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The Cauchy problem for viscous shallow water equations

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Abstract

In this paper we study the Cauchy problem for viscous shallow water equations. We work in the Sobolev spaces of index $s > 2$ to obtain local solutions for any initial data, and global solutions for small initial data.

1. Introduction

We consider in this work the following Cauchy problems for viscous shallow water equations:

$$h(u_t + (u \cdot \nabla)u) - \nu \nabla \cdot (h \nabla u) + h \nabla h = 0, \tag{1.1}$$

$$h_t + \operatorname{div}(hu) = 0, \tag{1.2}$$

$$u|_{t=0} = u_0, \quad h|_{t=0} = h_0; \tag{1.3}$$

where $h(x,t)$ is the height of fluid surface, $u(x,t) = (u^1(x,t), u^2(x,t))^t$ is the horizontal velocity field, $x = (x_1, x_2) \in \mathbb{R}^2$ and $0 < \nu < 1$ is the viscous coefficient.

These equations form a quasi-linear hyperbolic-parabolic system. For the initial data $h_0(x)$, we shall consider small perturbations of some positive constant $\bar{h}_0$. And we will study the Cauchy problem (1.1)-(1.3) in Sobolev function spaces. The main theorem of this paper is the following :

Theorem 1.1 *Let* $s > 0, u_0, h_0 - \bar{h}_0 \in H^{2+s}(\mathbb{R}^2), \|h_0 - \bar{h}_0\|_{H^{2+s}} << \bar{h}_0$. *Then there exist a positive time* T *and a unique solution* (u,h) *of Cauchy problem (1.1)-(1.3) such that*

$$u, h - \bar{h}_0 \in L^\infty([0,T]; H^{2+s}), \quad \nabla u \in L^2([0,T]; H^{2+s}).$$

Furthermore, there exists a constant c *such that if* $\|h_0 - \bar{h}_0\|_{H^{2+s}} + \|u_0\|_{H^{2+s}} \leq c$ *then we can choose* $T = +\infty$.

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In [2], it was obtained the local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for the shallow water equations using Lagrangian coordinates and Hölder space estimates with initial data in $C^{2+\alpha}$. Kloeden [5] and Sundbye [10] proved global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem using Sobolev space estimates and the energy method of Matsumura and Nishida [7, 8, 9]. Sundbye [11] proved also the existence and uniqueness of classical solutions to the Cauchy problem using the method of [7, 8, 9]. However, these results only consider the case of small initial data. In general, the problem of existence of solutions for large initial data is difficult, because its strong non-linear nature. In this paper, we use the Littlewood-Paley decomposition theory (see [1, 3]) for Sobolev spaces to obtain a losing energy estimate in H^{s+2} for any $s > 0$, which allows us to get the local existence of solution for all initial data. Moreover, we also improve the global existence of solution and regularity for small initial data. From this result of global existence, we can obtain some decay estimate as in [6, 12] with the method of Green function but, for brevity, we leave it for the future.

The structure of the paper is the following:

In section 2 we recall Littlewood-Paley theory for Sobolev spaces. In section 3 we prove the first part of the main theorem: local existence of solution for all size of the initial data. In section 4 we prove the global existence of solution for small initial data. Finally, in section 5 we prove the losing energy estimates for the nonlinear terms.

2. Littlewood-Paley theory

Let us recall Sobolev spaces and Littlewood-Paley theory (see, for example, Bony [1] and Chemin [3]). There exist functions φ and ψ in $C_0^\infty(\mathbb{R}^d)$ such that Supp $\varphi \subset \mathcal{C} = \{\xi; 5/6 \le |\xi| \le 12/5\}$, Supp $\psi \subset B = \{|\xi| \le 2\}$,

$$\forall \xi \in \mathbb{R}^d \setminus \{0\},\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \ \text{ and } \ \forall \xi \in \mathbb{R}^d,\ \psi(\xi) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}\xi) = 1.$$

