{p-1-q} |\varphi_{\varepsilon,x_j}|^q W_{\varepsilon,x_i} \\
&\quad \le C \int_{D \setminus B_{\sigma' d_{ij}}(x_j)} e^{-(p-1-\theta)|\tilde{y} - x_j|/\varepsilon} e^{-|\tilde{y} - x_i|/\varepsilon} \\
&\quad \le C e^{-d_{ij}/\varepsilon} \int_{D \setminus B_{\sigma' d_{ij}}(x_j)} e^{-(p-2-\theta)|\tilde{y} - x_j|/\varepsilon} \\
&\quad \le C \varepsilon^{N-m+1} e^{-d_{ij}/\varepsilon - (p-2-\theta)d_{ij}/\varepsilon} \\
&\quad = C \varepsilon^{N-m+1} e^{-(1+\sigma)d_{ij}/\varepsilon}.
\end{aligned} \tag{2.2}$$

Using Lemma 2.1 again, we obtain

$$\begin{aligned}
&\int_{B_{\sigma' d_{ij}}(x_j)} |z_1|^{m-1} W_{\varepsilon,x_j}^{p-1-q} |\varphi_{\varepsilon,x_j}|^q W_{\varepsilon,x_i} \\
&\quad \le C e^{-q d_j/\varepsilon} e^{-(1-\sigma')d_{ij}/\varepsilon} \int_{B_{\sigma' d_{ij}}(x_j)} W_{\varepsilon,x_j}^{p-1-q} \\
&\quad \le C \varepsilon^{N-m+1} e^{-q d_j/\varepsilon} e^{-(1-\sigma')d_{ij}/\varepsilon} \\
&\quad \le C \varepsilon^{N-m+1} \Big(e^{-(2+\sigma)d_j/\varepsilon} + e^{-(1+\sigma)|x_i - x_j|/\varepsilon} \Big).
\end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we prove this lemma. ∎

Lemma 2.3. *For $x_j \in \partial D$ and $x_i \in D$, there is a constant $c_0 > 0$, such that*

$$\Big| \int_\Omega W_{\varepsilon,x_j}^{p-1} P_{\varepsilon,\Omega} W_{\varepsilon,x_i} \Big| = c_0 U\Big(\frac{|x_i - x_j|}{\varepsilon} \Big) \varepsilon^{N-m+1} + O(\varepsilon^{N-m+2}).$$

Proof. By (1.3), (1.4) and Lemma 2.1, we have

$$\begin{aligned}
&\int_\Omega W^{p-1}_{\varepsilon,x_j} P_{\varepsilon,\Omega} W_{\varepsilon,x_i} \\
&= \int_\Omega \xi \bar{W}^{p-1}_{\varepsilon,x_j} P_{\varepsilon,\Omega} W_{\varepsilon,x_i} + O(e^{-\sigma/\varepsilon})\varepsilon^{N-m+1} \\
&= \int_\Omega \Big(-\varepsilon^2 \Delta W_{\varepsilon,x_j} + W_{\varepsilon,x_j} - \tilde{f}_{\varepsilon,x_j}(y)\Big) P_{\varepsilon,\Omega} W_{\varepsilon,x_i} + O(e^{-\sigma/\varepsilon})\varepsilon^{N-m+1} \\
&= \int_{\partial\Omega} \Big(-\varepsilon^2 \frac{\partial W_{\varepsilon,x_j}}{\partial n} P_{\varepsilon,\Omega} W_{\varepsilon,x_i} + \varepsilon^2 \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_i}}{\partial n} W_{\varepsilon,x_j}\Big) \\
&\qquad + O\big(e^{-d_j/\varepsilon}\varepsilon^{N-m+2} + \varepsilon^{N-m+2}\big) + \int_\Omega \Big(-\varepsilon^2 \Delta P_{\varepsilon,\Omega} W_{\varepsilon,x_i} + P_{\varepsilon,\Omega} W_{\varepsilon,x_i}\Big) W_{\varepsilon,x_j} \\
&= \int_{\partial\Omega} -\varepsilon^2 \frac{\partial W_{\varepsilon,x_j}}{\partial n} P_{\varepsilon,\Omega} W_{\varepsilon,x_i} + \int_\Omega \xi \bar{W}^{p-1}_{\varepsilon,x_i} W_{\varepsilon,x_j} + O\big(e^{-d_j/\varepsilon}\varepsilon^{N-m+2} + \varepsilon^{N-m+2}\big) \\
&= \int_{\partial\Omega} -\varepsilon^2 \frac{\partial W_{\varepsilon,x_j}}{\partial n} P_{\varepsilon,\Omega} W_{\varepsilon,x_i} + \int_\Omega W^{p-1}_{\varepsilon,x_i} W_{\varepsilon,x_j} + O\big(e^{-d_j/\varepsilon}\varepsilon^{N-m+2} + \varepsilon^{N-m+2}\big).
\end{aligned}$$

Since $x_j \in \partial\Omega$, we have

$$\Big|\frac{\partial W_{\varepsilon,x_j}}{\partial n}\Big| \le C\varepsilon e^{-(1-\theta)|\tilde{y}-x_j|/\varepsilon}.$$

As a result, from Lemma 2.1, we conclude

$$\Big|\int_\Omega W^{p-1}_{\varepsilon,x_j} P_{\varepsilon,\Omega} W_{\varepsilon,x_i}\Big| = c_0 U\Big(\frac{|x_i - x_j|}{\varepsilon}\Big)\varepsilon^{N-m+1} + O(\varepsilon^{N-m+2}).$$

■

Lemma 2.4. *Let $q \in [1, p-1)$ be a constant and suppose that $x_j \in \overline{D}$, $x_h \in \overline{D}$ and $x_i \in \overline{D}$ are different mutually. Then there is a constant $\sigma > 0$, such that*

$$\begin{aligned}
&\Big|\int_\Omega P_{\varepsilon,\Omega}\, W^{p-1-q}_{\varepsilon,x_j} P_{\varepsilon,\Omega} W^q_{\varepsilon,x_h} P_{\varepsilon,\Omega} W_{\varepsilon,x_i}\Big| \\
&\qquad \le C\varepsilon^{N-m+1}\Big(e^{-(1+\sigma)|x_i-x_j|/\varepsilon} + e^{-(1+\sigma)|x_i-x_h|/\varepsilon} + e^{-(1+\sigma)|x_j-x_h|/\varepsilon}\Big).
\end{aligned}$$

Proof. We only prove the case $x_j \in \partial D$, $x_h \in \partial D$ and $x_h \in D$, the remaining cases are similar.

Without loss of generality, suppose that

$$d_{ih} := |x_h - x_i| \le \min\{|x_i - x_j|, |x_h - x_j|\}.$$

Let σ' be a small constant. By Lemma 2.1, we deduce

$$
\begin{aligned}
&\Big|\int_\Omega P_{\varepsilon,\Omega}W_{\varepsilon,x_j}^{p-1-q}P_{\varepsilon,\Omega}W_{\varepsilon,x_h}^{q}P_{\varepsilon,\Omega}W_{\varepsilon,x_i}\Big|\le C\int_D W_{\varepsilon,x_j}^{p-1-q}W_{\varepsilon,x_h}^{q}e^{-(1-\theta)|\tilde y-x_i|/\varepsilon}\\
&=\int_{D\cap B_{\sigma' d_{ih}(x_j)}} W_{\varepsilon,x_j}^{p-1-q}W_{\varepsilon,x_h}^{q}e^{-(1-\theta)|\tilde y-x_i|/\varepsilon}\\
&\quad + C\int_{D\setminus B_{\sigma' d_{ih}(x_j)}} W_{\varepsilon,x_j}^{p-1-q}W_{\varepsilon,x_h}^{q}e^{-(1-\theta)|\tilde y-x_i|/\varepsilon}\\
&\le Ce^{-(1-\sigma')|x_h|/\varepsilon}e^{-(1-\theta)|x_i-x_j|/\varepsilon}\int_D W_{\varepsilon,x_j}^{p-1-q}\\
&\quad + Ce^{-(1-\theta)|x_h-x_i|}e^{-\sigma d_{ih}/\varepsilon}\int_D W_{\varepsilon,x_j}^{p-1-q-\theta'}\\
&\le C\varepsilon^{N-m+1}\Big(e^{-(1+\sigma)|x_i-x_j|/\varepsilon}+e^{-(1+\sigma)|x_i-x_h|/\varepsilon}+e^{-(1+\sigma)|x_j-x_h|/\varepsilon}\Big).
\end{aligned}
$$

