Rev. Mat. Iberoamericana **21** (2005), no. 2, 483–510

Dyadic BMO on the bidisk

Óscar Blasco and Sandra Pott

Abstract

We give several new characterizations of the dual of the dyadic Hardy space $H^{1,d}(\mathbb{T}^2)$, the so-called dyadic BMO space in two variables and denoted $\mathrm{BMO}^d_{\mathrm{prod}}$. These include characterizations in terms of Haar multipliers, in terms of the "symmetrised paraproduct" Λ_b, in terms of the rectangular BMO norms of the iterated "sweeps", and in terms of nested commutators with dyadic martingale transforms. We further explore the connection between $\mathrm{BMO}^d_{\mathrm{prod}}$ and John-Nirenberg type inequalities, and study a scale of rectangular BMO spaces.

1. Introduction

Throughout the paper $\mathcal{D}$ denotes the set of dyadic intervals in the unit circle $\mathbb{T}$. In the case of the bicircle $\mathbb{T}^2$, $\mathcal{D}_1$ denotes the dyadic intervals in the first, $\mathcal{D}_2$ the dyadic intervals in the second variable. We write $\mathcal{R} = \mathcal{D}_1 \times \mathcal{D}_2$ for the dyadic rectangles, $|I|$ for the length of $I \in \mathcal{D}$ and $|R|$ for the area of $R \in \mathcal{R}$, $(h_I)_{I\in\mathcal{D}}$ stands for the Haar basis in $L^2(\mathbb{T})$ and $(h_R)_{R\in\mathcal{R}}$ for the product Haar basis of $L^2(\mathbb{T}^2)$.

Here

$$h_I(t) = \frac{1}{|I|^{1/2}} \left(\chi_{I^+}(t) - \chi_{I^-}(t)\right)$$

for each dyadic interval $I \in \mathcal{D}$, where I^- denotes the left half of I, and I^+ denotes the right half of I. For each dyadic rectangle $R = I \times J \in \mathcal{R}$, h_R is defined by

$$h_R(s,t) = h_I(s) h_J(t).$$

We denote by $\mathcal{H}_{00}$ the space of all functions in $L^2(\mathbb{T}^2)$ which have a finite expansion in the product Haar basis.

2000 Mathematics Subject Classification: 42B30, 47B35.
Keywords: BMO on the bidisk, Carleson measures, Haar multipliers.

Given $g \in L^2(\mathbb{T})$, we use the notation $g_I = \langle g, h_I \rangle$ and $m_I g = \frac{1}{|I|}\int_I g(t)dt$. Similarly, given $f \in L^2(\mathbb{T}^2)$, we use the notation $f_R = \langle f, h_R \rangle$, $f_I(s) = \langle f(\cdot, s), h_I \rangle$, $m_I f(s) = \frac{1}{|I|}\int_I f(t,s)dt$, $f_J(t) = \langle f(t,\cdot), h_J \rangle$ and $m_J f(t) = \frac{1}{|J|}\int_J f(t,s)ds$. Therefore

$$f(t,s) = \sum_{R \in \mathcal{R}} f_R h_R(t,s) = \sum_{I \in \mathcal{D}} f_I(s) h_I(t) = \sum_{J \in \mathcal{D}} f_J(t) h_J(s).$$

Let $P_I g = (g - m_I g)\chi_I$ for $g \in L^2(\mathbb{T})$. Observe that P_I is the orthogonal projection on the subspace spanned by the Haar functions $h_{I'}$, $I' \in \mathcal{D}$, $I' \subseteq I$. If $g = \sum_{I \in \mathcal{D}} g_I h_I$, then

$$P_I g = \sum_{I' \in \mathcal{D}, I' \subseteq I} h_{I'} g_{I'}. \tag{1.1}$$

Similarly, for each measurable set $\Omega \subseteq \mathbb{T}^2$, let P_Ω be the orthogonal projection on the subspace spanned by the Haar functions $h_{R'}$, $R' \in \mathcal{R}$, $R' \subseteq \Omega$. In particular, for each dyadic rectangle $R = I \times J \in \mathcal{R}$ and for $f = \sum_{R' \in \mathcal{R}} h_{R'} f_{R'} \in L^2(\mathbb{T}^2)$, we have

$$P_R f = P_I \otimes P_J f = \sum_{R' \in \mathcal{R}, R' \subseteq R} h_{R'} f_{R'}.$$

It is easy to see that for $R \in \mathcal{R}$ and $f \in L^2(\mathbb{T}^2)$,

$$P_R f = (f - m_I f - m_J f + m_{I \times J})\, \chi_{I \times J}. \tag{1.2}$$

Recall that $g \in L^2(\mathbb{T})$ is said to belong to dyadic BMO, to be denoted $\mathrm{BMO}^d(\mathbb{T})$, if

$$\sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I |g(t) - m_I g|^2 dt \right)^{1/2} < \infty.$$

By John-Nirenberg's lemma, this is equivalent to

$$\sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I |g(t) - m_I g|^p dt \right)^{1/p} < \infty$$

for any $1 \le p < \infty$.

Hence $g \in \mathrm{BMO}^d(\mathbb{T})$ if and only if there exists a constant C such that for all $I \in \mathcal{D}$

$$\sum_{I' \in \mathcal{D}, I' \subseteq I} |g_{I'}|^2 \le C|I|^{1/2},$$

or equivalently

$$\sup_{I \in \mathcal{D}} \frac{1}{|I|^{1/p}} \|P_I g\|_p < \infty$$

for $1 \le p < \infty$.

The space BMO appears in many different contexts. We shall use that $\mathrm{BMO^d}(\mathbb{T}) = (H^{1,d}(\mathbb{T}))^*$ where $H^{1,d}$ is defined in terms of the dyadic square functions $\mathcal{S}$,

$$\mathcal{S}\, g = \left(\sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|} |g_I|^2 \right)^{1/2}.$$

That is,

$$H^{1,d}(\mathbb{T}) = \{g \in L^1(\mathbb{T}) : \mathcal{S}\, g \in L^1(\mathbb{T})\}.$$

Using Carleson measures, this gives rise to a description of $\mathrm{BMO^d}$ in terms of symbols g for which the dyadic paraproduct π_g,

$$\pi_g(f) = \sum_{I\in\mathcal{D}} g_I m_I f h_I$$

or its adjoint operator Δ_g, $\Delta_g(f) = \sum_{I\in\mathcal{D}} g_I f_I \frac{\chi_I}{|I|}$, is bounded on $L^2(\mathbb{T})$ (or equivalently, on $L^p(\mathbb{T})$ for $1 < p < \infty$).

The situation in two variables it is rather different and much more delicate. One main reason for the difficulties encountered in the multi-variable theory is the failure of the naive generalization of the Carleson Embedding Theorem to several variables (see [C], [Fef]). The reader is referred to [ChFef2] for an overview on the theory and an outline of the main differences.

Several new results (e. g. [FS] and [PS]) further exhibit the differences between certain BMO spaces on the polydisk defined by multi-variable versions of the different yet equivalent characterizations of $\mathrm{BMO}(\mathbb{T})$.

A function $f \in L^2(\mathbb{T}^2)$ is said to belong to the rectangular dyadic BMO space, to be denoted $\mathrm{BMO}^d_{\mathrm{rect}}$, if

$$\sup_{R=I\times J\in\mathcal{R}} \left(\frac{1}{|R|} \int_R |f(t,s) - m_I f(s) - m_J f(t) + m_{I\times J} f|^2 dt ds \right)^{1/2} < \infty. \tag{1.3}$$

Or equivalently,

$$\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect}}} = \sup_{R\in\mathcal{R}} \frac{1}{|R|^{1/2}} \|P_R \varphi\|_2.$$

We will also consider a p-version of the dyadic rectangular norm for $1 \le p < \infty$,

$$\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect},p}} = \sup_{R\in\mathcal{R}} \frac{1}{|R|^{1/p}} \|P_R \varphi\|_p. \tag{1.4}$$

Here, $\|\cdot\|_{\mathrm{BMO}^d_{\mathrm{rect}}} = \|\cdot\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect},2}}$. In the one-variable case, the corresponding norms are of course all equivalent because of John-Nirenberg's lemma.

Let us start by defining $\mathrm{BMO}^d_{\mathrm{prod}}(\mathbb{T}^2)$ as the dual of $H^{1,d}(\mathbb{T}^2)$, the space of functions $f \in L^1(\mathbb{T}^2)$ such that $\mathcal{S}(f) \in L^1(\mathbb{T}^2)$, where

$$\mathcal{S}(f) = \left(\sum_{R\in\mathcal{R}} |f_R|^2 h_R^2 \right)^{1/2}.$$

Although $\mathrm{BMO}^d_{\mathrm{prod}}(\mathbb{T}^2)$ cannot be characterized by (1.3) [Fef], it was shown by Bernard in the dyadic case [Be] and also by Chang and R. Fefferman in a continuous version [ChFef1] that $\mathrm{BMO}^d_{\mathrm{prod}}(\mathbb{T}^2)$ can also be described as the space of functions $\varphi \in L^2(\mathbb{T}^2)$ for which there exists $C > 0$ such that

$$\|\varphi\|_{\mathrm{prod}} = \sup_{\Omega\subset\mathbb{T}^2} \frac{1}{|\Omega|^{1/2}} \|P_\Omega \varphi\|_2 < \infty, \tag{1.5}$$

where the supremum is taken over all measurable sets $\Omega \subseteq \mathbb{T}^2$. This immediately implies $\mathrm{BMO}^d_{\mathrm{prod}} \subseteq \mathrm{BMO}_{\mathrm{rect},2}$.

The connection between both spaces can be also seen from the description of $\mathrm{BMO}^d_{\mathrm{prod}}$ in terms of the boundedness of the dyadic paraproduct in two variables, defined by $\pi_b^{(1,2)}(f) = \sum_{R\in\mathcal{R}} b_R m_R f h_R$.

It follows from Chang's generalization of the Carleson Embedding Theorem (see [Ch]) that $b \in \mathrm{BMO}^d_{\mathrm{prod}}$ if and only if the double paraproduct $\pi_b^{(1,2)}$ is bounded on $L^2(\mathbb{T}^2)$. In our paper the following fact will be rather crucial:

$$||\varphi||_{\mathrm{prod}} \approx ||\pi_\varphi^{(1,2)}||. \tag{1.6}$$

An similar characterization for $\mathrm{BMO}^d_{\mathrm{rect}}$ was proved in [PS], Proposition 3.3.1, namely that $b \in \mathrm{BMO}^d_{\mathrm{rect}}$ if and only if $\pi_b^{(1,2)}$ maps $L^2(\mathbb{T})\hat{\otimes}L^2(\mathbb{T})$ boundedly into $L^2(\mathbb{T}^2)$, where $L^2(\mathbb{T})\hat{\otimes}L^2(\mathbb{T})$ stands for the projective tensor product. This also implies that $\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \mathrm{BMO}^d_{\mathrm{rect}}$ (see [Fef] for an alternative approach).