Let us note that if $|j - j'| \ge 2$, then Supp $\varphi(2^{-j}\cdot) \cap$ Supp $\varphi(2^{-j'}\cdot) = \emptyset$. We define the following operators of localization in Fourier space

$$\dot{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\hat{u}(\cdot)) = 2^{jd}\int_{\mathbb{R}^d} f(2^j y)u(x-y)dy, \ \text{for } j \in \mathbb{Z},$$

and

$$\Delta_{-1}u = \mathcal{F}^{-1}(\psi(\cdot)\hat{u}(\cdot)),\ \ \Delta_j = \dot{\Delta}_j, \text{ for } j \in \mathbb{N},$$

where $\hat{u}$ denote the Fourier transformation of u, and $f = \mathcal{F}^{-1}(\varphi)$. So that for $u \in \mathcal{S}'$, we have that $\Delta_j u, \Delta_{-1} u \in C^\infty \cap L^2$. Then the Sobolev space is defined, for $s \in \mathbb{R}$, as follows:

$$H^s(\mathbb{R}^2) = \Big\{ u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{H^s}^2 = \sum_{j=-1}^{\infty} 2^{2js} \|\Delta_j u\|_{L^2}^2 < +\infty \Big\}.$$

In the low vertical frequencies estimates, we have to use the homogeneous Sobolev spaces,

$$\dot{H}^s(\mathbb{R}^2) = \Big\{ u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{\dot{H}^s}^2 = \sum_{j \in \mathbb{Z}} 2^{2js} \|\dot{\Delta}_j u\|_{L^2}^2 < +\infty \Big\}.$$

For $d = 2$, we have that $H^{2+s}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ for any $s > -1$, and

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C_s \|f\|_{H^{2+s}(\mathbb{R}^2)},$$

where C_s is Sobolev constant in $\mathbb{R}^2$. We have also that, for any $q \ge 0$,

$$\|\Delta_q f\|_{L^\infty} \le C \|\nabla(\Delta_q f)\|_{L^2},$$

and

$$\|\Delta_q f\|_{L^2} \le C 2^{-q} \|\nabla(\Delta_q f)\|_{L^2}.$$

We set $S_q(u) = \sum_{-1 \le p \le q-2} \Delta_p u$, then $S_q : H^s \to H^{+\infty}$,

$$\Delta_p(S_q(u)\Delta_q u) = 0, \text{ if } |p-q| \ge 4; \text{ and } \|S_q(\nabla u)\|_{L^\infty} \le 2^q \|S_q u\|_{L^\infty}$$

For the product of two functions, we have the decomposition:

$$\begin{aligned} uv &= \sum_q S_{q-1}(u)\Delta_q v + \sum_q S_{q-1}(v)\Delta_q u + \sum_{|p-q|<2} \Delta_p u \Delta_q v \\ &= T_u v + T_v u + R(u,v), \end{aligned}$$

where T_u is a linear operator. We have:

if $u \in L^\infty$, then for all $s \in \mathbb{R}$,

$$\|T_u\|_{\mathcal{L}(H^s,\, H^s)} \le C \|u\|_{L^\infty};$$

if $u \in H^\tau, \tau < d/2$, then for all $s \in \mathbb{R}$,

$$\|T_u\|_{\mathcal{L}(H^s,\, H^{s+\tau-d/2})} \le C \|u\|_{H^\tau};$$

if $u \in H^{s_1}, v \in H^{s_2}, s_1 + s_2 - d/2 > 0$, then

$$\|R(u,v)\|_{H^{s_1+s_2-d/2}} \le C \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.$$