■

Proposition 2.1. *Suppose that $x_j \in D\cap B_\delta(\bar x)$ for $j = 1,\cdots\cdot,k_1$ and $x_j \in \partial D\cap B_\delta(\bar x)$ for $j = k_1+1,\dots,k_1+k_2$. Then*

$$
\begin{aligned}
I_\varepsilon\Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) =& A\varepsilon^{N-m+1}\sum_{j=1}^{k_1} x_{j,1}^{m-1}+\frac12 A\varepsilon^{N-m+1}\sum_{j=k_1+1}^{k_1+k_2} x_{j,1}^{m-1}\\
&+\frac12\sum_{j=1}^{k_1}\tau_{\varepsilon,x_j}-c_0\varepsilon^{N-m+1}\sum_{1\le i<j\le k_1+k_2}U\Big(\frac{|x_i-x_j|}{\varepsilon}\Big)\\
&+\varepsilon^{N-m+1}O\Big(\sum_{j=1}^{k_1}e^{-(2+\sigma)d_j/\varepsilon}+\sum_{j\ne i}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}+\varepsilon\Big),
\end{aligned}
$$

where $c_0 > 0$ is a constant, $\sigma > 0$ is a small constant,

$$
A=\Big(\frac12-\frac1p\Big)c_{m-1}\int_{\mathbb{R}^{N-m+1}}U^p
$$

(c_{m-1} is the area of the unit sphere in $\mathbb{R}^m$), and for $j = 1,\dots,k_1$,

$$
\tau_{\varepsilon,x_j}=\int_\Omega \xi\ \bar W_{\varepsilon,x_j}^{p-1}\varphi_{\varepsilon,x_j}
$$

satisfying

$$
\begin{aligned}
(2.4)\qquad & C_1\varepsilon^{N-m+1}\ e^{-(2+\theta)d_j/\varepsilon}+\varepsilon^{N-m+1}O\big(\varepsilon e^{-d_j/\varepsilon}\big)\\
&\le -\tau_{\varepsilon,x_j}\le C_2\varepsilon^{N-m+1}e^{-(2-\theta)d_j/\varepsilon}+\varepsilon^{N-m+1}O\big(\varepsilon e^{-d_j/\varepsilon}\big).
\end{aligned}
$$

Proof.

$$
\begin{aligned}
I_\varepsilon\Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) &= I_\varepsilon\Big(\sum_{j=1}^{k_1} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) + I_\varepsilon\Big(\sum_{j=k_1+1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) \\
&+ \sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2} \big\langle P_{\varepsilon,\Omega}W_{\varepsilon,x_i}, P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big\rangle_\varepsilon \\
&- \frac{1}{p}\int_\Omega\Big(\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^p - \Big(\sum_{j=1}^{k_1}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^p - \Big(\sum_{j=k_1+1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+\Big). \qquad (2.5)
\end{aligned}
$$

Using the inequality

$$
(a+b)_+^p - a_+^p - b_+^p - pa_+^{p-1}b - pb_+^{p-1}a = \begin{cases} O(|a|^{p/2}|b^{p/2}), & 2 < p \le 4, \\ O(|a|^{p-2}b^2 + |b|^{p-2}a^2), & p > 4, \end{cases}
$$

we obtain

$$
\begin{aligned}
\frac{1}{p}\int_\Omega&\Big(\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^p - \Big(\sum_{j=1}^{k_1}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^p - \Big(\sum_{j=k_1+1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+\Big) \\
&= \int_\Omega\Big(\sum_{i=1}^{k_1}P_{\varepsilon,\Omega}W_{\varepsilon,x_i}\Big)_+^{p-1}\Big(\sum_{j=k_1+1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) \\
&\quad + \int_\Omega\Big(\sum_{i=1}^{k_1}P_{\varepsilon,\Omega}W_{\varepsilon,x_i}\Big)_+\Big(\sum_{j=k_1+1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^{p-1} \\
&\quad + O\Big(\int_\Omega\Big(\sum_{i=1}^{k_1}|P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|\Big)^{1+\sigma}\Big(\sum_{j=k_1+1}^{k_1+k_2}|P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|\Big)^{1+\sigma}\Big), \qquad (2.6)
\end{aligned}
$$

where $\sigma > 0$ is a constant.

Using Lemma 2.1, Lemma 2.2 and Lemma 2.4, we see that,

$$
\begin{aligned}
&\Big|\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}\big\langle P_{\varepsilon,\Omega}W_{\varepsilon,x_i}, P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big\rangle_\varepsilon \qquad (2.7) \\
&\qquad - \int_\Omega\Big(\sum_{i=1}^{k_1}P_{\varepsilon,\Omega}W_{\varepsilon,x_i}\Big)_+^{p-1}\Big(\sum_{j=k_1+1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\Big| \\
&= \sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}\Big\{\int_\Omega\Big|\xi\bar W_{\varepsilon,x_i}^{p-1} + \tilde f_{\varepsilon,x_i} - \Big(P_{\varepsilon,\Omega}W_{\varepsilon,x_i}\Big)_+^{p-1}\Big|\Big|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big| \\
&\qquad + O\Big(\varepsilon^{N-m+1}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}\Big)\Big\}
\end{aligned}
$$

$$\le C\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}\Big\{\int_\Omega W_{\varepsilon,x_i}^{p-1-q}|\varphi_{\varepsilon,x_i}|^q W_{\varepsilon,x_j}+O\Big(\varepsilon^{N-m+1}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}\Big)\Big\}$$
$$+O(\varepsilon^{N-m+2})$$
$$=\varepsilon^{N-m+1}O\Big(\sum_{i=1}^{k_1}e^{-(2+\sigma)d_i/\varepsilon}+\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}+\varepsilon\Big).$$

Using Lemma 2.1,

$$O\Big(\int_\Omega\Big(\sum_{i=1}^{k_1}|P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|\Big)^{1+\sigma}\Big(\sum_{j=k_1+1}^{k_1+k_2}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|\Big)^{1+\sigma}\Big)$$
$$\le C\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}\int_\Omega\Big(W_{\varepsilon,x_i}^{1+\sigma}W_{\varepsilon,x_j}^{1+\sigma}+|\varphi_{\varepsilon,x_i}|^{1+\sigma}W_{\varepsilon,x_j}^{1+\sigma}+W_{\varepsilon,x_i}^{1+\sigma}|\varphi_{\varepsilon,x_j}|^{1+\sigma}$$
$$+|\varphi_{\varepsilon,x_i}|^{1+\sigma}|\varphi_{\varepsilon,x_j}|^{1+\sigma}\Big)$$
$$=\varepsilon^{N-m+1}O\Big(\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}\Big). \tag{2.8}$$

From Lemma 2.1, Lemma 2.3 and Lemma 2.4, we see that

$$\int_\Omega\Big(\sum_{i=1}^{k_1}P_{\varepsilon,\Omega}W_{\varepsilon,x_i}\Big)_+\Big(\sum_{j=k_1+1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^{p-1} \tag{2.9}$$
$$=\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}\Big(P_{\varepsilon,\Omega}W_{\varepsilon,x_i}\Big)_+W_{\varepsilon,x_j}^{p-1}+\varepsilon^{N-m+1}O\Big(\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}+\varepsilon\Big)$$
$$=c_0\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}U\Big(\frac{|x_i-x_j|}{\varepsilon}\Big)+\varepsilon^{N-m+1}O\Big(\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}+\varepsilon\Big).$$