We shall try to better understand the difference between both spaces. Two approaches are used to this end. First we observe that John-Nirenberg type inequalities do not hold in $\mathrm{BMO}^d_{\mathrm{rect}}$, in the sense that the 2-norm in the definition of $\mathrm{BMO}^d_{\mathrm{rect}}$ cannot be replaced by any other p-norm. This solves a question left open in [FS]. Secondly, we analyse the behaviour of the sweep of functions in the $\mathrm{BMO}^d_{\mathrm{prod}}$ and in the $\mathrm{BMO}^d_{\mathrm{rect},p}$ spaces.

Our main new tool will be characterizations of $\mathrm{BMO}^d_{\mathrm{prod}}$ in terms of Haar multipliers. Recall that sequence of functions $(\phi_R)_{R\in\mathcal{R}}$ is called a Haar multiplier (see (2.16) or [Per]) on $L^p(\mathbb{T}^2)$, if the map

$$f = \sum_{R\in\mathcal{R}} f_R h_R(t,s) \mapsto \sum_{R\in\mathcal{R}} \phi_R(t,s) f_R h_R(t,s)$$

defines a bounded operator on $L^p(\mathbb{T}^2)$.

We shall say that $b \in \mathrm{BMO}^d_{\mathrm{mult}}$ if $\{(P_R b)\}_{R\in\mathcal{R}}$ defines a Haar multiplier on $L^2(\mathbb{T}^2)$.

Using the characterization of $\mathrm{BMO}^d_{\mathrm{prod}}$ in terms of dyadic paraproducts, we observe that $b \in \mathrm{BMO}^d_{\mathrm{prod}}$ if the operator $\Delta_b = (\pi^{(1,2)}_{\bar{b}})^*$ given by $\Delta_b(f) = \sum_R b_R h_R f_R h_R$ defines a bounded operator on $L^2(\mathbb{T}^2)$. Hence $b \in \mathrm{BMO}^d_{\mathrm{prod}}$ if and only if $(b_R h_R)_{R\in\mathcal{R}}$ is a Haar multiplier on $L^2(\mathbb{T}^2)$.

On the other hand, letting the Haar multiplier $(P_{R'}b)_{R'\in\mathcal{R}}$ act on h_R, we see that $\frac{1}{|R|^{1/2}}\|P_R b\|_2 \le \|b\|_{\mathrm{mult}}$, implying that $\mathrm{BMO}^d_{\mathrm{mult}} \subseteq \mathrm{BMO}^d_{\mathrm{rect}}$.

We shall get a description of $\mathrm{BMO}^d_{\mathrm{mult}}$ in terms of the boundedness of the operator

$$\Lambda_b = \pi^{(1,2)}_b + (\pi^{(1,2)}_b)^* + \Delta_{\pi_b} + (\Delta_{\pi_b})^*,$$

where Δ_{π_b} (see Definition 2.4) is an operator combining the one-variable paraproduct π and its adjoint. This will allow us to prove that $\mathrm{BMO}^d_{\mathrm{mult}}(\mathbb{T}^2) = \mathrm{BMO}^d_{\mathrm{prod}}(\mathbb{T}^2)$ (see Theorem 2.8).

On the other hand, $\mathrm{BMO}^d_{\mathrm{rect}}$ can also be described using Λ_b. We show that $\mathrm{BMO}^d_{\mathrm{rect}}$ can be characterized in terms of "average boundedness" of Λ_b or in terms of its boundedness from $L^2(\mathbb{T})\hat{\otimes}L^2(\mathbb{T})$ into $L^2(\mathbb{T}^2)$.

The paper is divided into three sections. The first one is devoted to the introduction of the space $\mathrm{BMO}^d_{\mathrm{mult}}$ and the proof of some of its properties. We see that $\mathrm{BMO}^d_{\mathrm{mult}}$ can be characterized as the space of symbols b for which the operator Λ_b is bounded, and that this space coincides with $\mathrm{BMO}^d_{\mathrm{prod}}$.

Section 3 deals with results on sweep functions. We prove the following formula connecting the boundedness of $\pi^{(1,2)}_b$ and Λ_{S_b} (see Lemma 3.2):

$$\pi^{(1,2)^*}_b \pi^{(1,2)}_b = \Lambda_{S_b} + D_b, \tag{1.7}$$

where D_b is bounded if $b \in \mathrm{BMO}^d_{\mathrm{rect},2}$.

This allows us to see that $b \in \mathrm{BMO}^d_{\mathrm{prod}}$ if and only if $S_b \in \mathrm{BMO}^d_{\mathrm{prod}}$. We also obtain a characterization of $\mathrm{BMO}^d_{\mathrm{prod}}$ in terms of nested commutators with dyadic martingale transforms, sharpening a result from [PS].

We further use the formula (1.7) to quantify the difference between the BMO spaces we have considered, and to get a characterization of $\mathrm{BMO}^d_{\mathrm{prod}}$ relying only upon the $\mathrm{BMO}^d_{\mathrm{rect}}$ norm of the n-fold sweeps.

Finally, in the last section, we apply the results from the previous ones together with interpolation to study the scale of spaces $\mathrm{BMO}^d_{\mathrm{rect},p}$ introduced in (1.4) and show that these spaces are pairwise distinct. As a corollary, we obtain that $\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \bigcap_{p\ge 1}\mathrm{BMO}^d_{\mathrm{rect},p}$.

2. BMO via Haar multipliers

Definition 2.1 *We shall say that* $b \in \mathrm{BMO}^d_{\mathrm{mult}}$ *if* $\{(P_R b)\}_{R\in\mathcal{R}}$ *defines a Haar multiplier on* $L^2(\mathbb{T}^2)$*, i.e. there exists* $C > 0$ *such that*

$$\Big\| \sum_{R\in\mathcal{R}} P_R b\, f_R h_R \Big\|_2 \leq C\|f\|_2$$

for all $f \in L^2(\mathbb{T}^2)$*. We define* $\|b\|_{\mathrm{mult}}$ *as the norm of the corresponding operator.*

Let us start by pointing out some simple facts about this space.

Given $I \in \mathcal{D}$ we write P_I for the operator on $L^2(\mathbb{T})$ given by (1.1), and $\tilde{P}_I = P_I \otimes \mathrm{id}$ for the corresponding projection on $L^2(\mathbb{T}^2)$,

$$\tilde{P}_I(f)(t,s) = \sum_{I'\in\mathcal{D}, I'\subseteq I} h_{I'}(t) f_{I'}(s).$$

Similarly, given $J \in \mathcal{D}_2$, we write $\tilde{P}_J$ for $\mathrm{id} \otimes P_J$.

Of course, $\tilde{P}_I(f)(t,s) = P_I(f(s,\cdot))(t)$ and $P_R f = \tilde{P}_J(\tilde{P}_I f)$ for $R = I \times J$.

Proposition 2.2

$$L^\infty(\mathbb{T}^2) \subset \mathrm{BMO}^d_{\mathrm{mult}}(\mathbb{T}^2) \tag{2.1}$$

$$\mathrm{BMO}^{\mathrm{d}}(\mathbb{T}) \otimes \mathrm{BMO}^{\mathrm{d}}(\mathbb{T}) \subseteq \mathrm{BMO}^d_{\mathrm{mult}}(\mathbb{T}^2) \tag{2.2}$$

Proof. Using (1.2), one easily obtains the following formula:

$$\sum_{R\in\mathcal{R}} P_R b f_R h_R = fb - \sum_{I\in\mathcal{D}} (m_I b) f_I h_I - \sum_{J\in\mathcal{D}} (m_J b) f_J h_J + \sum_{R\in\mathcal{R}} m_R b f_R h_R.$$

Now (2.1) follows from this expression together with

$$\Big\| \sum_{J\in\mathcal{D}} (m_J b) f_J h_J \Big\|^2_{L^2(\mathbb{T}^2)} = \sum_{J\in\mathcal{D}} ||m_J b\, f_J||^2_{L^2(\mathbb{T})},$$

since

$$\Big\| \sum_{J\in\mathcal{D}} (m_J b) f_J h_J \Big\|^2_{L^2(\mathbb{T}^2)} \leq ||b||^2_\infty \sum_{J\in\mathcal{D}} ||f_J||^2_{L^2(\mathbb{T})} = ||b||^2_\infty ||f||^2_2,$$

and the trivial estimates for the terms bf and $\sum_{R\in\mathcal{R}} m_R b f_R h_R$.

To see (2.2), note first that for $b_1 \in \mathrm{BMO}^{\mathrm{d}}$ and $f \in L^2(\mathbb{T})$,

$$\sum_{I\in\mathcal{D}} P_I b_1 f_I h_I = (\pi_{b_1} + \Delta_{b_1}) f = (\pi_{b_1} + (\pi_{\bar{b}_1})^*) f. \tag{2.3}$$

Therefore $(P_I b_1)_{I\in\mathcal{D}}$ defines a bounded Haar multiplier on $L^2(\mathbb{T})$.

Now let $b(t,s) = b_1(t)b_2(s)$ with $b_1, b_2 \in \mathrm{BMO^d}(\mathbb{T})$. Then $P_R(b) = P_I(b_1)P_J(b_2)$ and therefore

$$\sum_{R\in\mathcal{R}} P_R b f_R h_R = \sum_{I\in\mathcal{D}} P_I b_1 \Bigg(\sum_{J\in\mathcal{D}} P_J b_2 f_J h_J\Bigg)_I h_I.$$

This yields

$$\begin{aligned}\Big\|\sum_{R\in\mathcal{R}} P_R b f_R h_R\Big\|_2^2 &= \int_{\mathbb{T}^2}\Bigg|\sum_{I\in\mathcal{D}} P_I b_1(t)\Bigg(\sum_{J\in\mathcal{D}} P_J b_2 f_J h_J\Bigg)_I (s) h_I(t)\Bigg|_2^2 dtds\\ &\leq C\|b_1\|^2_{\mathrm{BMO}}\int_{\mathbb{T}}\sum_{I\in\mathcal{D}}\Bigg|\sum_{J\in\mathcal{D}}(P_J b_2 f_J h_J)_I(s)\Bigg|^2 ds\\ &\leq C^2\|b_1\|^2_{\mathrm{BMO}}\|b_2\|^2_{\mathrm{BMO}}\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}}|f_{I\times J}|^2\end{aligned}$$

with some absolute constant $C > 0$. ■

As announced in the introduction, we first relate this space to $\mathrm{BMO}^d_{\mathrm{prod}}$. For this purpose we introduce the dyadic paraproducts in two variables (see [PS]):

Definition 2.3 *Given $b \in L^2(\mathbb{T}^2)$, we write*

$$\pi_b^{(1,2)}(f) = \sum_{R\in\mathcal{R}} b_R m_R f h_R$$

and

$$\Delta_b^{(1,2)}(f) = (\pi_{\bar b}^{(1,2)})^*(f) = \sum_{R\in\mathcal{R}} b_R f_R \frac{\chi_R}{|R|}.$$

The formula

$$\langle \pi_b^{(1,2)}(f), g\rangle = \langle f, \Delta_{\bar b}^{(1,2)}(g)\rangle = \int_{\mathbb{T}^2} b\Bigg(\sum_{R\in\mathcal{R}} m_R(f)\bar g_R h_R\Bigg) dtds \tag{2.4}$$

for $f, g \in \mathcal{H}_{00}$ completely describe the action of the operators $\pi_b^{(1,2)}$and $\Delta_b^{(1,2)}$.

Let us now define the following mixed operators (see [PS]).