For the nonlinear composition, if $F \in C^\infty(I)$ such that $F(0) = 0$, $u \in H^\tau(\mathbb{R}^2)$, $\tau > 1$ with $u(x) \in I$ for all $x \in \mathbb{R}^2$, then there exists a function of one variable B_0 depending only on τ, F, I such that

$$\|F(u)\|_{H^\tau} \le B_0(\|u\|_{L^\infty})\|u\|_{H^\tau}. \tag{2.1}$$

In our equation, we have the products of 3 functions, so that we need the following precise estimates:

$$\begin{aligned} |(ab,c)_{L^2}| \le & \ C\|a\|_{L^\infty}\|b\|_{L^2}\|c\|_{L^2}, \\ |(ab,c)_{L^2}| \le & \ C\|a\|_{\dot H^{1/2}}\|b\|_{L^2}\|c\|_{\dot H^{1/2}}, \\ \|a\|^2_{\dot H^{1/2}} \le & \ \|a\|_{L^2}\|\nabla a\|_{L^2}. \end{aligned} \tag{2.2}$$

For the detail of those results, we send to the reference [3].

In the proof of main theorem, we need to estimate the nonlinear term in the equations, using the so-called "Losing energy estimates".

Lemma 2.1 *Let $\tau > 1$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $v, \nabla v, g, \nabla g \in H^\tau$, we have*

$$\left| \int_{\mathbb{R}^2} \Delta_k((v\cdot\nabla)g)\Delta_k g\, dx \right| \le C_0\, d_k^2\, 2^{-2k\tau}\|v\|_{H^{\tau+1}}\|g\|^2_{H^\tau},$$

with $\{d_k\} \in \ell^2$ and $\|\{d_k\}\|_{\ell^2} \le 1$.

Lemma 2.2 *(a) Let $\tau > 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, v, g, u, \nabla u \in H^\tau$, with $\|g\|_{L^\infty} \le 1/4$, we have*

$$\left| \int_{\mathbb{R}^2} \Delta_k(\frac{\nabla f}{1+g}\nabla v)\Delta_k u\, dx \right| \le C_0 d_k^2 2^{-2k\tau}\|f\|_{H^\tau}\|v\|_{H^\tau}(1+\|g\|_{H^\tau})\|u\|_{H^{\tau+1}},$$

where $\|\{d_k\}\|_{\ell^2} \le 1$.

(b) Let $1 < \tau < 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, g, u, \nabla u, v, \nabla v \in H^\tau$, with $\|g\|_{L^\infty} \le 1/4$, we have

$$\left| \int_{\mathbb{R}^2} \Delta_k(\frac{\nabla f}{1+g}\nabla v)\Delta_k u\, dx \right| \le C_0 d_k^2 2^{-2k\tau}\|f\|_{H^\tau}(1+\|g\|_{H^\tau})U_\tau(u,v),$$

with $\|\{d_k\}\|_{\ell^2} \le 1$, and

$$U_\tau(u,v) =: \|\nabla v\|_{L^\infty}\|u\|_{H^{\tau+1}} + \|\nabla v\|_{H^\tau}(\|\nabla u\|_{H^1} + \|u\|_{H^\tau}).$$

Lemma 2.3 *(a) Let $\tau > 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, v, u, \nabla u, g_1, g_2 \in H^\tau$, with $\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty} \le 1/4$, we have*

$$\Big|\int_{\mathbb{R}^2} \Delta_k\Big(\frac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f \nabla v\Big)\Delta_k u\, dx\Big| \le C_0\, d_k^2\, 2^{-2k\tau}\|f\|_{H^\tau}\|v\|_{H^\tau}\|g_1-g_2\|_{H^\tau}\|u\|_{H^{\tau+1}},$$

with $\|\{d_k\}\|_{\ell^2} \le 1$.

(b) Let $1 < \tau < 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, v, g_1, g_2, u, \nabla u, v, \nabla v \in H^\tau$, with $\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty} \le 1/4$, we have

$$\Big|\int_{\mathbb{R}^2} \Delta_k\Big(\frac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f \nabla v\Big)\Delta_k u\, dx\Big| \le C_0\, d_k^2\, 2^{-2k\tau}\|f\|_{H^\tau}\|g_1-g_2\|_{H^\tau}U_\tau(u,v),$$

with $\|\{d_k\}\|_{\ell^2} \le 1$, and $U_\tau(u,v)$ is as in Lemma 2.2, (b).