Now combining (2.5)-(2.9), we obtain that

$$I_\varepsilon\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)$$
$$=I_\varepsilon\Big(\sum_{j=1}^{k_1}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)+I_\varepsilon\Big(\sum_{j=k_1+1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)-c_0\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}U\Big(\frac{|x_i-x_j|}{\varepsilon}\Big)$$
$$+\varepsilon^{N-m+1}O\Big(\sum_{i=1}^{k_1}e^{-(2+\sigma)d_i/\varepsilon}+\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_1+k_2}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}+\varepsilon\Big).$$

Employing similar arguments as in [10], we conclude that

$$\begin{aligned} I_\varepsilon\Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) &= A\varepsilon^{N-m+1}\sum_{j=1}^{k_1} x_{j,1}^{m-1} + \frac{1}{2}A\varepsilon^{N-m+1}\sum_{j=k_1+1}^{k_1+k_2} x_{j,1}^{m-1} \\ &+ \frac{1}{2}\sum_{j=1}^{k_1}\tau_{\varepsilon,x_j} - c_0\varepsilon^{N-m+1}\sum_{1\le i<j\le k_1+k_2} U\Big(\frac{|x_i-x_j|}{\varepsilon}\Big) \\ &+ \varepsilon^{N-m+1}O\Big(\sum_{j=1}^{k_1} e^{-(2+\sigma)d_j/\varepsilon} + \sum_{j\neq i} e^{-(1+\sigma)|x_i-x_j|/\varepsilon} + \varepsilon\Big), \end{aligned}$$

and τ_{ε,x_j} satisfies (2.4). ■

3. Proof of the main result

Let

$$\begin{aligned} D_\varepsilon^* = \Big\{x = (x_1,\dots,x_{k_1+k_2}) :\ & x_i \in D, e^{-2d_i/\varepsilon} \le \varepsilon^{1-\tilde\theta}, i = 1,\dots,k_1; \\ & x_j \in \partial D,\ j = k_1+1,\dots,k_1+k_2; \\ & x_i \in B_\delta(\bar x),\ e^{-|x_i-x_j|/\varepsilon} \le \varepsilon^{1-\tilde\theta}, \\ & \qquad i,j = 1,\cdots,k_1+k_2, i\neq j\Big\}, \end{aligned}$$

where $\tilde\theta > 0$ is a fixed small constant, $d_i = d(x_i,\partial D)$.

Define

$$J(x,\omega) = I_\varepsilon\Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j} + \omega\Big), \quad \forall x \in D_\varepsilon^*,\ \omega\in H_s. \tag{3.1}$$

Let

$$\begin{aligned} E_{\varepsilon,x,k_1+k_2} = \Big\{\omega \in H_s : & \Big\langle \omega, \frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{j,l}}\Big\rangle_{D,\varepsilon} = 0, \\ & \qquad j = 1,\cdots,k_1, l = 1\cdots,N-m+1; \\ & \Big\langle \omega, \frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial \tau_{j,l}}\Big\rangle_{D,\varepsilon} = 0, \\ & \qquad j = k_1+1,\cdots,k_1+k_2, l = 1,\cdots,N-m\Big\}, \end{aligned}$$

where

$$\langle u,v\rangle_{D,\varepsilon} = \int_D z_1^{m-1}\big(\varepsilon^2 DuDv + uv\big)\,dz,$$

$\tau_{j,l}$ $(l = 1,\dots,N-m)$ denotes the $N-1$ tangent vectors of $\partial\Omega$ at $x_j \in \partial\Omega$, $(j = k_1+1,\dots,k_1+k_2)$.

Lemma 3.1. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0]$ and $\delta \in (0, \delta_0]$, (x, ω) is a critical point of J in $D^*_\varepsilon \times E_{\varepsilon,x,k_1+k_2}$ if and only if*

$$u = \sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_j} + \omega_\varepsilon$$

is a critical point of I_ε in H_s.

The proof of Lemma 3.1 can be completed with the same arguments as in [3], we omit it here.

We notice that (x, ω) is a critical point of J in $D^*_\varepsilon \times E_{\varepsilon,x,k_1+k_2}$ if and only if there are scalars $A_{j,l} \in \mathbb{R}$, $j = 1, \dots, k_1 + k_2$, $l = 1, \dots, N - m + 1$, such that

$$\frac{\partial J}{\partial x_{j,l}} = \sum_{h=1}^{N-m+1} A_{j,h} \Big\langle \frac{\partial^2 P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial x_{j,h} \partial x_{j,l}}, \omega \Big\rangle_{D,\varepsilon}, \qquad j = 1, \dots, k_1, l = 1, \dots, N - m + 1, \tag{3.2}$$

$$\frac{\partial J}{\partial \tau_{j,l}} = \sum_{h=1}^{N-m} A_{j,h} \Big\langle \frac{\partial^2 P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial \tau_{j,h} \partial \tau_{j,l}}, \omega \Big\rangle_{D,\varepsilon}, \qquad j = k_1, \dots, k_1 + k_2, l = 1, \dots, N - m, \tag{3.3}$$

$$\frac{\partial J}{\partial w} = \sum_{j=1}^{k_1} \sum_{l=1}^{N-m+1} A_{j,l} \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial x_{j,l}} + \sum_{j=k_1+1}^{k_1+k_2} \sum_{l=1}^{N-m} A_{j,l} \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial \tau_{j,l}}. \tag{3.4}$$

In order to prove Theorem 1.1, we show first that for $x = (x_1, \dots, x_{k_1+k_2}) \in D^*_\varepsilon$ given, ε small enough, there exist $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k_1+k_2}$ and scalars $A_{j,l}$, $j = 1, \dots, k_1 + k_2$, $l = 1, \dots, N - m + 1$, such that (3.4) is satisfied and the mapping $x \to \omega_{\varepsilon,x}$ is C^1. We then show that for sufficiently small ε, there exists a point $x_\varepsilon \in D^*_\varepsilon$, such that $(x_\varepsilon, \omega) \in D^*_\varepsilon \times E_{\varepsilon,x,k_1+k_2}$ and (3.2), (3.3) are satisfied with these scalars $A_{j,l}$.

We expand $J(x, \omega)$ near $\omega = 0$ as follows:

$$J(x, \omega) = J(x, 0) + h_{\varepsilon,x}(\omega) + \frac{1}{2} Q_{\varepsilon,x}(\omega) - R_{\varepsilon,x}(\omega),$$

where

$$h_{\varepsilon,x}(\omega) = \sum_{j=1}^{k_1+k_2} \int_\Omega \big(\varepsilon^2 D P_{\varepsilon,\Omega} W_{\varepsilon,x_j} D\omega + P_{\varepsilon,\Omega} W_{\varepsilon,x_j} \omega\big) - \int_\Omega \Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big)_+^{p-1} \omega, \tag{3.5}$$

$$Q_{\varepsilon,x}(\omega) = \int_\Omega \big(\varepsilon^2|D\omega|^2 + \omega^2\big) - (p-1)\int_\Omega \big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-2}\omega^2, \tag{3.6}$$

and

$$\begin{aligned} R_{\varepsilon,x}(\omega) &= \int_\Omega F\Big(y, \sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j} + \omega\Big) - \int_\Omega F\Big(y, \sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) \\ &\quad - \int_\Omega \big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\omega - \frac{1}{2}(p-1)\int_\Omega \big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-2}\omega^2. \end{aligned} \tag{3.7}$$

Lemma 3.2. *There are constants $C > 0$ and $\sigma > 0$, such that*

$$\begin{aligned} &\big|\, h_{\varepsilon,x}(\omega)\big| \\ &\le C\varepsilon^{(N-m+1)/2}\Big(\sum_{j=1}^{k_1} e^{-(1+\sigma)d_j/\varepsilon} + \sum_{j\neq i} e^{-(1+\sigma)|x_i-x_j|/(2\varepsilon)} + \varepsilon^{(1+2\sigma)/2}\Big)\|\omega\|_\varepsilon. \end{aligned}$$