Definition 2.4 *Given $b \in L^2(\mathbb{T}^2)$, we define the operators π_{Δ_b} and Δ_{π_b} by*

$$\langle \pi_{\Delta_b}(f), g\rangle = \langle f, \Delta_{\pi_{\bar b}}(g)\rangle = \int_{\mathbb{T}^2} b\Bigg(\sum_{I\times J\in\mathcal{R}} m_I(f_J) m_J(\bar g_I) h_{I\times J}\Bigg) dtds \tag{2.5}$$

for $f, g \in \mathcal{H}_{00}$.

We write

$$\Lambda_b = \pi_b^{(1,2)} + \Delta_b^{(1,2)} + \Delta_{\pi_b} + \pi_{\Delta_b}.$$

Clearly we have the following expressions:

$$\pi_b^{(1,2)}(f)(t,s) = \sum_{I\in\mathcal{D}} \pi_{b_I}(m_I f)(s)h_I(t) \tag{2.6}$$

$$\Delta_b^{(1,2)}(f)(t,s) = \sum_{I\in\mathcal{D}} \Delta_{b_I}(f_I)(s)h_I^2(t) \tag{2.7}$$

$$\Delta_{\pi_b}(f)(t,s) = \sum_{I\in\mathcal{D}} \pi_{b_I}(f_I)(s)h_I^2(t) \tag{2.8}$$

$$\pi_{\Delta_b}(f)(t,s) = \sum_{I\in\mathcal{D}} \Delta_{b_I}(m_I f)(s)h_I(t). \tag{2.9}$$

Lemma 2.5 *Let $R = I \times J \in \mathcal{R}$ and denote $R_+ = I^+ \times J^+ \cup I^- \times J^-$ and $R_- = I^+ \times J^- \cup I^- \times J^+$. Then*

$$\pi_b^{(1,2)}(h_R) = (P_{R_+}(b) + P_{R_-}(b))h_R = (P_{R_+}(b) - P_{R_-}(b))|R|^{-1/2} \tag{2.10}$$

Proof. Using that $m_{R'}(h_R) \neq 0$ only if $I' \subsetneq I$ and $J' \subsetneq J$ and that in this case $m_{R'}(h_R) = h_R(x_{R'})$, where $x_{R'} = (t_{I'}, s_{J'})$ is the center of R, we obtain that

$$\pi_b^{(1,2)}(h_R) = \sum_{I'\subsetneq I, J'\subsetneq J} b_{R'} h_R(x_{R'})h'_R.$$

Observe that $h_R(x_{R'}) = h_R(t,s) = \frac{1}{|R|^{1/2}}$ for $R' \subset R_+$ and $(t,s) \in R^+$. Similarly $h_R(x_{R'}) = h_R(t,s) = -\frac{1}{|R|^{1/2}}$ for $R' \subset R_+$ and $(t,s) \in R^-$. This gives (2.10). ∎

Corollary 2.6 *Let $b \in L^2(\mathbb{T}^2)$. Then $b \in \mathrm{BMO}^d_{\mathrm{prod}}$ if and only if $(P_{R_+}(b) + P_{R_-}(b))h_R)_{R\in\mathcal{R}}$ is a Haar multiplier on $L^2(\mathbb{T}^2)$.*

Lemma 2.7 $\Lambda_b(f) = \sum_{R\in\mathcal{R}} P_R(b) f_R h_R.$

Proof. Note that for $\phi, g \in \mathcal{H}_{00}$, we have

$$\phi g = \pi_\phi(g) + \Delta_\phi(g) + \pi_g(\phi). \tag{2.11}$$

As in (2.3), one obtains

$$\sum_{J\in\mathcal{D}} P_J(\phi) g_J h_J = \pi_\phi(g) + \Delta_\phi(g). \tag{2.12}$$

Combining (2.12) with the formulas in (2.6)- (2.9) we get

$$\begin{aligned}\Lambda_b(f)(t,s) &= \sum_{I\in\mathcal{D}}(\pi_{b_I}+\Delta_{b_I})(m_I f)(s)h_I(t) + \sum_{I\in\mathcal{D}}(\pi_{b_I}+\Delta_{b_I})(f_I)(s)h_I^2(t)\\
&= \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}} P_J(b_I)(s)(m_I f)_J h_J(s)h_I(t) + \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}} P_J(b_I)(s)(f_I)_J h_J(s)h_I^2(t)\\
&= \sum_{J\in\mathcal{D}}(\sum_{I\in\mathcal{D}}(\tilde{P}_J b)_I(s)m_I(f_J)h_I(t))h_J(s) + \sum_{J\in\mathcal{D}}(\sum_{I\in\mathcal{D}}(\tilde{P}_J b)_I(s)(f_J)_I h_I^2(t))h_J(s)\\
&= \sum_{J\in\mathcal{D}}\left(\pi_{\tilde{P}_J b(s,\cdot)}(f_J)\right)(t)h_J(s) + \sum_{J\in\mathcal{D}}\left(\Delta_{\tilde{P}_J b(s,\cdot)}(f_J)\right)(t)h_J(s)\\
&= \sum_{I\times J\in\mathcal{R}} \tilde{P}_I(\tilde{P}_J(b))(t,s)f_{I\times J}h_{I\times J}(t,s)\\
&= \sum_{R\in\mathcal{R}} P_R(b)(t,s)f_R h_R(t,s).\end{aligned}$$

■

We now are ready to prove our characterization of $\mathrm{BMO}^d_{\mathrm{prod}}$ in terms of Haar multipliers.

Theorem 2.8 $\mathrm{BMO}^d_{\mathrm{prod}} = \mathrm{BMO}^d_{\mathrm{mult}}$.

Proof. To see that $\mathrm{BMO}^d_{\mathrm{prod}} \subseteq \mathrm{BMO}^d_{\mathrm{mult}}$, it suffices to see that the boundedness of $\pi_b^{(1,2)}$ implies the boundedness of Δ_{π_b}. This was proved in [PS], we include here a proof for the sake of completeness.

By (2.5) and the characterization of $\mathrm{BMO}^d_{\mathrm{prod}}$ as the dual of $H^{1,d}$, the space of functions with integrable square function, we simply need to show that

$$F = \sum_{I\times J\in\mathcal{R}} m_I(f_J)m_J(g_I)h_{I\times J}(t,s)$$

belongs to $H^{1,d}$. Note that

$$\mathcal{S}(F)(t,s) = \left(\sum_{I\times J\in\mathcal{R}} |m_I(f_J)|^2|m_J(g_I)|^2 h^2_{I\times J}(t,s)\right)^{1/2}.$$

Therefore

$$\begin{aligned}\mathcal{S}(F)(t,s) &\le \left(\sum_{J\in\mathcal{D}}\sum_{I\in\mathcal{D}}(g_I^*(s))^2h_J^2(s)(f_J^*(t))^2h_I^2(t)\right)^{1/2}\\
&= \left(\sum_{I\in\mathcal{D}}(g_I^*(s))^2h_I^2(t)\right)^{1/2}\left(\sum_{J\in\mathcal{D}}(f_J^*(t))^2h_J^2(s)\right)^{1/2},\end{aligned}$$

and hence

$$\int_{\mathbb{T}^2} \mathcal{S}(F)(t,s)dtds \le$$

$$\le \Big(\int_{\mathbb{T}^2}\sum_{I\in\mathcal{D}}(g_I^*(s))^2h_I^2(t)dtds\Big)^{1/2}\Big(\int_{\mathbb{T}^2}\sum_{J\in\mathcal{D}}(f_J^*(t))^2h_J^2(s)dtds\Big)^{1/2}$$

$$= \Big(\sum_{I\in\mathcal{D}}\int_{\mathbb{T}}(g_I^*(s))^2ds\Big)^{1/2}\Big(\sum_{J\in\mathcal{D}}\int_{\mathbb{T}}(f_J^*(t))^2dt\Big)^{1/2}$$

$$\le C\Big(\sum_{I\in\mathcal{D}}||g_I||_2^2\Big)^{1/2}\Big(\sum_{J\in\mathcal{D}}||f_J||_2^2\Big)^{1/2} = C||g||_2||f||_2.$$

To prove the reverse inclusion $\mathrm{BMO}^d_{\mathrm{mult}} \subseteq \mathrm{BMO}^d_{\mathrm{prod}}$, we shall use the characterization of $\mathrm{BMO}^d_{\mathrm{prod}}$ given in (1.6).

It is clear that for each measurable set Ω, we have $P_\Omega(b) = P_\Omega(\pi_b^{(1,2)}(\chi_\Omega))$. We shall show now that

$$P_\Omega(\pi_b^{(1,2)}(\chi_\Omega)) = P_\Omega(\Lambda_b(\chi_\Omega)).$$

Let $R\in\mathcal{R}$ and $R\subseteq\Omega$. Then by (2.10)

$$\langle\Delta_b^{(1,2)}(\chi_\Omega),h_R\rangle = \langle\chi_\Omega,\pi_b^{(1,2)}(h_R)\rangle = |R|^{-1/2}\langle\chi_\Omega,P_{R^+}b - P_{R^-}b\rangle = 0.$$

This shows that $P_\Omega(\Delta_b^{(1,2)}(\chi_\Omega)) = 0$.

On the other hand, we also have for $R = I\times J\subseteq\Omega$ that

$$\pi_{\Delta_b}(h_R) = \sum_{I'\subsetneq I} b_{I'\times J}m_{I'}(h_I)\chi_J h_{I'}.$$

Using that $\langle\chi_\Omega,\chi_J h_{I'}\rangle = 0$ for all $I'\subseteq I$, we obtain $P_\Omega(\pi_{\Delta_b}(\chi_\Omega)) = 0$.

Similarly, $P_\Omega(\Delta_{\pi_b}(\chi_\Omega)) = 0$. Finally,

$$\begin{aligned} ||P_\Omega(b)|| &= ||P_\Omega(\pi_b^{(1,2)}(\chi_\Omega))|| = ||P_\Omega(\Lambda_b(\chi_\Omega))|| \\ &\le ||\Lambda_b(\chi_\Omega))|| \le ||\Lambda_b|||\Omega|^{1/2}. \end{aligned}$$

■

As a consequence of Thm 2.8, we can sharpen Thm 7.7.2 from [PS] and characterize $\mathrm{BMO}^d_{\mathrm{prod}}$ in terms of the boundedness of nested commutators with dyadic martingale transforms. This can be understood as a dyadic analogue of the characterization of the continuous product BMO space $\mathrm{BMO}_{\mathrm{prod}}$ as the space of functions for which the nested commutator

$$[H_1,[H_2,b]] : L^2(\mathbb{T}^2)\to L^2(\mathbb{T}^2)$$

is bounded, where H_1 resp. H_2 denote the Hilbert transform in the first resp. second variable on $L^2(\mathbb{T}^2)$. The latter was proved in [FS] and [LF].

Let Σ_1, Σ_2 be the spaces of all sequences of signs indexed by the elements of $\mathcal{D}_1$, $\mathcal{D}_2$, $\Sigma_1 = \{0,1\}^{\mathcal{D}_1}$, $\Sigma_1 = \{0,1\}^{\mathcal{D}_2}$, and let $d\sigma_1$ denote the natural product probability measure on Σ_1, which assigns measure 2^{-n} to each cylindrical set of length n. Let $d\sigma_2$ denote the corresponding measure on Σ_2. Let $\Sigma = \Sigma_1 \times \Sigma_2$, with $d\sigma$ denoting the product measure, and $\mathcal{R} = \mathcal{D}_1 \times \mathcal{D}_2$ as before.