In the proof of existence of global solutions, we need the following high vertical frequencies estimates:

Lemma 2.4 *Let $\tau > 0$, then there exists $M > 0, C_0 > 0$ such that for all $h, u, v, \nabla h, \nabla u \in H^\tau, M \le k < +\infty$, with $\|h\|_{L^\infty} \le 1/4$, we have*

$$\Big|\int_{\mathbb{R}^2} \Delta_k(\frac{1}{1+h}\nabla h \nabla u)\Delta_k v\, dx\Big| \le C_0\, d_k^2\, 2^{-2k\tau}(1+\|h\|_{H^{\tau+1}})\|\triangle u\|_{H^\tau}\|\nabla h\|_{H^\tau}\|v\|_{H^\tau},$$

with $\|\{d_k\}\|_{\ell^2} \le 1$.

Lemma 2.5 *Let $\tau > 0$, then there exists $M > 0, C_0 > 0$ such that for all $h \in H^{\tau+1}$ with $\|h\|_{L^\infty} \le 1/4$, and $u \in H^{\tau+2}, M \le k < +\infty$, we have*

$$\Big|\int_{\mathbb{R}^2} \Delta_k(\mathrm{div}(hu))\Delta_k(\Delta h)\, dx\Big| \le C_0\, d_k^2\, 2^{-2k\tau}\|\nabla h\|_{H^\tau}(\|\nabla h\|_{H^\tau}^2 + \|\nabla u\|_{H^{\tau+1}}^2),$$

with $\|\{d_k\}\|_{\ell^2} \le 1$.

We will prove these five lemmas in the last section.

3. The local existence of solution

In order to study the local existence of solution, we define the function set $(f,g)\in\mathcal{X}([t_1,t_2],\sigma,E_1,E_2)$ if

$$(f,g)\in L^\infty([t_1,t_2],H^\sigma(\mathbb{R}^2)),\quad \nabla f\in L^2([t_1,t_2],H^\sigma(\mathbb{R}^2))$$

and

$$\|f\|^2_{L^\infty([t_1,t_2],H^\sigma(\mathbb{R}^2))}+\nu\|\nabla f\|^2_{L^2([t_1,t_2],H^\sigma(\mathbb{R}^2))}\le E_1^2$$
$$\|g\|_{L^\infty([t_1,t_2],H^\sigma(\mathbb{R}^2))}\le E_2.$$

The main result of this section is the following local existence theorem for any initial data:

Theorem 3.1 *Let $s>0,(u_0,h_0-\bar h_0)\in H^{s+2}(\mathbb{R}^2)$ with $\|h_0-\bar h_0\|_{H^{2+s}}\le\frac{\bar h_0}{4C_s}$, then there exist a positive time T and a solution*

$$(u,h-\bar h_0)\in\mathcal{X}([0,T],s+2,E_1,E_2)$$

for the Cauchy problem (1.1)-(1.3). Here

$$E_1=2\|u_0\|_{H^{s+2}},\quad E_2=2\|h_0-\bar h_0\|_{H^{s+2}}$$

and C_s is the Sobolev constant.

For the sake of convenience, we take $\bar h_0=1$. Changing h by $1+h$ in (1.1)-(1.3), we have

$$u_t+(u\cdot\nabla)u-\nu\tfrac{\nabla\cdot((1+h)\nabla u)}{1+h}+\nabla h=0,\tag{3.1}$$
$$h_t+\mathrm{div}u+\mathrm{div}(hu)=0,\tag{3.2}$$
$$u(x,0)=u_0(x),h(x,0)=h_0(x).\tag{3.3}$$

We suppose now that $h_0\in H^{s+2}(\mathbb{R}^2),\|h_0\|_{H^{2+s}}\le\frac{1}{4C_s}$, and $E_1=2\|u_0\|_{H^{s+2}}$, $E_2=2\|h_0\|_{H^{s+2}}$.