Proof. We have

$$\begin{aligned} h_{\varepsilon,x}(\omega) &= \sum_{j=1}^{k_1+k_2} \int_\Omega \Big(\xi \bar{W}_{\varepsilon,x_j}^{p-1} + \tilde{f}_{\varepsilon,x_j}(y)\Big)\omega - \int_\Omega \big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\omega \\ &= \sum_{j=1}^{k_1+k_2} \int_\Omega \Big(\xi \bar{W}_{\varepsilon,x_j}^{p-1} - W_{\varepsilon,x_j}^{p-1}\Big)\omega + \sum_{j=1}^{k_1+k_2} \int_\Omega \Big(W_{\varepsilon,x_j}^{p-1} - \big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\Big)\omega \\ &\quad + \int_\Omega \Big(\sum_{j=1}^{k_1+k_2} (P_{\varepsilon,\Omega}W_{\varepsilon,x_j})_+^{p-1} - \big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\Big)\omega \\ &\quad + \sum_{j=1}^{k_1+k_2} \int_\Omega \tilde{f}_{\varepsilon,x_j}(y)\omega \\ &= O\Big(\sum_{j=1}^{k_1+k_2} \int_\Omega W_{\varepsilon,x_j}^{(1+2\sigma)/2}\varphi_{\varepsilon,x_j}^{(1+2\sigma)/2}|\omega|\Big) + O\big(e^{-\delta'/\varepsilon}\big)\|\omega\|_\varepsilon \\ &\quad + \sum_{j=1}^{k_1+k_2} \int_\Omega \tilde{f}_{\varepsilon,x_j}(y)\omega \\ &\quad + O\Big(\sum_{j\neq i} \int_\Omega |P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|^{(p-1)/2}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|^{(p-1)/2}|\omega|\Big). \end{aligned} \tag{3.8}$$

From Lemma 2.1 we obtain

$$\int_\Omega W_{\varepsilon,x_j}^{(1+2\sigma)/2}\varphi_{\varepsilon,x_j}^{(1+2\sigma)/2}|\omega| \le \Big(\int_\Omega W_{\varepsilon,x_j}^{(1+2\sigma)}\varphi_{\varepsilon,x_j}^{(1+2\sigma)}\Big)^{1/2}\|\omega\|_\varepsilon$$

$$\le \begin{cases} C\Big(e^{-(1+2\sigma)d_j/\varepsilon-(1+2\sigma)(1-\theta)d_j/\varepsilon}\displaystyle\int_\Omega W_{\varepsilon,x_j}^{(1+2\sigma)\theta}\Big)^{1/2}\|\omega\|_\varepsilon & (x_j\in D)\\ C\varepsilon^{(N-m+2+2\sigma)/2}\|\omega\|_\varepsilon & (x_j\in\partial D)\end{cases} \tag{3.9}$$

$$\le \begin{cases} C\varepsilon^{(N-m+1)/2}e^{-(1+\sigma)d_j/\varepsilon}\|\omega\|_\varepsilon & (x_j\in D)\\ C\varepsilon^{(N-m+2+2\sigma)/2}\|\omega\|_\varepsilon & (x_j\in\partial D),\end{cases}$$

where in the second inequality, we have used $|\tilde y - x_j| + d(\tilde y,\partial D) \ge d_j$ for $x_j \in D$.

Again by using Lemma 2.1, we deduce that for $i \ne j$,

$$\int_\Omega |P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|^{(1+2\sigma)/2}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|^{(1+2\sigma)/2}|\omega| \tag{3.10}$$

$$\le \Big(\int_\Omega |P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|^{p-1}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|^{p-1}\Big)^{1/2}\|\omega\|_\varepsilon$$

$$\le \begin{cases} C\varepsilon^{(N-m+1)/2}e^{-(1+\sigma)|x_i-x_j|/(2\varepsilon)}\|\omega\|_\varepsilon & (x_i,\, x_j\in\Omega)\\ C\big(\varepsilon^{(N-m+1)/2}e^{-(1+\sigma)|x_i-x_j|/(2\varepsilon)}+\varepsilon^{(N-m+p)/2}\big)\|\omega\|_\varepsilon & (x_i\in\Omega,\, x_j\in\partial\Omega)\\ C\big(\varepsilon^{(N-m+1)/2}e^{-(1+\sigma)|x_i-x_j|/(2\varepsilon)}+\varepsilon^{(N-m+2p-1)/2}\big)\|\omega\|_\varepsilon & (x_i,\, x_j\in\partial\Omega).\end{cases}$$

On the other hand,

$$\Big|\int_\Omega \tilde f_{\varepsilon,x_j}(y)\omega\Big| \le \Big(\int_\Omega |\tilde f_{\varepsilon,x_j}(y)|^2\Big)^{1/2}\|\omega\|_\varepsilon \le C\varepsilon^{1+(N-m+1)/2}\|\omega\|_\varepsilon. \tag{3.11}$$

Combining (3.8)-(3.11), we obtain the result. ■

Let $Q_{\varepsilon,x}$ be the bounded linear map $E_{\varepsilon,x,k_1+k_2}$ to $E_{\varepsilon,x,k_1+k_2}$, such that

$$\big\langle Q_{\varepsilon,x}\omega_1,\omega_2\big\rangle_\varepsilon = \int_\Omega\big(\varepsilon^2 D\omega_1 D\omega_2+\omega_1\omega_2\big)-(p-1)\int_\Omega\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^{p-2}\omega_1\omega_2,$$

for $\omega_1,\ \omega_2\in E_{\varepsilon,x,k_1+k_2}$. Then we have

Lemma 3.3. *There are constants $\varepsilon_0>0$ and $\rho>0$, such that for each $\varepsilon\in(0,\varepsilon_0]$, and $x\in D_\varepsilon^*$,*

$$\|Q_{\varepsilon,x}\omega\|_\varepsilon \ge \rho\|\omega\|_\varepsilon,\quad \omega\in E_{\varepsilon,x,k_1+k_2}.$$

Proof. Suppose to the contrary that Lemma 3.3 does not hold, then there exist $\varepsilon_n \to 0$, $x_n = (x_{n,1}, \dots, x_{n,k_1+k_2}) \in D^*_{\varepsilon_n}$ and $\omega_n \in E_{\varepsilon_n,x_n,k_1+k_2}$, such that

$$\|Q_{\varepsilon_n,x_n}\omega_n\|_{\varepsilon_n} = o_n(1)\|\omega_n\|_{\varepsilon_n},$$

that is

$$\int_\Omega (\varepsilon_n^2 D\omega_n D\varphi + \omega_n\varphi) - (p-1)\int_\Omega \Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon_n,\Omega}W_{\varepsilon_n,x_{n,j}}\Big)_+^{p-2}\omega_n\varphi = o_n(1)\|\omega_n\|_{\varepsilon_n}\|\varphi\|_{\varepsilon_n}, \quad \forall\, \varphi \in E_{\varepsilon_n,x_n,k_1+k_2}. \tag{3.12}$$

Assume without loss of generality that

$$\|\omega_n\|_{\varepsilon_n} = \varepsilon_n^{\frac{N-m+1}{2}}. \tag{3.13}$$

For each fixed $j \in \{1, \dots, k_1 + k_2\}$, let $\tilde{\omega}_{n,j}(\tilde{y}) = \omega_n(\varepsilon_n\tilde{y} + x_{n,j})$. Since $x_{n,j,1} > c > 0$, by (3.12),

$$\int_{B_R} (|D\tilde{\omega}_{n,j}|^2 + |\tilde{\omega}_{n,j}|^2) \le C,$$

for any $R > 0$ large, where $C > 0$ is in dependent of R, $B_R = B_R(0)$ for $j = 1, \dots, k_1$ and $B_R = B_R(0) \cap \mathbb{R}^{N-m+1}_+$ for $j = k_1 + 1, \dots, k_1 + k_2$, $B_R(0)$ is the ball in $\mathbb{R}^{N-m+1}$ with radius R and centered at the origin.

Thus there is a subsequence (still denoted by $\{n\}$) and an $\omega \in H^1(\mathbb{R}^{N-m+1})$, such that for any $R > 0$,

$$\tilde{\omega}_{n,j} \to \omega, \text{ weakly in } H^1(B_R), \qquad \text{and} \qquad \tilde{\omega}_{n,j} \to \omega, \text{ strongly in } L^2(B_R).$$

Moreover, by the nondegeneracy properties of U which were stated in Section 1 and using the similar arguments to [10, 6](see also [3, 8]), we deduce that $\omega \equiv 0$.