For $\sigma_1 = (\sigma_1(I))_{I \in \mathcal{D}_1} \in \Sigma_1$, $\sigma_2 = (\sigma_2(J))_{J \in \mathcal{D}_2} \in \Sigma_2$, let $T_{\sigma_1}, T_{\sigma_2}$ denote the dyadic martingale transforms

$$T_{\sigma_1} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2), \quad f = \sum_{I \times J \in \mathcal{R}} f_{I \times J} h_{I \times J} \mapsto \sum_{I \times J \in \mathcal{R}} \sigma_1(I) f_{I \times J} h_{I \times J},$$

$$T_{\sigma_2} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2), \quad f = \sum_{I \times J \in \mathcal{R}} f_{I \times J} h_{I \times J} \mapsto \sum_{I \times J \in \mathcal{R}} \sigma_2(J) f_{I \times J} h_{I \times J}.$$

Theorem 2.9 *Let $b \in L^2(\mathbb{T}^2)$. Then the following are equivalent:*

(i) $b \in \mathrm{BMO}^d_{\mathrm{prod}}$

(ii) *The nested commutators*

$$[T_{\sigma_1}, [T_{\sigma_2}, b]] : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) \tag{2.13}$$

are uniformly bounded for all $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$.

(iii) *The nested commutators $[T_{\sigma_1}, [T_{\sigma_2}, b]] : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ are bounded on average, in the sense that the map*

$$\Phi_b : L^2(\mathbb{T}^2) \to L^2(\Sigma_1 \times \Sigma_2 \times \mathbb{T}^2), f \mapsto [T_{\sigma_1}, [T_{\sigma_2}, b]] f$$

is bounded.

In this case, we have

$$\|b\|_{\mathrm{BMO}^d_{\mathrm{prod}}} \approx \|\Lambda_b\| \leq \|\Phi_b\| \leq \sup_{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2} \|[T_{\sigma_1}, [T_{\sigma_2}, b]]\| \leq 4 \|\Lambda_b\|. \tag{2.14}$$

Proof. We use the ideas of the proofs of Thm 3.4, Cor 4.1 in [GPTV], adapted to the two-variable case, and of Thm 7.7.2 in [PS].

From [PS, p. 493], we know that

$$[T_{\sigma_1}, [T_{\sigma_2}, b]] = [T_{\sigma_1}, [T_{\sigma_2}, \Lambda_b]].$$

Therefore

$$\sup_{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2} \|[T_{\sigma_1}, [T_{\sigma_2}, b]]\| \leq 4 \|\Lambda_b\|.$$

The second inequality in (2.14) is obvious.

Finally, for $f \in L^2(\mathbb{T}^2)$ one has

$$
\begin{aligned}
\|\Phi_b f\|^2 &= \int\int_{\Sigma_1\times\Sigma_2} \|[T_{\sigma_1},[T_{\sigma_2},b]]f\|^2_{L^2(\mathbb{T}^2)} d\sigma_1 d\sigma_2 \\
&= \int\int_{\Sigma_1\times\Sigma_2} \|\sum_{I\times J\in\mathcal{R}} \sigma_1(I)\sigma_2(J)[\tilde P_I,[\tilde P_J,b]]f\|^2_{L^2(\mathbb{T}^2)} d\sigma_1 d\sigma_2 \\
&= \sum_{I\times J\in\mathcal{R}} \|[\tilde P_I,[\tilde P_J,b]]f\|^2_{L^2(\mathbb{T}^2)} \\
&= \sum_{I\times J\in\mathcal{R}} \|[\tilde P_I,[\tilde P_J,\Lambda_b]]f\|^2_{L^2(\mathbb{T}^2)} \qquad (2.15)\\
&= \sum_{I\times J\in\mathcal{R}} \|(\tilde P_I\tilde P_J\Lambda_b - \tilde P_I\Lambda_b\tilde P_J - \tilde P_J\Lambda_b\tilde P_I + \Lambda_b\tilde P_J\tilde P_I)f\|^2_{L^2(\mathbb{T}^2)} \\
&\geq \sum_{I\times J\in\mathcal{R}} \|\tilde P_I\tilde P_J\Lambda_b f\|^2 = \|\Lambda_b f\|^2,
\end{aligned}
$$

since $\tilde P_I\Lambda_b\tilde P_I = 0$ and $\tilde P_J\Lambda_b\tilde P_J = 0$. This proves the first inequality in (2.14). ∎

The martingale transformation approach is also interesting in the study of $\mathrm{BMO}^d_{\mathrm{rect}}$. Although Λ_b is in general not bounded for $b \in \mathrm{BMO}^d_{\mathrm{rect}}$, the space $\mathrm{BMO}^d_{\mathrm{rect}}$ can be characterized in terms of "average boundedness" of Λ_b, and also in terms of the boundedness of Λ_b from $L^2(\mathbb{T})\hat\otimes L^2(\mathbb{T})$ into $L^2(\mathbb{T}^2)$. For $\sigma = (\sigma_1,\sigma_2) \in \Sigma$, let $T_\sigma = T_{\sigma_1}T_{\sigma_2} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$.

Theorem 2.10 *For $\varphi \in \mathcal{H}_{00}$, $\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect}}}$ is equal to the norm of the operator*

$$\Psi_\varphi : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2\times\Sigma), f \mapsto \Lambda_\varphi T_\sigma f.$$

Proof. Let $f \in L^2(\mathbb{T}^2)$ and $\varphi \in \mathcal{H}_{00}$. From Lemma 2.7 we have

$$\Lambda_\varphi f = \sum_{R\in\mathcal{R}} P_R\varphi f_R h_R. \qquad (2.16)$$

Thus

$$
\begin{aligned}
\int_\Sigma\int_{\mathbb{T}^2} \|\Lambda_\varphi T_\sigma f\|^2 dt ds d\sigma &= \int_{\mathbb{T}^2}\int_\Sigma \left\|\sum_{R\in\mathcal{R}} \sigma(R)(P_R\varphi)(t,s))f_R h_R(t,s)\right\|^2 d\sigma dt ds \\
&= \int_{\mathbb{T}^2} \sum_{R\in\mathcal{R}} |f_R|^2 \frac{\chi_R(t,s)}{|R|} |(P_R\varphi)(t,s)|^2 dt ds \\
&= \sum_{R\in\mathcal{R}} |f_R|^2 \frac{1}{|R|}\|P_R\varphi\|^2.
\end{aligned}
$$

Thus the operator norm of Ψ_φ is $\sup_{R\in\mathcal{R}} \frac{1}{|R|^{1/2}}\|P_R\varphi\| = \|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect}}}$. ∎

Proposition 2.11 *If $b \in \mathrm{BMO}^d_{\mathrm{rect}}$ then Λ_b maps $L^2(\mathbb{T})\hat{\otimes}L^2(\mathbb{T})$ into $L^2(\mathbb{T}^2)$.*

Proof. Assume $f(t,s) = f_1(t)f_2(s)$ with $||f_1|| = ||f_2|| = 1$. Then we have

$$\sum_{R\in\mathcal{R}} P_R b\, f_R h_R = \sum_{I\in\mathcal{D}} \tilde{P}_I(\sum_{J\in\mathcal{D}} \tilde{P}_J b(f_2)_J h_J)(t)(f_1)_I h_I(t)$$

Writing $g(t,s) = \sum_{J\in\mathcal{D}} P_J(b(t,\cdot))(s)(f_2)_J h_J(s)$, we obtain

$$\Big\| \sum_{R\in\mathcal{R}} P_R b f_R h_R \Big\|_2^2 = \int_{\mathbb{T}}\int_{\mathbb{T}} \Big| \sum_{I\in\mathcal{D}} P_I(g(\cdot,s))(t)(f_1)_I h_I(t) \Big|^2 dtds.$$

Now let us consider g as a function in t taking values in the Hilbert space $L^2(\mathbb{T})$. Recall that as in the scalar case, the Haar multiplier norm of $(P_I g)_{I\in\mathcal{D}}$ is controlled by the vector $\mathrm{BMO^d}(\mathbb{T})$ norm of g given by

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}} \|P_I g\|_{L^2(\mathbb{T},L^2(\mathbb{T}))}.$$

Thus

$$\sup_{||f_1||=1} \int_{\mathbb{T}} \Big\| \sum_{I\in\mathcal{D}} P_I(g(\cdot,s))(t)(f_1)_I h_I(t) \Big\|^2_{L^2(\mathbb{T})} dt \le C \sup_I \frac{1}{|I|} ||P_I g||^2_{L^2(\mathbb{T},L^2(\mathbb{T}))}.$$

Notice now that

$$P_I\Big(\sum_{J\in\mathcal{D}} P_J(b(t,\cdot))(s)(f_2)_J h_J(s)\Big) = \sum_{J\in\mathcal{D}} P_{I\times J}(b)(t,s)(f_2)_J h_J(s).$$

On the other hand, applying a corresponding argument to the function $(\tilde{P}_I b)(t,s) = \sum_{J\in\mathcal{D}}(P_{I\times J} b)(t,s)h_J(s)$ understood as a function in s which takes values in $L^2(\mathbb{T})$, we obtain for $||f_2||_2 = 1$

$$\begin{aligned} ||P_I g||^2_{L^2(\mathbb{T},L^2(\mathbb{T}))} &= \int_{\mathbb{T}^2} \Big| \sum_{J\in\mathcal{D}} P_{I\times J}(b)(t,s)(f_2)_J h_J(s) \Big|^2 dtds \\ &= \int_{\mathbb{T}} \Big\| \sum_{J\in\mathcal{D}} P_{I\times J}(b)(\cdot,s)(f_2)_J h_J(s) \Big\|^2_{L^2(\mathbb{T})} ds \\ &\le C \sup_J \frac{1}{|J|} \|P_{I\times J}(b)\|^2_{L^2(\mathbb{T},L^2(\mathbb{T}))} \le C\|I\|\|b\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}. \end{aligned}$$

This finishes the proof of the proposition. ∎

3. Sweeps of functions in BMO

Let us now recall that the (dyadic) sweep of a function $\varphi \in L^2(\mathbb{T}^2)$ is defined as follows:

$$S_\varphi = \sum_{R\in\mathcal{R}} |\varphi_R|^2 \frac{\chi_R}{|R|},$$

i.e. $S_\varphi = \mathcal{S}(\varphi)^2$.

We list some properties of the sweep which will be relevant for our purposes, the proofs of which are elementary and left to the reader.

Proposition 3.1

(i) $S_\varphi(t,s) = \sum_{I\in\mathcal{D}} S_{\varphi_I}(s)\frac{\chi_I(t)}{|I|}$.

(ii) $S_\varphi = \Delta_\varphi^{(1,2)}(\varphi)$.

(iii) $P_\Omega(S_\varphi) = P_\Omega(S_{P_\Omega\varphi})$.

(iv) *If* $p > \frac{1}{2}$ *then* $\varphi \in L^{2p}(\mathbb{T}^2)$ *if and only if* $S_\varphi \in L^p(\mathbb{T}^2)$.