The proof of Theorem 3.1 involves the method of successive approximations. Let us define the sequence $\{u_n,h_n\}$ by the following linear systems:

$$(u_1,h_1)=S_2(u_0,h_0),$$
$$\partial_t u_{n+1}-\nu\triangle u_{n+1}=G_1(u_n,h_n),\tag{3.4}$$
$$\partial_t h_{n+1}+u_n\nabla h_{n+1}=G_2(u_n,h_n),\tag{3.5}$$
$$(u_{n+1},h_{n+1})|_{t=0}=S_{n+2}(u_0,h_0),\tag{3.6}$$

where

$$\begin{aligned}G_1(u_n,h_n)=&\ \tfrac{\nu}{1+h_n}\nabla h_n\nabla u_n-u_n\nabla u_n+\nabla h_n\\ G_2(u_n,h_n)=&\ -(1+h_n)\mathrm{div}u_n.\end{aligned}$$

Since S_q are smooth operators, the initial data $S_{n+2}(u_0, h_0)$ are smooth functions. If $(u_n, h_n) \in \mathcal{X}([0,T], s+2, E_1, E_2)$ and smooth, we have

$$\|h_n\|_{L^\infty} \le C_s\|h_n\|_{H^{2+s}} \le C_s E_2 = 2C_s\|h_0\|_{H^{2+s}} \le \frac{2C_s}{4C_s} \le \frac{1}{2}\ ,$$

then $G_1(u_n, h_n)$ and $G_2(u_n, h_n)$ are also smooth functions. Note that (3.4) is the heat equation for u_{n+1}, while (3.5) is the transport equation for h_{n+1}. Therefore, the existence of smooth solutions for the Cauchy problems (3.4)-(3.6) is evident. We denote by P_n the application from (u_n, h_n) to (u_{n+1}, h_{n+1}) the solution of problem (3.4)-(3.6).

Now the proof of Theorem 3.1 consists in two steps: "Estimates for big norms" and "convergence for small norms".

Estimates for big norms

Proposition 3.1 *Suppose that* $(u_0, h_0) \in H^{s+2}(\mathbb{R}^2)$ *for* $s > 0$ *and* $\|h_0\|_{H^{s+2}} \le \frac{1}{4C_s}$*, then there exists a positive time* T_1 *such that for any* $n \in \mathbb{N}$*,* P_n *is an application from* $\mathcal{X}([0,T_1], s+2, E_1, E_2)$ *to* $\mathcal{X}([0,T_1], s+2, E_1, E_2)$ *for* $E_1 = 2\|u_0\|_{H^{s+2}}, E_2 = 2\|h_0\|_{H^{s+2}}$.

Proof. For the sake of convenience, we suppose that $1 \le E_1$ (the proof for $E_1 < 1$ is easy), and remark that $0 < E_2 < 1, 0 < \nu < 1$. We take now

$$T_1 = \min\left\{\left(\frac{12}{5}K\right)^{-2}, \frac{\nu E_2^2}{16C_0^2E_1^4}\right\},$$

where $K = \|\mathcal{F}^{-1}(\varphi)\|_{L^1}$. We prove the proposition by induction. Firstly, $(u_1, h_1) = S_2(u_0, h_0)$, then

$$\|u_1\|_{H^{s+2}} \le \|u_0\|_{H^{s+2}}, \qquad \|h_1\|_{H^{s+2}} \le \|h_0\|_{H^{s+2}},$$

$$\nu\int_0^{T_1}\|\nabla u_1\|^2_{H^{s+2}}d\tau \le \nu T_1(\frac{12}{5}K)^2\|u_0\|^2_{H^{s+2}} \le \|u_0\|^2_{H^{s+2}}.$$

Thus $(u_1, h_1) \in \mathcal{X}([0,T_1], s+2, E_1, E_2)$.