Now, for $j = 1, \dots, k_1+k_2$, let $B_{j,R} = \{y \in \Omega : (|y'|, y'') \in B_{\varepsilon R}(x_{n,j}) \cap D\}$. Then,

$$\begin{aligned}
&\int_\Omega \Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon_n,\Omega}W_{\varepsilon_n,x_{n,j}}\Big)^{p-2}\omega_n^2 dy \\
&= \int_{\cup_{j=1}^{k_1+k_2} B_{j,R}} \Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon_n,\Omega}W_{\varepsilon_n,x_{n,j}}\Big)^{p-2}\omega_n^2 dy + \int_{\Omega\setminus\cup_{j=1}^{k_1+k_2} B_{j,R}} \Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon_n,\Omega}W_{\varepsilon_n,x_{n,j}}\Big)^{p-2}\omega_n^2 dy \\
&\le C\int_{\cup_{j=1}^{k_1+k_2} B_{j,R}} \omega_n^2 dx + o_R(1)\|\omega_n\|^2_{\varepsilon_n} = o(\varepsilon_n^{N-m+1}) + o_R(1)\varepsilon_n^{N-m+1},
\end{aligned}$$

where $o_R(1) \to 0$ as $R \to \infty$.

Hence from (3.12), we have

$$o(\varepsilon_n^{N-m+1}) = \|w_n\|_{\varepsilon_n}^2 + o(\varepsilon_n^{N-m+1}) + o_R(1)\varepsilon_n^{N-m+1},$$

which is impossible.

As a result, we complete the proof. ■

Let

$$S_\varepsilon = \Big\{\omega : \omega \in H_s(\Omega),\ |\omega| \le \sum_{j=1}^{k_1+k_2} e^{-\alpha|\tilde{y}-x_j|/\varepsilon}\Big\},$$

where $\alpha > 0$ is a small constant.

Lemma 3.4. *For any $\omega \in S_\varepsilon$ with $\|\omega\|_\varepsilon \le \varepsilon^{(N-m+1)/2}$, we have*

$$R_{\varepsilon,x}(\omega) = \varepsilon^{N-m+1} O\big(\varepsilon^{-p^*(N-m+1)/2}\|\omega\|_\varepsilon^{p^*}\big), \tag{3.14}$$

$$\big\langle R'_{\varepsilon,x}(\omega), \xi\big\rangle_\varepsilon = \varepsilon^{(N-m+1)/2} O\big(\varepsilon^{-(p^*-1)(N-m+1)/2}\|\omega\|_\varepsilon^{p^*-1}\big)\|\xi\|_\varepsilon \tag{3.15}$$

and

$$R''_{\varepsilon,x}(\omega)(\xi_1,\xi_2) = O\big(\varepsilon^{-(p^*-2)(N-m+1)/2}\|\omega\|_\varepsilon^{p^*-2}\big)\|\xi_1\|_\varepsilon\|\xi_2\|_\varepsilon, \tag{3.16}$$

where $p^* = \min\{p, 3\}$.

Proof. The proof of this lemma is similar to the proof of Lemma 3.3 in [10]. Thus, we omit it. ■

Proposition 3.1. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there exists a C^1-map $\omega_{\varepsilon,x} : D_\varepsilon^* \to H_s$ such that $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k_1+k_2}$, (3.4) holds for some constants A_{jl}. Moreover, we have*

$$\|\omega_{\varepsilon,x}\|_\varepsilon \le C\varepsilon^{\sigma+(N-m+2)/2}, \tag{3.17}$$

where $\sigma > 0$ is a constant.

Proof. By Lemma 3.2, we know that there is a $h_{\varepsilon,x} \in E_{\varepsilon,x,k_1+k_2}$, such that

$$\big\langle h_{\varepsilon,x}, \omega\big\rangle_\varepsilon = h_{\varepsilon,x}(\omega), \quad \forall\, \omega \in E_{\varepsilon,x,k_1+k_2}.$$

Thus, solving (3.17) is equivalent to solving

$$h_{\varepsilon,x} + Q_{\varepsilon,x}\omega + R'_{\varepsilon,x}(\omega) = 0, \quad \text{in } E_{\varepsilon,x,k_1+k_2}. \tag{3.18}$$

By Lemma 3.3, $Q_{\varepsilon,x}$ is invertible. So we can write (3.18) as

$$\omega = G_{\varepsilon,x}\omega =: -Q_{\varepsilon,x}^{-1}h_{\varepsilon,x} - Q_{\varepsilon,x}^{-1}R'_{\varepsilon,x}(\omega). \tag{3.19}$$

Let

$$\tilde{S}_\varepsilon = \Big\{\omega : \omega \in H_s(\Omega),\ |\omega| \le \varepsilon^\alpha \sum_{j=1}^{k_1+k_2} e^{-\alpha|\tilde{y}-x_j|/\varepsilon}, \|\omega\|_\varepsilon \le \varepsilon^{(N-m+2)/2}\Big\},$$

where $\alpha > 0$ is a small constant.

Now, we prove that $G_{\varepsilon,x}$ is a contraction map from $\tilde{S}_s$ to $\tilde{S}_\varepsilon$.

By (3.16), we see that for any $\omega_1, \omega_2 \in \tilde{S}_\varepsilon$,

$$\|G_{\varepsilon,x}\omega_1 - G_{\varepsilon,x}\omega_2\|_\varepsilon \le C\|R'_{\varepsilon,x}(\omega_1) - R'_{\varepsilon,x}(\omega_2)\|_\varepsilon \le C\varepsilon^{(p^*-2)/2}\|\omega_1 - \omega_2\|_\varepsilon. \tag{3.20}$$

Thus, $G_{\varepsilon,x}$ is a contraction map. Moreover, by Lemma 3.2 and (3.15) ,

$$\|G_{\varepsilon,x}\omega\|_\varepsilon \le C\|h_{\varepsilon,x}\|_\varepsilon + C\|R'_{\varepsilon,x}(\omega)\|_\varepsilon \le C\varepsilon^{\sigma_1+(N-m+2)/2} \le \varepsilon^{(N-m+2)/2}. \tag{3.21}$$

To finish the proof of $G_{\varepsilon,x}\omega \in \tilde{S}_\varepsilon$, we need to prove

$$|G_{\varepsilon,x}\omega| \le \varepsilon^\alpha \sum_{j=1}^{k_1+k_2} e^{-\alpha|\tilde{y}-x_j|/\varepsilon}.$$

Let $\omega_1 = G_{\varepsilon,x}\omega$. Then, we have

$$Q_{\varepsilon,x}\omega_1 = -h_{\varepsilon,x} - R'(\omega), \quad \text{in } E_{\varepsilon,x,k},$$

which is equivalent to

$$Q_{\varepsilon,x}\omega_1 + h_{\varepsilon,x} + R'(\omega) = \sum_{j=1}^{k_1}\sum_{h=1}^{N-m+1} A_{jh}\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{j,h}} + \sum_{j=k_1+1}^{k_1+k_2}\sum_{h=1}^{N-m} A_{jh}\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial \tau_{j,h}} \tag{3.22}$$

for some $A_{jh} \in \mathbb{R}$.