(v) *If* $S_\varphi \in L^\infty$ *then* $\varphi \in \mathrm{BMO}^d_{\mathrm{prod}}$.

(vi) $||S_\varphi||_2 \le C||\varphi||_{\mathrm{BMO}^d_{\mathrm{prod}}}||\varphi||_2$.

Here it is the basic result relating the boundedness of $\pi_b^{(1,2)}$ and Λ_{S_b}.

Lemma 3.2 *Let* $b \in \mathcal{H}_{00}$. *Then*

$$\pi_b^{(1,2)*}\pi_b^{(1,2)} = \Lambda_{S_b} + D_b,$$

where D_b *is a linear operator on* $L^2(\mathbb{T}^2)$ *with* $\|D_b\| \le C\|b\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}$, *and* $C > 0$ *is an absolute constant.*

Proof. Let $R = I \times J$, $R' = I' \times J' \in \mathcal{R}$.

First, observe that

$$(3.1)\quad \left\langle \pi_b^{(1,2)*}\pi_b^{(1,2)} h_R, h_{R'} \right\rangle$$

$$= \left\langle \sum_{I''\times J''\in\mathcal{D}_1\times\mathcal{D}_2} h_{I''\times J''}\, b_{I''\times J''}\, m_{I''\times J''}(h_R), \sum_{I''\times J''\in\mathcal{D}_1\times\mathcal{D}_2} h_{I''\times J''}\, b_{I''\times J''}\, m_{I''\times J''}(h_{R'}) \right\rangle$$

$$= \sum_{I''\times J''\in\mathcal{D}_1\times\mathcal{D}_2, I''\subsetneq I, J''\subsetneq J} |b_{I''\times J''}|^2\, m_{I''}(h_I)m_{I''}(h_{I'})m_{J''}(h_J)m_{J''}(h_{J'}).$$

We now do a kind of triangular truncation with respect to the indices I, I', J, J'.

(i) $I \supsetneq I'$, $J \supsetneq J'$.

$$\begin{aligned}\left\langle \pi_{S_b}^{(1,2)} h_R, h_{R'} \right\rangle &= \langle S_b, h_{R'} \rangle m_{R'}(h_R) \\ &= \left\langle \sum_{I''\times J'' \in \mathcal{D}_1\times\mathcal{D}_2} \frac{\chi_{I''\times J''}}{|I''||J''|} |b_{I''\times J''}|^2, h_{R'} \right\rangle m_{R'}(h_R) \\ &= \sum_{I''\times J'' \in \mathcal{D}_1\times\mathcal{D}_2} |b_{I''\times J''}|^2\, m_{I''}(h_{I'})\, m_{J''}(h_{J'})\; m_{I'}(h_I)\, m_{J'}(h_J).\end{aligned}$$

This is nonzero only if $I' \subsetneq I$ and $J' \subsetneq J$. In this case, we get contributions only for $I'' \subsetneq I'$ and $J'' \subsetneq J'$, and the expression agrees with (3.1).

(ii) $I \subsetneq I'$, $J \subsetneq J'$. Observe that

$$\langle \Delta_{S_b}^{(1,2)} h_R, h_{R'} \rangle = \langle h_R, \pi_{S_b}^{(1,2)} h_{R'} \rangle = \langle \pi_{S_b}^{(1,2)} h_{R'}, h_R \rangle.$$

As shown above, this equals $\langle \pi_b^{(1,2)^*} \pi_b^{(1,2)} h_{R'}, h_R \rangle$ if $I' \supsetneq I$ and $J' \supsetneq J$, and is 0 otherwise.

(iii) $I \supsetneq I'$, $J \subsetneq J'$.

$$\begin{aligned}\left\langle \pi_{\Delta_{S_b}} h_R, h_{R'} \right\rangle &= \left\langle \sum_{I''\times J'' \in \mathcal{D}_1\times\mathcal{D}_2} S_{b I''\times J''} h_{I''} \frac{\chi_{J''}}{|J''|} m_{I''}(h_{R J''}), h_{R'} \right\rangle \\ &= \left\langle \sum_{I'' \in \mathcal{D}_1} S_{b I''\times J} h_{I''} \frac{\chi_J}{|J|} m_{I''}(h_I), h_{R'} \right\rangle \\ &= S_{b I'\times J} m_{I'}(h_J) m_J(h_{I'}) = \langle S_b, h_{I'\times J} \rangle m_{I'}(h_I) m_J(h_{J'}) \\ &= \left\langle \sum_{I''\times J'' \in \mathcal{D}_1\times\mathcal{D}_2} \frac{\chi_{I''\times J''}}{|I''||J''|} |b_{I''\times J''}|^2, h_{I'\times J} \right\rangle m_{I'}(h_I) m_J(h_{J'}) \\ &= \sum_{I''\times J'' \in \mathcal{D}_1\times\mathcal{D}_2} |b_{I''\times J''}|^2 m_{I''}(h_{I'}) m_{J''}(h_J) m_{I'}(h_I) m_J(h_{J'}).\end{aligned}$$

This is nonzero only for $I' \subsetneq I$ and $J' \supsetneq J$. In this case, the sum has only contributions for $I'' \subsetneq I'$ and $J'' \subsetneq J$, and agrees with (3.1).

(iv) $I' \supsetneq I$ and $J' \subsetneq J$. Note that $\langle \Delta_{\pi_{S_b}} h_R, h_{R'} \rangle = \langle \pi_{\Delta_{S_b}} h_{R'}, h_R \rangle$. As shown above, this is only nonzero for $I' \supsetneq I$ and $J' \subsetneq J$, and agrees with (3.1) in this case.

(v) $I' = I$ or $J = J'$. Let $f \in L^2(\mathbb{T}^2)$. Then

$$\sum_{I\in\mathcal{D}_1}\sum_{J,J'\in\mathcal{D}_2} \langle \pi_b^{(1,2)^*}\pi_b^{(1,2)} h_{I\times J} f_{I\times J}, h_{I\times J'} f_{I\times J'}\rangle$$
$$= \sum_{I\in\mathcal{D}_1}\frac{1}{|I|}\sum_{I''\subsetneq I}\sum_{J''\in\mathcal{D}_2} |b_{I''\times J''}|^2 f_{I\times J} f_{I\times J'} m_{J''}(h_J) m_{J''}(h_{J'})$$
$$= \sum_{I\in\mathcal{D}_1} \|\pi_{b^I} f_I\|^2,$$

where for each I, f_I stands for the one-variable function

$$\sum_{J\in\mathcal{D}_2} h_J f_{I\times J},$$

and b^I for the function

$$\sum_{J\in\mathcal{D}_2} h_J \frac{1}{|I|^{1/2}}\Big(\sum_{I''\subsetneq I} |b_{I''\times J}|^2\Big)^{1/2}.$$

It is easy to see that

$$\|b^I\|_{\mathrm{BMO^d}} \le \|b\|_{\mathrm{BMO}^d_{\mathrm{rect}}} \qquad \text{for all } I \in \mathcal{D}_1.$$

Thus the above sum is bounded by $c\,\|b\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}\|f\|^2$, where c is an absolute constant.

The same estimate holds for the terms corresponding to $J = J'$.

Now we have counted the terms corresponding to $I = I'$, $J = J'$ twice and need to estimate them separately. Let $f \in L^2(\mathbb{T}^2)$. Then

$$\text{(3.2)}\quad \sum_{I\in\mathcal{D}_1}\sum_{J\in\mathcal{D}_2}\langle \pi_b^{(1,2)^*}\pi_b^{(1,2)} h_{I\times J} f_{I\times J}, h_{I\times J} f_{I\times J}\rangle$$
$$= \sum_{I\in\mathcal{D}_1}\sum_{J\in\mathcal{D}_2}\frac{1}{|I||J|}\sum_{I''\subsetneq I, J''\subsetneq J} |b_{I''\times J''}|^2 |f_{I\times J}|^2 \le \|b\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}\|f\|^2.$$

Defining D_b now by

$$\text{(3.3)}\quad \langle D_b f, f\rangle = \sum_{I\times J, I'\times J'\in\mathcal{R}, I'=I} \langle \pi_b^{(1,2)^*}\pi_b^{(1,2)} h_{I\times J} f_{I\times J}, h_{I'\times J'} f_{I'\times J'}\rangle +$$
$$+ \sum_{I\times J, I'\times J'\in\mathcal{R}, I'\ne I, J'=J} \langle \pi_b^{(1,2)^*}\pi_b^{(1,2)} h_{I\times J} f_{I\times J}, h_{I'\times J'} f_{I'\times J'}\rangle,$$

we obtain the statement of the lemma. ■

Now we are ready to state the main result of this section.

Theorem 3.3 *Let* $b \in \mathrm{BMO}^d_{\mathrm{rect},2}$. *Then* $b \in \mathrm{BMO}^d_{\mathrm{prod}}$ *if and only if* $S_b \in \mathrm{BMO}^d_{\mathrm{prod}}$.

Proof. We will first show that there exist $C > 0$ such that

$$\|S_b\|_{\mathrm{prod}} \leq C\|b\|^2_{\mathrm{prod}}. \tag{3.4}$$

Indeed, by Chang's Theorem [Ch], [ChFef2] it is sufficient to show that there exists a constant $C > 0$ with

$$\|P_\Omega S_b\|_2 \leq C\|b\|^2_{\mathrm{prod}}|\Omega|^{1/2}$$

for all $\Omega \subseteq \mathbb{T}^2$ measurable (see (1.5)). Using Proposition 3.1, we obtain

$$\|P_\Omega S_b\|_2 = \|P_\Omega S_{P_\Omega b}\|_2 \leq \|S_{P_\Omega b}\|_2 \leq \|P_\Omega b\|_{\mathrm{prod}}\|P_\Omega b\|_2 \leq \|b\|^2_{\mathrm{prod}}|\Omega|^{1/2} \tag{3.5}$$

For the converse, assume that $S_b \in \mathrm{BMO}^d_{\mathrm{prod}}$. Then Λ_{S_b} is bounded by Theorem 2.8. Now Lemma 3.2 finishes the proof. ■

Remark. The first implication can also be shown with the John-Nirenberg Theorem for product BMO, which was proved in [ChFef1] (for a dyadic version, see [T]).

The sweep can be understood as a bilinear map. For $f, g \in \mathcal{H}_{00}$, let

$$S_{f,g} = \sum_{R \in \mathcal{R}} \frac{\chi_R}{|R|} f_r g_r,$$

so $S_f = S_{f,\bar{f}}$.

Corollary 3.4 $S : \mathrm{BMO}^d_{\mathrm{prod}} \times \mathrm{BMO}^d_{\mathrm{prod}} \to \mathrm{BMO}^d_{\mathrm{prod}}$ *is bounded.*

Proof. The Cauchy-Schwarz inequality gives the pointwise inequality

$$S_{f,g} \leq (S_f)^{1/2}(S_g)^{1/2} \qquad \text{for } f, g \in \mathcal{H}_{00}.$$

Let $\Omega \subseteq \mathbb{T}^2$ be measurable. Using an adaption of 3.1(iii), we see that

$$\begin{aligned}\|P_\Omega S_{f,g}\|_2 = \|P_\Omega S_{P_\Omega f, P_\Omega g}\|_2 &\leq \|S_{P_\Omega f, P_\Omega g}\|_2 \leq \|(S_{P_\Omega f})^{1/2}(S_{P_\Omega g})^{1/2}\|_2 \\ &\leq \|S_{P_\Omega f}\|_2^{1/2}\|S_{P_\Omega g}\|_2^{1/2} \leq |\Omega|^{1/2}\|f\|_{\mathrm{prod}}\|g\|_{\mathrm{prod}}\end{aligned}$$

by (3.5). ■

Another application of Lemma 3.2 yields the following result.