Now, we assume that $(u_n, h_n) \in \mathcal{X}([0,T_1], s+2, E_1, E_2)$ is valid and prove that $P_n(u_n, h_n) = (u_{n+1}, h_{n+1}) \in \mathcal{X}([0,T_1], s+2, E_1, E_2)$ is also valid.

Applying the operator Δ_k to the equations (3.4), (3.5), multiplying the first by $\Delta_k u_{n+1}$, and the second by $\Delta_k h_{n+1}$, integration over $\mathbb{R}^2$ yields

$$\partial_t\|\Delta_k u_{n+1}\|^2_{L^2} + 2\nu\|\nabla\Delta_k u_{n+1}\|^2_{L^2} = 2\int_{\mathbb{R}^2}\Delta_k G_1(u_n, h_n)\Delta_k u_{n+1}dx,$$

$$\partial_t\|\Delta_k h_{n+1}\|^2_{L^2} - 2\int_{\mathbb{R}^2}\Delta_k(u_n\nabla h_{n+1})\Delta_k h_{n+1}dx = 2\int_{\mathbb{R}^2}\Delta_k G_2(u_n, h_n)\Delta_k h_{n+1}dx.$$

Using Lemma 2.1, Lemma 2.2 (a) and hypotheses on (u_n, h_n), we obtain

$$\begin{aligned}(3.7)\qquad &\partial_t\|\,\Delta_k u_{n+1}\|_{L^2}^2 + 2\nu\|\Delta_k(\nabla u_{n+1})\|_{L^2}^2 \le C_0\, d_k^2\, 2^{-2k(s+2)}\\ &\quad\times(\|h_n\|_{H^{s+2}}\|\nabla u_{n+1}\|_{H^{s+2}} + V_1(t)(\|u_{n+1}\|_{H^{s+2}} + \|\nabla u_{n+1}\|_{H^{s+2}})),\end{aligned}$$

$$\begin{aligned}(3.8)\qquad &\partial_t\|\,\Delta_k h_{n+1}\|_{L^2}^2 \le C_0\, d_k^2\, 2^{-2k(s+2)}\\ &\quad\times(\|u_n\|_{H^{s+2}}\|h_{n+1}\|_{H^{s+2}}^2 + (1+\|h_n\|_{H^{s+2}})\|\nabla u_n\|_{H^{s+2}}\|h_{n+1}\|_{H^{s+2}}),\end{aligned}$$

where

$$V_1(t) = \|h_n(t)\|_{H^{s+2}}\|u_n(t)\|_{H^{s+2}}(1+\|h_n(t)\|_{H^{s+2}}) + \|u_n(t)\|_{H^{s+2}}^2 \le \frac{3}{4}E_1^2.$$

Multiplying (3.7) and (3.8) by $2^{2k(s+2)}$, and taking the sum over k gives respectively

$$\begin{aligned}\partial_t\|u_{n+1}\|_{H^{s+2}}^2 + \nu\|\nabla u_{n+1}\|_{H^{s+2}}^2 &\le \|u_{n+1}\|_{H^{s+2}}^2 + 2C_0^2E_1^4\nu^{-1},\\ \partial_t\|h_{n+1}\|_{H^{s+2}}^2 &\le \frac{\nu E_2^2}{4E_1^2}\|\nabla u_n\|_{H^{s+2}}^2 + \frac{5C_0^2E_1^2}{\nu E_2^2}\|h_{n+1}\|_{H^{s+2}}^2.\end{aligned}$$