We claim that there is a $\sigma > 0$, such that

$$|A_{jh}| \le C\varepsilon^{\sigma+3/2}, \quad j = 1, \cdots, k,\ h = 1, \cdots, N-m+1. \tag{3.23}$$

In fact, taking the scalar product in H^s of (3.22) with

$$\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{j,l}} \qquad j = 1, \dots, k_1, \quad l = 1, \dots, N-m+1$$

$$\text{and} \quad \frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial \tau_{j,l}} \qquad j = k_1, \dots, k_1+k_2, \quad l = 1, \dots, N-m$$

respectively, we get a quasi-diagonal linear system with $A_{j,l}$ as unknown. Obviously, by Lemma 4.1 in Appendix, there exists $\varepsilon^* > 0$, such that if $\varepsilon < \varepsilon^*$, the coefficient matrix of this linear system is invertible, which means

$$\begin{aligned} |A_{jh}| &\le C\varepsilon^{1-(N-m+1)/2}\big(\|\omega_1\|_\varepsilon + \|h_{\varepsilon,x}\|_\varepsilon + \|R'(\omega)\|_\varepsilon\big) \\ &\le C\varepsilon^{1-(N-m+1)/2}\varepsilon^{\frac{1}{2}+\sigma+(N-m+1)/2} \le C\varepsilon^{\sigma+3/2}. \end{aligned}$$

Rewrite (3.22) as

$$
\begin{aligned}
&-\varepsilon^2\Delta\omega_1+\omega_1-(p-1)\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^{p-2}\omega_1\\
&\quad=-\sum_{j=1}^{k_1+k_2}\Big(\xi\bar W_{\varepsilon,x_j}^{p-1}+\tilde f_{\varepsilon,x_j}(y)\Big)+\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^{p-1}\\
&\qquad-\Big(f\Big(y,\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}+\omega\Big)-f\Big(y,\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\\
&\qquad-f'\Big(y,\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\omega\Big)\\
&\qquad+\sum_{j=1}^{k_1}\sum_{h=1}^{N-m+1}A_{jh}\frac{\partial g_{\varepsilon,x_j}(y)}{\partial x_{j,h}}+\sum_{j=k_1+1}^{k_1+k_2}\sum_{h=1}^{N-m}A_{jh}\frac{\partial g_{\varepsilon,x_j}(y)}{\partial \tau_{j,h}}=:G_{\varepsilon,x}(y),
\end{aligned}
\tag{3.24}
$$

where $f(y,t)$ is the function defined in (1.6), and $g_{\varepsilon,x_j}(y)=\xi W_{\varepsilon,x_j}^{p-1}+\tilde f_{\varepsilon,x_j}(y)$.

Since $\omega\in\tilde S_\varepsilon$, we see $|\omega|\le\frac12$ in $\Omega\setminus B$. Thus

$$
\begin{aligned}
&\Big|f\Big(y,\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}+\omega\Big)-f\Big(y,\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\\
&\qquad\qquad-f'\Big(y,\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\Big|\le C|\omega|^{p^*-1}.
\end{aligned}
\tag{3.25}
$$

Direct calculations lead to

$$
\begin{aligned}
&-\sum_{j=1}^{k_1+k_2}\xi\bar W_{\varepsilon,x_j}^{p-1}+\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^{p-1}\\
&\quad=-\sum_{j=1}^{k_1+k_2}\Big(W_{\varepsilon,x_j}^{p-1}-\big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\Big)+\sum_{j=1}^{k_1+k_2}O\big(e^{-\delta'/\varepsilon}W_{\varepsilon,x_j}\big)\\
&\qquad+\Big(\sum_{j=1}^{k_1+k_2}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)_+^{p-1}-\sum_{j=1}^{k_1+k_2}\big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\\
&\quad=O\Big(\sum_{j=1}^{k}e^{-d_j/\varepsilon}U^{p-2}\Big(\frac{|\tilde y-x_j|}{\varepsilon}\Big)+\sum_{j=k_1+1}^{k_1+k_2}\varepsilon U^{p-2}\Big(\frac{|\tilde y-x_j|}{\varepsilon}\Big)\\
&\qquad+\sum_{j=1}^{k_1+k_2}e^{-\delta'/\varepsilon}W_{\varepsilon,x_j}+\sum_{i\ne j}|P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|^{(1+\sigma)/2}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|^{(1+\sigma)/2}\Big).
\end{aligned}
\tag{3.26}
$$

But by Lemma 2.1, we see for $x_i \in \bar{D}$

$$|P_{\varepsilon,\Omega} W_{\varepsilon,x_i}| \le C e^{-(1-\theta)|\tilde{y}-x_j|},$$

so, we deduce that

$$\begin{aligned}
&|P_{\varepsilon,\Omega} W_{\varepsilon,x_i}|^{(1+\sigma)/2} |P_{\varepsilon,\Omega} W_{\varepsilon,x_j}|^{(1+\sigma)/2} \\
&\quad \le C e^{-(1+\sigma)(1-\theta)|x_i-x_j|/(4\varepsilon)} e^{-(1-\theta)(1+\sigma)|\tilde{y}-x_i|/(4\varepsilon)} e^{-(1-\theta)(1+\sigma)|\tilde{y}-x_j|/(4\varepsilon)} \\
&\quad \le C e^{-(1+\sigma')|x_i-x_j|/(4\varepsilon)} \big(e^{-(1+\sigma')|\tilde{y}-x_i|/(2\varepsilon)} + e^{-(1+\sigma')|\tilde{y}-x_j|/(2\varepsilon)} \big),
\end{aligned} \tag{3.27}$$

for some $\sigma' > 0$. Combining (3.26) and (3.27), we are led to

$$\begin{aligned}
&- \sum_{j=1}^{k_1+k_2} \xi \bar{W}_{\varepsilon,x_j}^{p-1} + \big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_j} \big)_+^{p-1} \\
&\le C \Big(\sum_{j=1}^{k_1} e^{-d_j/\varepsilon} U^{p-2} \Big(\frac{|\tilde{y}-x_j|}{\varepsilon} \Big) \\
&\quad + \sum_{j=k_1+1}^{k_1+k_2} \varepsilon U^{p-2} \Big(\frac{|\tilde{y}-x_j|}{\varepsilon} \Big) + \sum_{j=1}^{k_1+k_2} e^{-\delta'/\varepsilon} W_{\varepsilon,x_j} \Big) \\
&\quad + C \sum_{i \ne j} e^{-(1+\sigma')|x_i-x_j|/(4\varepsilon)} \big(e^{-(1+\sigma')|\tilde{y}-x_i|/(2\varepsilon)} + e^{-(1+\sigma')|\tilde{y}-x_j|/(4\varepsilon)} \big).
\end{aligned} \tag{3.28}$$

Direct calculations show that for $x_i \in D$, $h = 1, \dots, N-m+1$ and $x_j \in \partial D$, $l = 1, \dots, N-m$

$$\Big| \frac{\partial g_{\varepsilon,x_i}(y)}{\partial x_{i,h}} \Big| \le C \varepsilon^{-1} U^{p-1} \Big(\frac{|\tilde{y}-x_i|}{\varepsilon} \Big) + C U \Big(\frac{|\tilde{y}-x_i|}{\varepsilon} \Big), \tag{3.29}$$

$$\Big| \frac{\partial g_{\varepsilon,x_j}(y)}{\partial \tau_{j,l}} \Big| \le C \varepsilon^{-1} U^{p-1} \Big(\frac{|\tilde{y}-x_j|}{\varepsilon} \Big) + C U \Big(\frac{|\tilde{y}-x_j|}{\varepsilon} \Big). \tag{3.30}$$

Combining (3.23), (3.25), (3.28), (3.29) and (3.30), we find that

$$\begin{aligned}
|G_{\varepsilon,x}(y)| &\le C \sum_{j=1}^{k_1+k_2} \Big(e^{-\delta'/\varepsilon} U \big(\frac{|\tilde{y}-x_j|}{\varepsilon} \big) + \varepsilon^{\sigma+1/2} U \big(\frac{|\tilde{y}-x_j|}{\varepsilon} \big) \Big) + |\omega|^{p^*-1} \\
&\quad + C \sum_{j=1}^{k_1} e^{-d_j/\varepsilon} U^{p-2} \big(\frac{|\tilde{y}-x_j|}{\varepsilon} \big) + C \sum_{j=k_1+1}^{k_1+k_2} \varepsilon U^{p-2} \big(\frac{|\tilde{y}-x_j|}{\varepsilon} \big) \\
&\quad + C \sum_{i \ne j} e^{-(1+\sigma')|x_i-x_j|/(4\varepsilon)} \big(e^{-(1+\sigma')|\tilde{y}-x_i|/(2\varepsilon)} + e^{-(1+\sigma')|\tilde{y}-x_j|/(4\varepsilon)} \big) \\
&\le C \sum_{j=1}^{k_1+k_2} \Big(\varepsilon^{\alpha} e^{-\alpha|\tilde{y}-x_j|/\varepsilon} \Big)^{p^*-1}.
\end{aligned} \tag{3.31}$$

if $\alpha > 0$ is small enough.