Theorem 3.5 *Let $\|\cdot\|_*$ be an positive homogeneous function of degree 1 on $\mathcal{H}_{00}$ such that*

(i) There exists $c>0$ such that $\|\cdot\|_{\mathrm{BMO}^d_{\mathrm{rect}}} \le c\|\cdot\|_*$

(ii) There exists $k>0$ such that $\|S_\varphi\|_* \le k\|\varphi\|_*^2$.

Then there exists a constant $\tilde{C}$ such that for all $\varphi \in \mathcal{H}_{00}$,

$$\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{prod}}} \le \tilde{C}\|\varphi\|_*.$$

Proof. Throughout this proof we simply write $\pi_\varphi = \pi_\varphi^{(1,2)}$. From Lemma 3.2 we have

$$\pi_\varphi^*\pi_\varphi = \pi_{S_\varphi}^{(1,2)} + \pi_{S_\varphi}^{(1,2)^*} + \pi_{\Delta_{S_\varphi}} + \pi_{\Delta_{S_\varphi}}{}^* + D_\varphi,$$

with $\|D_\varphi\| \le C\|\varphi\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}$.

Let

$$E_n = \mathrm{span}\{h_{I\times J} : I \in \mathcal{D}_1, J \in \mathcal{D}_2, |I|, |J| \ge 2^{-n}\},$$

let P_n be the orthogonal projection onto E_n in $L^2(\mathbb{T}^2)$, and let

$$c(n) = \sup\{\|\pi_\varphi|_{E_n}\|, \|\varphi\|_* \le 1\}.$$

A trivial estimate shows that $c(n) < \infty$ for each $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, choose $f_n \in E_n$ and $\varphi \in \mathcal{H}_{00}$ with $\|\varphi\|_* = 1$, $\|f_n\| = 1$ and $\|\pi_\varphi f_n\| \ge (1-\varepsilon)c(n)$. Then

$$\begin{aligned}(3.6)\qquad (1-\varepsilon)^2c(n)^2 &\le \|\pi_\varphi f_n\|^2 = \langle \pi_\varphi^*\pi_\varphi f_n, f_n\rangle \\ &= \langle \pi_{S_\varphi}^{(1,2)} f_n, f_n\rangle + \langle \pi_{S_\varphi}^{(1,2)^*} f_n, f_n\rangle + \\ &\quad + \langle \pi_{\Delta_{S_\varphi}} f_n, f_n\rangle + \langle \pi^*_{\Delta_{S_\varphi}} f_n, f_n\rangle + \langle D_\varphi f_n, f_n\rangle.\end{aligned}$$

By definition of $c(n)$, the first two terms can be estimated by

$$c(n)\|S_\varphi\|_* \le c(n)k.$$

For the next two terms, we have to remark that that

$$(3.7)\quad \langle \pi_{\Delta_{S_\varphi}} f_n, f_n\rangle = \langle \pi_{\Delta_{P_nS_\varphi}} f_n, f_n\rangle \le \|\pi_{\Delta_{P_nS_\varphi}}\| \le \tilde{c}\|\pi^{(1,2)}_{P_nS_\varphi}\| \le \tilde{c}\|\pi^{(1,2)}_{S_\varphi}|_{E_n}\|$$

Here, we use as in the proof of Thm 2.8 that there exists a constant $\tilde{c}$ such that $\|\pi_{\Delta_b}\| \le \tilde{c}\|\pi_b^{(1,2)}\|$ for all $b \in \mathcal{H}_{00}$ (see [PS], Thm 7.7.2).

The last term is easily controlled by

$$\langle D_\varphi f_n, f_n\rangle \le C\|\varphi\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}.$$

Altogether, we obtain that

$$\|\pi_{\varphi}^{(1,2)}|_{E_n}\|^2 \leq 4\tilde{c}\|\pi_{S_\varphi}^{(1,2)}|_{E_n}\| + C\|\varphi\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}. \tag{3.8}$$

With

$$\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect}}} \leq c\|\varphi\|_*$$

and

$$\|\pi_{S_\varphi}^{(1,2)}|_{E_n}\| \leq c(n)\|S_\varphi\|_* \leq kc(n)\|\varphi\|_*^2,$$

it follows that

$$(1-\varepsilon)^2 c(n)^2 \leq 4\tilde{c}kc(n) + c^2 C \tag{3.9}$$

Thus

$$c(n) \leq \sqrt{4\tilde{c}^2k^2 + Cc^2} + 2\tilde{c}k.$$

With

$$\tilde{C} = \sqrt{4\tilde{c}^2k^2 + Cc^2} + 2\tilde{c}k,$$

it follows that $\|\pi_\varphi\| \leq \tilde{C}\|\varphi\|_*$. ■

We can now characterize $\mathrm{BMO}^d_{\mathrm{prod}}$ in terms of the $\mathrm{BMO}^d_{\mathrm{rect},2}$-norm.

Theorem 3.6 *Let $\varphi \in \mathrm{BMO}^d_{\mathrm{rect}}(\mathbb{T}^2)$. Then $\varphi \in \mathrm{BMO}^d_{\mathrm{prod}}$ if and only if $(\|S_\varphi^{(n)}\|^{1/2^n}_{\mathrm{BMO}^d_{\mathrm{rect}}})_{n\in\mathbb{N}}$ is bounded, where $S_\varphi^{(n)}$ is the n-fold sweep of φ, defined recursively by $S_\varphi^{(n)} = S_{S_\varphi}^{(n-1)}$.*

Proof. By Theorem 3.3, we have for each $n \in \mathbb{N}$

$$\|S_\varphi^{(n)}\|_{\mathrm{BMO}^d_{\mathrm{rect}}} \leq \|S_\varphi^{(n)}\|_{\mathrm{BMO}^d_{\mathrm{prod}}} \leq C\cdot C^2\cdots C^{2^{n-1}}\|\varphi\|^{2^n}_{\mathrm{BMO}^d_{\mathrm{prod}}} \leq C^{2^n}\|\varphi\|^{2^n}_{\mathrm{BMO}^d_{\mathrm{prod}}},$$

and consequently

$$\|S_\varphi^{(n)}\|^{1/2^n}_{\mathrm{BMO}^d_{\mathrm{rect}}} \leq C\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{prod}}}.$$

Conversely, the map

$$\varphi \mapsto \sup_{n\in\mathbb{N}} \|S_\varphi^{(n)}\|^{1/2^n}_{\mathrm{BMO}^d_{\mathrm{rect}}}$$

clearly defines a positive homogeneous function on $\mathcal{H}_{00}$ with satisfies conditions in Theorem 3.5. ■

Another consequence of Theorem 3.5 is

Corollary 3.7 *S does not map $\mathrm{BMO}^d_{\mathrm{rect}} \times \mathrm{BMO}^d_{\mathrm{rect}}$ boundedly into $\mathrm{BMO}^d_{\mathrm{rect}}$.*

Proof. We know that the $\|\cdot\|_{\mathrm{prod}}$ norm cannot be controlled by the $\|\cdot\|_{\mathrm{rect}}$ norm. So Condition (ii) in Theorem 3.5 cannot hold, and in particular S does not map $\mathrm{BMO}^d_{\mathrm{rect}} \times \mathrm{BMO}^d_{\mathrm{rect}}$ boundedly into $\mathrm{BMO}^d_{\mathrm{rect}}$. ■

4. The scale $\mathrm{BMO}^d_{\mathrm{rect},p}$

Recall that for $1 \le p < \infty$, a function $\varphi \in L^2(\mathbb{T}^2)$ is said to belong to $\mathrm{BMO}^d_{\mathrm{rect},p}$ if

$$\|\varphi\|_{\mathrm{rect},p} = \sup_{R\in\mathcal{R}} \frac{1}{|R|^{1/p}} \|P_R\varphi\|_p < \infty.$$

Note that $\mathrm{BMO}^d_{\mathrm{rect},p_2} \subseteq \mathrm{BMO}^d_{\mathrm{rect},p_1}$ for $p_1 \le p_2$.

The reader should also be aware that functions in $\mathrm{BMO}^d_{\mathrm{rect},p}$ are actually in $L^p(\mathbb{T}^2)$, due to the identities $m_I(f) = m_I(P_{I\times\mathbb{T}}f)$ and $m_J(f) = m_J(P_{\mathbb{T}\times J}f)$.

The following proposition characterizes the behaviour of the $\mathrm{BMO}^d_{\mathrm{rect},p}$ norms under the sweep.

Proposition 4.1 *Let $p > \frac{1}{2}$ and let $C_p = \|\,\mathcal{S}\,\|_{L^{2p}\to L^{2p}}$. Then*

$$\|S_\varphi\|_{\mathrm{rect},p} \le 4C_p^2 \|\varphi\|^2_{\mathrm{rect},2p}.$$

Proof. Since $P_R(S_\varphi) = P_R(S_{P_R\varphi})$ and $\|P_R(g)\|_p \le 4\|g\|_p$, we obtain

$$\|S_\varphi\|_{\mathrm{rect},p} \le 4 \sup_{R\in\mathcal{R}} \frac{1}{|R|^{1/p}} \|S_{P_R(\varphi)}\|_p \le 4C_p^2 \sup_{R\in\mathcal{R}} \frac{1}{|R|^{1/p}} \|P_R(\varphi)\|^2_{2p}.$$

This gives the result. ■

It is known that $\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \mathrm{BMO}^d_{\mathrm{rect},2}$. Indeed, this is basically the content of Carleson's original counterexample [C] (for the continuous case, see [Fef]). As pointed out in [Fef], the example in [C] implies that $\mathrm{BMO}^d_{\mathrm{rect},2} \not\subseteq L^4(\mathbb{T}^2)$.

We shall improve this by showing that actually $\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \mathrm{BMO}^d_{\mathrm{rect},p}$ for all p. We will show that for any $p_2 > p_1 \ge 1$, $\mathrm{BMO}^d_{\mathrm{rect},p_1} \not\subseteq L^{p_2}(\mathbb{T}^2)$ and therefore in particular $\mathrm{BMO}^d_{\mathrm{rect},p_1} \not\subseteq \mathrm{BMO}^d_{\mathrm{rect},p_2}$. For the case $p_1 = 1$, $p_2 = 2$, this answers a question posed in [FS].

As a corollary, we show that

$$\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \bigcap_{p\ge 1} \mathrm{BMO}^d_{\mathrm{rect},p}\,.$$

Theorem 4.2 *Let $p \ge 2$. Then* $\mathrm{BMO}^d_{\mathrm{prod}} \subseteq \mathrm{BMO}^d_{\mathrm{rect},p}$.