With the same technique in [10] and using the theory of L^p-estimate and Schauder estimate on elliptic equation, (3.31) yields

$$|\omega_1| \le C\varepsilon^{(p^*-1)\alpha}, \quad \text{in } B. \tag{3.32}$$

Let

$$a_\varepsilon(y) = \Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big)^{p-2}\eta,$$

where η is a C^1 function, such that $\eta = 0$ if $y \in B$. It is easy to see that $a_\varepsilon(y) \to 0$ uniformly in Ω as $\varepsilon \to 0$. From (3.32), we have

$$-\varepsilon^2\Delta\omega_1 + (1-(p-1)a_\varepsilon)\omega_1 = G_{\varepsilon,x}(y) + O(\varepsilon^{(\tilde p-1)\alpha})\Big(\sum_{j=1}^{k_1+k_2} W_{\varepsilon,x_j}\Big)^{p-2}. \tag{3.33}$$

Let $\tilde\omega \in H$ be the solution of

$$\begin{cases} -\varepsilon^2\Delta\tilde\omega + \frac{1}{2}\tilde\omega = \Big|G_{\varepsilon,x}(y) + O(\varepsilon^{(\tilde p-1)\alpha})\Big(\sum_{j=1}^{k_1+k_2} W_{\varepsilon,x_j}\Big)^{p-2}\Big|, & \text{in } \Omega \\ \dfrac{\partial\tilde\omega}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.34}$$

By the maximum principle, we have $\tilde\omega > 0$.

Let $v = \omega_1 - \tilde\omega$. Then

$$-\varepsilon^2\Delta v + \frac{1}{2}v \le -(\frac{1}{2} - (p-1)a_\varepsilon)\omega_1.$$

Multiplying the above relation by v_+ and integrating by part, we obtain

$$\int_\Omega \big(\varepsilon^2|Dv_+|^2 + \frac{1}{2}v_+^2\big) \le -(\frac{1}{2} - (p-1)a_\varepsilon)\int_\Omega v_+\omega_1 \le 0.$$

Thus, $v_+ = 0$. That is, $\omega_1 \le \tilde\omega$. Similarly, $-\omega_1 \le \tilde\omega$. Hence,

$$|\omega_1| \le \tilde\omega. \tag{3.35}$$

Now, by the estimate (3.31) and the arguments used in the proof of Lemma 2.1, it is easy to verify

$$|\omega_1(y)| \le \tilde\omega(y) \le \varepsilon^\alpha \sum_{j=1}^{k_1+k_2} e^{-\alpha|y-x_j|/\varepsilon}. \tag{3.36}$$

Therefore, the contraction mapping theorem yields that there exists an $\omega_{\varepsilon,x} \in \tilde S_{\varepsilon,x}$, such that

$$\omega_{\varepsilon,x} = G_{\varepsilon,x}\omega_{\varepsilon,x}.$$

Moreover, by (3.21),

$$\|\omega_{\varepsilon,x}\|_\varepsilon \le C\varepsilon^{\sigma+(N-m+2)/2}.$$

∎

In the following, we will choose $x_\varepsilon \in D^*_\varepsilon$, such that (3.2) and (3.3) are satisfied with $A_{j,l}$.

Proof of Theorem 1.1. By our assumption on Ω, we can deduce that there is a constant $\delta' \in (0, \delta)$, such that

$$\max_{z'' \in B_\delta(\bar{x}) \setminus B_{\delta'}(\bar{x})} \psi(z'') < \max_{z'' \in B_\delta(\bar{x})} \psi(z'').$$

Define

$$D^* = \{z = (z_1, z'') : z_1 \in (\psi(z'') - \gamma, \psi(z'')), z'' \in B_{\delta'}(\bar{x})\},$$

where $\gamma > 0$ is a small constant.

Define

$$D_\varepsilon = \big\{x = (x_1, \cdots, x_{k_1+k_2}) : \ x \in D^*_\varepsilon, \ x_j \in D^*, j = 1, \cdots, k_1\big\}. \tag{3.37}$$

Let

$$K(x) = J(x, \omega_{\varepsilon,x}), \quad x \in D_\varepsilon.$$

Consider the following problem:

$$\max_{x \in \overline{D}_\varepsilon} K(x). \tag{3.38}$$

Let $x_\varepsilon \in \overline{D}_\varepsilon$ be a maximum point of (3.38). We will prove that x_ε is an interior point of D_ε. Thus, x_ε is a critical point of $K(x)$.

It follows from Propositions 3.1 and 2.1 that for any $x \in D_\varepsilon$,

$$\begin{aligned} K(x) &= J(x, 0) + O(\varepsilon^{N-m+2+\sigma}) \\ &= A\varepsilon^{N-m+1} \sum_{j=1}^{k_1} x_{j,1}^{m-1} + \frac{1}{2} A\varepsilon^{N-m+1} \sum_{j=k_1+1}^{k_1+k_2} x_{j,1}^{m-1} \\ &\quad + \frac{1}{2} \sum_{j=1}^{k_1} \tau_{\varepsilon, x_j} - c_0 \varepsilon^{N-m+1} \sum_{i<j} U\big(\frac{|x_i - x_j|}{\varepsilon}\big) \\ &\quad + \varepsilon^{N-m+1} O\big(\sum_{j=1}^{k_1} e^{-(2+\sigma) d_j/\varepsilon} + \sum_{j \neq i} e^{-(1+\sigma)|x_i - x_j|/\varepsilon} + \varepsilon\big). \end{aligned} \tag{3.39}$$

Let $\bar{x}_{\varepsilon,j} = (\bar{x}_{\varepsilon,j,1}, x''_{\varepsilon,j}) \in D$, $\bar{x}_{\varepsilon,j,1} = \bar{x}_1 - Lj\varepsilon |\ln \varepsilon| \psi(\bar{x}'')$, $\bar{x}''_{\varepsilon,j} = \bar{x}''$, $\bar{x}_\varepsilon = (\bar{x}_{\varepsilon,1}, \cdots, \bar{x}_{\varepsilon,k})$, $j = 1, \ldots, k_1$, and $\bar{x}_{\varepsilon,j} = (\bar{x}_{\varepsilon,j,1}, x''_{\varepsilon,j}) \in \partial D$, $\bar{x}_{\varepsilon,j,1} = \bar{x}_1 - Lj\varepsilon|\ln \varepsilon|$, $j = k_1, \ldots, k_1 + k_2$, where $L > 0$ are large constants. Then $\bar{x}_\varepsilon \in D_\varepsilon$. Moreover,

$$\tau_{\varepsilon, \bar{x}_j} = O(\varepsilon^{N-m+3}),$$

for $L > 0$ large. It is easy to see that for $j \neq i$,

$$e^{-|x_i - x_j|/\varepsilon} = O(\varepsilon^{N-m+3}).$$

So, from (3.39), we obtain

$$K(\bar{x}_\varepsilon) = A(k_1 + k_2/2)\varepsilon^{N-m+1}\bar{x}_1^{m-1} + \varepsilon^{N-m+1}O\big(\varepsilon|\ln\varepsilon|\big). \tag{3.40}$$

Note that for any $x \in D_\varepsilon$, we have

$$\tau_{\varepsilon,x_j} < 0,\ j = 1, \dots, k_1; \qquad x_{j,1} \le \bar{x}_1,\ j = 1, \dots, k_1 + k_2.$$

Suppose that $x_\varepsilon \in \partial D_\varepsilon$.