Moreover

$$\|\varphi\|_{\mathrm{rect},p} \le C\|\varphi\|^{1-2/p}_{\mathrm{prod}} \|\varphi\|^{2/p}_{\mathrm{rect},2} \tag{4.1}$$

Proof. Let us first show that $\mathrm{BMO}^d_{\mathrm{prod}} \subseteq L^p(\mathbb{T}^2)$ and

$$||\varphi||_p \le C||\varphi||^{1-2/p}_{\mathrm{prod}}||\varphi||^{2/p}_2 . \tag{4.2}$$

For $p = 2^k$, $k \in \mathbb{N}$, we shall prove (4.2) by induction.

It is obvious for $k = 1$. For $k = 2$ we have

$$||\varphi||^2_4 = ||S_\varphi||_2 = ||\Delta^{(1,2)}_\varphi(\varphi)||_2 \le C||\varphi||_{\mathrm{prod}}||\varphi||_2. \tag{4.3}$$

Assume it holds for $p_k = 2^k$.

$$||\varphi||^2_{p_{k+1}} = ||S_\varphi||_{p_k} \le C||S_\varphi||^{1-2/p_k}_{\mathrm{prod}}||S_\varphi||^{2/p_k}_2.$$

Now from (3.4) and (4.3) we obtain

$$||\varphi||_{p_{k+1}} \le C||\varphi||^{1-2/p_k}_{\mathrm{prod}}||\varphi||^{2/p_k}_4 \le C||\varphi||^{1-2/p_{k+1}}_{\mathrm{prod}}||\varphi||^{2/p_{k+1}}_2.$$

Now the general case follows by interpolation.

Given $p > 2$ and $p \ne 2^k$ for any $k \in \mathbb{N}$, find $m \in \mathbb{N}$ such that $2^{m-1} < p < 2^m$. Write

$$\frac{1}{p} = \frac{1-\theta_m}{2^{m-1}} + \frac{\theta_m}{2^m}.$$

Now apply the previous case combined with

$$||\varphi||_p \le ||\varphi||^{1-\theta_m}_{2^{m-1}}||\varphi||^{\theta_m}_{2^m}.$$

Let us use (4.2) to obtain the desired estimate for the $\mathrm{BMO}^d_{\mathrm{rect},p}$-norm. Given $R \in \mathcal{R}$ we have

$$||P_R\varphi||_p \le C||P_R\varphi||^{1-2/p}_{\mathrm{prod}}||P_R\varphi||^{2/p}_2 \le C||\varphi||^{1-2/p}_{\mathrm{prod}}||\varphi||^{2/p}_{\mathrm{rect},2}|R|^{1/p}.$$

This finishes the proof. ∎

Proposition 4.3 *Let* $2 < p$. *There exists* $\phi \in \mathrm{BMO}^d_{\mathrm{rect},2} \setminus L^p(\mathbb{T}^2)$.

In particular $\mathrm{BMO}^d_{\mathrm{prod}} \subseteq \mathrm{BMO}^d_{\mathrm{rect},p} \subsetneq \mathrm{BMO}^d_{\mathrm{rect},2}$.

Proof. We shall find a sequence φ_N such that $\sup_N ||\varphi_N||_{\mathrm{rect},2} < \infty$ but $\sup_N ||\varphi_N||_p = \infty$. A standard argument then gives the existence of ϕ.

From Carleson's construction [C] we know that for each $N \in \mathbb{N}$ there exists a collection of dyadic rectangles Φ_N such that

$$\sum_{R\in\Phi_N} |R| \;=\; 1 \tag{4.4}$$

$$\Big|\bigcup_{R\in\Phi_N} R\Big| \;<\; \frac{1}{N} \tag{4.5}$$

$$\sum_{R\in\Phi_N, R\subseteq R'} |R| \;\le\; C|R'|, \qquad R' \in \mathcal{R} \tag{4.6}$$

where C is a constant independent of N.

Defining

$$\varphi_N = \sum_{R\in\Phi_N} |R|^{1/2} h_R$$

we have that

$$||\varphi_N||_2 = 1, \qquad ||\varphi_N||_{\text{rect},2} \le C$$

but, since $supp(\varphi_N) \subset \cup_{R\in\Phi_N} R$,

$$||\varphi_N||_p \ge \Big| \bigcup_{R\in\Phi_N} R\Big|^{\frac{1}{p}-\frac{1}{2}} \ge N^{\frac{1}{2}-\frac{1}{p}}.$$

∎

We now can answer in the negative the above mentioned question of C. Sadosky and S. Ferguson posed in [FS].

Corollary 4.4 *There exists* $\phi \in \mathrm{BMO}^d_{\text{rect},1} \setminus \bigcup_{p>1} L^p(\mathbb{T}^2)$. *In particular, for each* $p > 1$, $\mathrm{BMO}^d_{\text{rect},p} \subsetneq \mathrm{BMO}^d_{\text{rect},1}$, *and the norms* $\|\cdot\|_{\text{rect},1}$ *and* $\|\cdot\|_{\text{rect},p}$ *are not equivalent.*

Proof. We use the sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ with

$$||\varphi_n||_{\text{rect},2} \le C \text{ and } ||\varphi_n||_p \ge n^{1/2-1/p}$$

for each $n \in \mathbb{N}$, $p > 2$ from Proposition 4.3.

Define

$$\phi = \sum_{n=1}^{\infty} \frac{1}{n^2} S_{\varphi_{2^n}}.$$

Then $\phi \in \mathrm{BMO}_{\text{rect},1}$ by Proposition 4.1, but $\|S_{\varphi_{2^n}}\|_p \approx \|\varphi_{2^n}\|^2_{2p} \ge 2^{n(1-1/p)}$ for each $n \in \mathbb{N}$, $p > 1$ and consequently $\phi \notin \cup_{p>1} L^p(\mathbb{T}^2)$. ∎

To differentiate the spaces $\mathrm{BMO}^d_{\text{rect},p}$ and $\mathrm{BMO}^d_{\text{prod}}$ we shall introduce the following coefficients.

Definition 4.5 *Let* $E_n = \operatorname{span}\{h_{I\times J} : I \in \mathcal{D}_1, J \in \mathcal{D}_2, |I|, |J| \ge 2^{-n}\}$ *and let* P_n *be the orthogonal projection onto* E_n *in* $L^2(\mathbb{T}^2)$.

For each $q \ge 1$ *and each* $n \in \mathbb{N}$

$$c(n, q) = \sup\{\|\pi_\varphi\| : \varphi \in E_n, \|\varphi\|_{\text{rect},q} \le 1\},$$

and for $p \ge q$,

$$a(n, p, q) = \sup\{\|\varphi\|_{\text{rect},p} : \varphi \in E_n, \|\varphi\|_{\text{rect},q} \le 1\},$$

We first analyse the behaviour of these constants. Of course we have

$$\begin{aligned}
&(4.7) \qquad & c(n,p_2) &\le c(n,p_1), \quad p_1 \le p_2 \\
&(4.8) \qquad & a(n,p,q_1) &\le a(n,p,q_2), \quad q_1 \le q_2 \\
&(4.9) \qquad & a(n,p_2,q) &\le a(n,p_1,q), \quad p_1 \le p_2.
\end{aligned}$$

If $p \ge q$, clearly

$$c(n,q) \le a(n,p,q)c(n,p). \tag{4.10}$$

Let us now extend Therorem 4.2.

Theorem 4.6 *Let* $p > q \ge 2$ *and* $\varphi \in \mathrm{BMO}^d_{\mathrm{prod}}$. *If* $q \le 2^k \le p$ *for some* $k \in \mathbb{N}$ *then*

$$||\varphi||_{\mathrm{rect},p} \le C_{p,q}||\varphi||_{\mathrm{prod}}^{1-q/p}||\varphi||_{\mathrm{rect},q}^{q/p}. \tag{4.11}$$

In particular, for $p > q_1 \ge q_2 \ge 2$ *we have*

$$a(n,p,q_2) \le C_p c(n,q_2)^{1-q_1/p} a(n,q_1,q_2)^{q_1/p}. \tag{4.12}$$

Proof. We shall see first that

$$||\varphi||_p \le C_{p,q}||\varphi||_{\mathrm{prod}}^{1-\theta}||\varphi||_q^{\theta} \tag{4.13}$$

for the above values of $\theta = q/p$. We do this in several steps.

First suppose that $q = 2^n$ for some $n \in \mathbb{N}$. Theorem 4.2 gives the case $n = 1$. Assume that the result is true for $n \ge 2$, and let us consider the case $q = 2^{n+1}$.

Let $p > 2^{n+1}$. Applying the induction assumption to S_φ for $p/2$, we get

$$\begin{aligned}
||\varphi||_p^2 &\approx ||S_\varphi||_{p/2} \\
&\le C||S_\varphi||_{\mathrm{prod}}^{1-2^{n+1}/p}||S_\varphi||_{2^n}^{2^{n+1}/p} \\
&\le C||\varphi||_{\mathrm{prod}}^{2(1-2^{n+1}/p)}||\varphi||_{2^{n+1}}^{2^{n+2}/p}.
\end{aligned}$$

This shows that

$$||\varphi||_p \le C||\varphi||_{\mathrm{prod}}^{1-2^{n+1}/p}||\varphi||_{2^{n+1}}^{2^{n+1}/p}.$$

Let us now proceed to the general case. We may assume that $q < 2^k \le p$ for some $k \in N$. We can apply the previous case for $n = k$ together with interpolation. Writing

$$\frac{1}{2^k} = \frac{1-\alpha}{q} + \frac{\alpha}{p},$$

we obtain

$$\begin{aligned} ||\varphi||_p &\le C||\varphi||_{\text{prod}}^{1-2^k/p}||\varphi||_{2^k}^{2^k/p} \\ &\le C||\varphi||_{\text{prod}}^{1-2^k/p}(||\varphi||_q^{1-\alpha}||\varphi||_p^{\alpha})^{2^k/p} \end{aligned}$$

Consequently

$$||\varphi||_p^{1-\alpha 2^k/p} \le C||\varphi||_{\text{prod}}^{1-2^k/p}||\varphi||_q^{(1-\alpha)2^k/p}.$$

Note that

$$(1-\alpha)\frac{2^k}{q} = 1 - \alpha\frac{2^k}{p}.$$

Hence we get with $\theta = q/p$ that

$$||\varphi||_p \le C||\varphi||_{\text{prod}}^{1-\theta}||\varphi||_q^{\theta}.$$

To finish the proof, note that for each $R \in \mathcal{R}$,

$$\begin{aligned} ||P_R\varphi||_p &\le C_{p,q}||P_R\varphi||_{\text{prod}}^{1-q/p}||P_R\varphi||_q^{q/p} \\ &\le C_{p,q}||\varphi||_{\text{prod}}^{1-q/p}||\varphi||_{\text{rect},q}^{q/p}|R|^{1/p}. \end{aligned}$$

■

Let us now establish a further connection between the constants introduced in 4.5.

Theorem 4.7 *There exist $K_1 > 0$ and $K_2 > 0$ such that for all $n \in \mathbb{N}$ and $p \ge 1$*

$$c^2(n, 2p) \le K_1 C_p^2 c(n, p) + K_2,$$

where $C_p = ||\,\mathcal{S}\,||_{L^{2p}\to L^{2p}}$.