If $e^{-2d(x_{\varepsilon,j},\partial D)/\varepsilon} = \varepsilon^{1-\tilde{\theta}}$ for some $j \in \{1, \dots, k_1\}$, then, by (3.39),

$$\begin{aligned} K(x_\varepsilon) &\le A(k_1 + k_2/2)\varepsilon^{N-m+1}\bar{x}_1^{m-1} - \frac{1}{2}\varepsilon^{N-m+1+(1+\theta)(1-\tilde{\theta})} + O(\varepsilon^{N-m+2}) \\ &< K(\bar{x}_\varepsilon), \end{aligned}$$

since $(1 - \tilde{\theta})(1 + \theta) < 1$ if $\theta > 0$ is small enough. This is a contradiction.

Suppose that there is a $j \in \{1, \dots, k_1\}$, such that $x_{\varepsilon,j}$ satisfies $x_{\varepsilon,j,1} = \psi(x''_{\varepsilon,j}) - \gamma$, or $x_{\varepsilon,j} \in \partial B_{\delta'}(\bar{x})$, or that there is a $j \in \{k_1 + 1, \dots, k_1 + k_2\}$, such that $x_j \in \partial D \cap \partial B_\delta(\bar{x})$, then $x_{\varepsilon,j,1} \le \bar{x}_1 - \beta$ for some small $\beta > 0$. So, by (3.39),

$$K(x_\varepsilon) \le A(k_1 + k_2/2)\varepsilon^{N-m+1}\bar{x}_1^{m-1} - c'\varepsilon^{N-m+1} + \varepsilon^{N-m+1}O\big(\varepsilon\big) < K(\bar{x}_\varepsilon),$$

where $c' > 0$ is a small constant. This is a contradiction.

Suppose that there are $i, j \in \{1, \dots, k_1+k_2\}$, $i \neq j$, such that $e^{-|x_i - x_j|/\varepsilon} = \varepsilon^{1-\tilde{\theta}}$. Then

$$K(x_\varepsilon) \le A(k_1 + k_2/2)\varepsilon^{N-m+1}\bar{x}_1^{m-1} - c_0\varepsilon^{N-m+2-\tilde{\theta}} + O(\varepsilon^{N-m+2}) < K(\bar{x}_\varepsilon).$$

This is also a contradiction.

So x_ε is an interior point of D_ε. As a result,

$$\begin{aligned} D_{x_{i,h}}K(x_\varepsilon) &= 0,\ i = 1, \dots, k_1;\ h = 1, \dots, N - m + 1; \\ D_{\tau_{j,l}}K(x_\varepsilon) &= 0,\ j = k_1 + 1, \dots, k_1 + k_2;\ l = 1, \dots, N - m, \end{aligned}$$

where

$$\begin{aligned} D_{x_{i,h}}K(x_\varepsilon) &= \frac{\partial J}{\partial x_{i,h}} + \Big\langle \frac{\partial J}{\partial \omega}, \frac{\partial \omega}{\partial x_{i,h}} \Big\rangle_{D,\varepsilon}; \\ D_{\tau_{j,l}}K(x_\varepsilon) &= \frac{\partial J}{\partial \tau_{j,l}} + \Big\langle \frac{\partial J}{\partial \omega}, \frac{\partial \omega}{\partial \tau_{j,l}} \Big\rangle_{D,\varepsilon}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\Big\langle \frac{\partial J}{\partial \omega}, \frac{\partial \omega}{\partial x_{i,h}} \Big\rangle_{D,\varepsilon} &= \sum_{k=1}^{N-m+1} A_{i,k} \Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_i}}{\partial x_{i,k}}, \frac{\partial \omega}{\partial x_{i,h}} \Big\rangle_{D,\varepsilon} \\ &= -\sum_{k=1}^{N-m+1} A_{i,k} \Big\langle \frac{\partial^2 P_{\varepsilon,\Omega} W_{\varepsilon,x_i}}{\partial x_{i,k} \partial x_{i,h}}, \omega \Big\rangle_{D,\varepsilon}, \\ \Big\langle \frac{\partial J}{\partial \omega}, \frac{\partial \omega}{\partial \tau_{j,l}} \Big\rangle_{D,\varepsilon} &= \sum_{k=1}^{N-m} A_{j,k} \Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial \tau_{j,k}}, \frac{\partial \omega}{\partial \tau_{j,l}} \Big\rangle_{D,\varepsilon} \\ &= -\sum_{k=1}^{N-m} A_{j,k} \Big\langle \frac{\partial^2 P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial \tau_{j,k} \partial \tau_{j,l}}, \omega \Big\rangle_{D,\varepsilon}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial J}{\partial x_{i,h}} &= \sum_{k=1}^{N-m+1} A_{i,k} \Big\langle \frac{\partial^2 P_{\varepsilon,\Omega} W_{\varepsilon,x_i}}{\partial x_{i,k} \partial x_{i,h}}, \omega \Big\rangle_{D,\varepsilon}, \\ \frac{\partial J}{\partial \tau_{j,l}} &= \sum_{k=1}^{N-m} A_{j,k} \Big\langle \frac{\partial^2 P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial \tau_{j,k} \partial \tau_{j,l}}, \omega \Big\rangle_{D,\varepsilon},\end{aligned}$$

which is exactly (3.2) and (3.3).

Using Lemma 3.1,

$$\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}$$

is a solution of (1.7). Since $f(y,t) = 0$ if $t \le 0$, we see that

$$\sum_{j=1}^{k} P_{\varepsilon,\Omega} W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}$$

is positive. But $\omega_{\varepsilon,x_\varepsilon} \in S_\varepsilon$. Thus $|\omega_{\varepsilon,x_\varepsilon}| \le \frac{1}{2}$ in $\Omega \setminus B$, which gives

$$f\Big(y, \sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}\Big) = \Big(\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}\Big)_+^{p-1}.$$

As a result,

$$\sum_{j=1}^{k_1+k_2} P_{\varepsilon,\Omega} W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}$$

is a solution of (1.1). ∎

4. Appendix

Lemma 4.1. *Suppose* $x_i, x_j \in D$, $x_i \neq x_j$ *and* $l, n = 1, \dots, N-m+1$, $x_h, x_k \in \partial D$, $x_h \neq x_k$, $r, \mu = 1, \dots, N-m$, *we have*

$$\Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial x_{j,l}}, \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial x_{j,l}} \Big\rangle_{D,\varepsilon} = C_1 \varepsilon^{N-m-1} + O(\varepsilon^{N-m-1} e^{-\frac{c}{\varepsilon}}),$$
$$\Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial x_{j,l}}, \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_i}}{\partial x_{i,n}} \Big\rangle_{D,\varepsilon} = O(\varepsilon^{N-m-1} e^{-\frac{|x_i - x_j|}{\varepsilon}} + \varepsilon^{N-m-1} e^{-\frac{c}{\varepsilon}}),$$
$$\Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial x_{j,l}}, \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial x_{j,n}} \Big\rangle_{D,\varepsilon} = o(\varepsilon^{N-m-1}), \quad l \neq n,$$
$$\Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_h}}{\partial \tau_{h,r}}, \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_h}}{\partial \tau_{h,r}} \Big\rangle_{D,\varepsilon} = C_2 \varepsilon^{N-m-1} + O(\varepsilon^{N-m-1} e^{-\frac{c}{\varepsilon}}),$$
$$\Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_h}}{\partial \tau_{h,r}}, \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_k}}{\partial \tau_{k,\mu}} \Big\rangle_{D,\varepsilon} = O(\varepsilon^{N-m-1} e^{-\frac{|x_h - x_k|}{\varepsilon}} + \varepsilon^{N-m-1} e^{-\frac{c}{\varepsilon}}),$$
$$\Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_h}}{\partial \tau_{h,r}}, \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_h}}{\partial \tau_{h,\mu}} \Big\rangle_{D,\varepsilon} = o(\varepsilon^{N-m-1}), \quad r \neq \mu,$$
$$\Big\langle \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_i}}{\partial x_{i,l}}, \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_h}}{\partial \tau_{h,\mu}} \Big\rangle_{D,\varepsilon} = O(\varepsilon^{N-m-1} e^{-\frac{|x_i - x_h|}{\varepsilon}} + \varepsilon^{N-m-1} e^{-\frac{c}{\varepsilon}}),$$

where $C_1, C_2 > 0$ *are constants.*

Proof. The proof is similar to [**?**] and [6] and we omit it here. ■

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