Proof. Write

$$\pi_\varphi^*\pi_\varphi = \Lambda_{S_\varphi} + D_\varphi,$$

with $\|D_\varphi\| \le C\|\varphi\|^2_{\text{BMO}^d_{\text{rect}}}$ as above.

For $n \in \mathbb{N}$, $p \ge 1$ and $\varepsilon > 0$, choose $f_n \in L^2(\mathbb{T}^2)$ and $\varphi \in E_n$ with $\|\varphi\|_{\text{rect},2p} = 1$, $\|f_n\|_2 = 1$ and $\|\pi_\varphi f_n\|_2 \ge (1-\varepsilon)c(n, 2p)$. Then

$$(1-\varepsilon)^2 c(n,2p)^2 \le \|\pi_\varphi f_n\|_2^2 = \langle \pi_\varphi^*\pi_\varphi f_n, f_n\rangle = \langle \Lambda_{S_\varphi} f_n, f_n\rangle + \langle D_\varphi f_n, f_n\rangle.$$

Therefore, we obtain that

$$(1-\varepsilon)^2 c(n,2p)^2 \le \|\Lambda_{S_\varphi}\| + C\|\varphi\|^2_{\text{BMO}^d_{\text{rect}}}. \tag{4.14}$$

Since $||\varphi||_{\mathrm{rect},2p} = 1$, Proposition 4.1 implies $||S_\varphi||_{\mathrm{rect},p} \le 4C_p^2$. Therefore, since $\|\varphi\|_{\mathrm{rect},2} \le \|\varphi\|_{\mathrm{rect},2p}$, it follows that

$$(1-\varepsilon)^2 c(n,2p)^2 \le 4C_p^2 c(n,p) + C. \tag{4.15}$$

Using (4.10) we get the second part. ■

Corollary 4.8 *Let* $p \ge 1$. *Then* $\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \mathrm{BMO}^d_{\mathrm{rect},p}$.

Proof. Observe first that Proposition 4.3 implies that

$$\lim_{n\to\infty} c(n,2) = \infty. \tag{4.16}$$

This shows that $\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \mathrm{BMO}^d_{\mathrm{rect},p}$ for any $1 \le p \le 2$.

On the other hand, the estimates (4.10) and (4.12) in the case $q = q_1 = q_2$ imply that if $p > q \ge 2$ with $q \le 2^k \le p$ for some k,

$$c^{q/p}(n,q) \le Cc(n,p), \tag{4.17}$$

where C is independent of n.

Hence $\mathrm{BMO}^d_{\mathrm{prod}} = \mathrm{BMO}^d_{\mathrm{rect},p}$ for some $p>2$ would imply $\sup_n c(n,p)<\infty$ and therefore $\sup_n c(n,2) < \infty$, contradicting (4.16). ■

The particular case $p = 4$ means that a question left open in [PS] can be answered in the negative. There, it was asked whether the condition

$$\begin{aligned} \|(\pi_b^{(1,2)})^*\pi_b^{(1,2)} h_{I'} f\|^2 &= \frac{1}{|I'|}\Bigg\| \sum_{I\times J\in\mathcal{D}_1\times\mathcal{D}_2, I\subsetneq I'} \frac{\chi_{I\times J}}{|I||J|}|b_{IJ}|^2 m_J f\Bigg\|^2_{L^2(\mathbb{T}^2)} \\ &\le C\|f\|_{L^2(\mathbb{T})} \qquad (f\in L^2(\mathbb{T}), I'\in\mathcal{D}_1) \end{aligned} \tag{4.18}$$

((27) and (28) in [PS]) already implies that $b \in \mathrm{BMO}^d_{\mathrm{prod}}$. Note that f here denotes a function in the second variable. We know from Prop 4.1 that $b \in \mathrm{BMO}_{\mathrm{rect},4}$ implies $S_b \in \mathrm{BMO}^d_{\mathrm{rect}}$. By Lemma 3.2,

$$\|(\pi_b^{(1,2)})^*\pi_b^{(1,2)} h_{I'} f\| = \|(\Lambda_{S_b} + D_b) h_{I'} f\|,$$

where D_b is bounded on $L^2(\mathbb{T}^2)$ and Λ_{S_b} maps $L^2(\mathbb{T})\hat{\otimes}L^2(\mathbb{T})$ boundedly into $L^2(\mathbb{T}^2)$ by Prop 2.11. Thus $b \in \mathrm{BMO}_{\mathrm{rect},4}$ implies (4.18). This condition is therefore not sufficient for $b \in \mathrm{BMO}^d_{\mathrm{prod}}$.

As pointed out in [PS], this has also consequences for the study of operator-valued Carleson measures, in the sense that a certain vector BMO condition of the sweep of an operator-valued measure does not imply boundedness of the corresponding vector Carleson embedding.

We can further show that even the intersection of all $\mathrm{BMO}^d_{\mathrm{rect},p}$ spaces is still bigger than $\mathrm{BMO}^d_{\mathrm{prod}}$.

Corollary 4.9

$$\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \bigcap_{p\geq 1} \mathrm{BMO}^d_{\mathrm{rect},p}\,.$$

Proof. Obviously

$$\bigcap_{p\geq 1} \mathrm{BMO}^d_{\mathrm{rect},p} = \bigcap_{p\in\mathbb{N}} \mathrm{BMO}^d_{\mathrm{rect},p}\,.$$

With the locally convex topology defined by the increasing sequence of seminorms $(\|\cdot\|_{\mathrm{rect},p})_{p\in\mathbb{N}}$, the latter is a metrizable locally convex linear space. Since each of the $\mathrm{BMO}^d_{\mathrm{rect},p}$ is complete in $\|\cdot\|_{\mathrm{rect},p}$, $\bigcap_{p\in\mathbb{N}} \mathrm{BMO}^d_{\mathrm{rect},p}$ is complete in this topology and therefore a Fréchet space. We know from Theorem 4.2 that

$$\mathrm{BMO}^d_{\mathrm{prod}} \subseteq \bigcap_{p\geq 1} \mathrm{BMO}^d_{\mathrm{rect},p},$$

and that the embedding is continuous with respect to the norm topology on $\mathrm{BMO}^d_{\mathrm{prod}}$ and the locally convex topology on $\bigcap_{p\geq 1} \mathrm{BMO}_{\mathrm{rect},p}$. Let us assume towards a contradiction that the embedding is surjective. Then the open mapping theorem implies that the locally convex topology on $\bigcap_{p\geq 1} \mathrm{BMO}^d_{\mathrm{rect},p}$ is normable with $\|\cdot\|_{\mathrm{prod}}$ and therefore contains a nonempty open neighbourhood of 0 which is bounded with respect to $\|\cdot\|_{\mathrm{prod}}$. Since the family $(\|\cdot\|_{\mathrm{rect},p})_{p\in\mathbb{N}}$ is increasing, this means that there exists $p \in \mathbb{N}$ and $\varepsilon > 0$ such that $\|b\|_{\mathrm{prod}} < 1$ whenever $\|b\|_{\mathrm{rect},p} < \varepsilon$, in contradiction to Corollary 4.8. ■

We will now separate the $\mathrm{BMO}^d_{\mathrm{rect},p}$ spaces. Note that for Corollary 4.4 means that

$$\lim_{n\to\infty} a(n,p,1) = \infty \tag{4.19}$$

for all $p > 1$.

Let us see that this holds in general.

Corollary 4.10 *Let $p > q \geq 1$. Then* $\mathrm{BMO}^d_{\mathrm{rect},p} \subsetneq \mathrm{BMO}^d_{\mathrm{rect},q}$.

Proof. We have to show that

$$\lim_{n\to\infty} a(n,p,q) = \infty.$$

It suffices to prove

$$\lim_{n\to\infty} a(n,q+\varepsilon,q) = \infty$$

for sufficiently small ε. For fixed $q > 1$, choose $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $q < q + \varepsilon < 2^k \leq 2q$.

Using Theorem 4.7, (4.10) and (4.17), we obtain constants C_1, C_2 and C_3 independent of n such that

$$\begin{aligned} c^2(n,2q) &\le C_1 c(n,q) \le C_1 C_2 a(n,q+\varepsilon,q) c(n,q+\varepsilon) \\ &\le C_1 C_2 C_3 a(n,q+\varepsilon,q) c(n,2q)^{\frac{2p}{p+\varepsilon}}. \end{aligned}$$

This shows that

$$c(n,2q)^{\frac{2\varepsilon}{p+\varepsilon}} \le C a(n,q+\varepsilon,q)$$

where C is independent of n. Now the result follows from Corollary 4.8. ■

Acknowledgement. We thank C. Sadosky for fruitful discussions.

References

[Be] Bernard, A.: Espaces H^1 de martingales à deux indices. Dualité avec les martingales de type “BMO” (French). *Bull. Sci. Math. (2)* **103** (1979), no. 3, 297–303.

[C] Carleson, L.: A counterexample for measures bounded on H^p for the bi-disc. Mittag-Leffler Report No. **7**, 1974.

[Ch] Chang, S.-Y. A.: Carleson measure on the bi-disc. *Ann. of Math. (2)* **109** (1979), no. 3, 613–620.

[ChFef1] Chang, S.-Y. A. and Fefferman, R.: A continuous version of duality of H^1 with BMO on the bidisc. *Ann. of Math. (2)* **112** (1980), no. 1, 179–201.

[ChFef2] Chang, S.-Y. A. and Fefferman, R.: Some recent developments in Fourier analysis and H^p theory on product domains. *Bull. Amer. Math. Soc. (N. S.)* **12** (1985), 1–43.

[Fef] Fefferman, R.: Bounded mean oscillation on the polydisc. *Ann. of Math. (2)* **110** (1979), 395–406.

[FS] Ferguson, S. and Sadosky, C.: Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures. *J. Anal. Math.* **81** (2000), 239–267.

[GPTV] Gillespie, T. A., Pott, S., Treil, S. and Volberg, A.: Logarithmic growth for matrix martingale transforms. *J. London Math. Soc. (3)* **64** (2001), 624–636.

[LF] Lacey, M. and Ferguson, S.: A characterization of product BMO by commutators. *Acta Math.* **189** (2002), no. 2, 143–160.

[Per] Pereyra, C. M.: Lecture notes on dyadic harmonic analysis. In *Second Summer School in Analysis and Mathematical Physics (Cuernavaca, 2000)*, 1–60. Contemp. Math. **289**. Amer. Math. Soc., Providence, 2001.

[PS] Pott, S. and Sadosky, C.: Bounded mean oscillation on the bidisk and Operator BMO. *J. Funct. Anal.* **189** (2002), 475–495.

[T] Tao, T.: Dyadic product H^1, BMO and Carleson's Counterexample. Short stories, `http://www.math.ucla.edu/~tao/`

Recibido: 7 de febrero de 2003
Revisado: 2 de octubre de 2003

Óscar Blasco
Department of Mathematics
Universitat de Valencia
Burjassot 46100 (Valencia), Spain
`Oscar.Blasco@uv.es`

Sandra Pott
Department of Mathematics
University of Glasgow
Glasgow G12 8QW, UK
`sp@maths.gla.ac.uk`

The first author gratefully acknowledges support by the Spanish grants PR2001-0134, BFM2002-04013 and by the LMS. The second author acknowledges support by EPSRC and by the Nuffield Foundation.