Integrating from 0 to t yields

$$\begin{aligned}\|u_{n+1}(t)\|_{H^{s+2}}^2 + \nu\int_0^t e^{t-\tau}\|\nabla u_{n+1}(\tau)\|_{H^{s+2}}^2 d\tau \le\\ \le \|u_{n+1}(0)\|_{H^{s+2}}^2 e^t + te^t 2C_0^2E_1^4\nu^{-1},\end{aligned}$$

$$\|h_{n+1}(t)\|_{H^{s+2}}^2 \le e^{t5C_0^2E_1^2\nu^{-1}E_2^{-2}}(\|h_{n+1}(0)\|_{H^{s+2}}^2 + \frac{\nu E_2^2}{4E_1^2}\int_0^t \|\nabla u_n(t')\|_{H^{s+2}}^2 dt').$$

By the definition of $(u_{n+1}, h_{n+1})|_{t=0}$ we know that

$$\begin{aligned}\|u_{n+1}(0)\|_{H^{s+2}} &\le \|u_0\|_{H^{s+2}},\\ \|h_{n+1}(0)\|_{H^{s+2}} &\le \|h_0\|_{H^{s+2}}.\end{aligned}$$

Thus, the choice of T_1 gives that

$$\begin{aligned}\|u_{n+1}(t)\|_{L^\infty([0,T_1],H^{s+2})}^2 + \nu\|\nabla u_{n+1}(\tau)\|_{L^2([0,T_1],H^{s+2})}^2 &\le E_1^2\\ \|h_{n+1}(t)\|_{L^\infty([0,T_1],H^{s+2})}^2 &\le E_2^2.\end{aligned}$$

We have proved proposition 3.1. ■

Convergence for small norm

Proposition 3.2 *Let $(u_0(x), h_0(x)) \in H^{s+2}(\mathbb{R}^2)$ for $s > 0$ and $\|h_0\|_{H^{s+2}} \leq \frac{1}{4C_s}$, then there exists a positive time $T_2(\leq T_1)$ which independent of n, such that $\{(u_n(x,t), h_n(x,t))\}$ is a Cauchy sequence in $\mathcal{X}([0,T_2], s+1, E_1, E_2)$ if $s \neq 1$, and in $\mathcal{X}([0,T_2], 2-\varepsilon, E_1, E_2)$ for all $1 > \varepsilon > 0$ if $s = 1$.*

Proof. From equations (3.4) and (3.5), we have

$$\partial_t(u_{n+1} - u_n) - \nu\triangle(u_{n+1} - u_n) = \sum_{j=1}^{6} F_j, \tag{3.9}$$

$$\partial_t(h_{n+1} - h_n) + u_n\nabla(h_{n+1} - h_n) = \sum_{j=1}^{3} J_j, \tag{3.10}$$

where

$$\begin{aligned}\sum_{j=1}^{6} F_j = &\frac{1}{1+h_n}\nabla h_n \nabla(u_n - u_{n-1}) \\ &+ \frac{1}{1+h_n}\nabla(h_n - h_{n-1})\nabla u_{n-1} + \Big(\frac{1}{1+h_n} - \frac{1}{1+h_{n-1}}\Big)\nabla h_{n-1}\nabla u_{n-1} \\ &- u_n\nabla(u_n - u_{n-1}) - (u_n - u_{n-1})\nabla u_{n-1} + \nabla(h_n - h_{n-1}),\end{aligned}$$

$$\sum_{j=1}^{3} J_j = (u_n - u_{n-1})\nabla h_n + (1+h_n)\mathrm{div}(u_n - u_{n-1}) + (h_n - h_{n-1})\mathrm{div} u_{n-1}.$$

As in the proof of Proposition 3.1, applying the operator Δ_k to the equations (3.9) and (3.10), multiplying the first by $\Delta_k(u_{n+1} - u_n)$, and the second by $\Delta_k(h_{n+1} - h_n)$, then integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned}&\partial_t\|\Delta_k(u_{n+1} - u_n)\|_{L^2}^2 + 2\nu\|\Delta_k(u_{n+1} - u_n)\|_{L^2}^2 = \sum_{j=1}^{6}\int_{\mathbb{R}^2}\Delta_k F_j \Delta_k(u_{n+1} - u_n)dx, \\ &\partial_t\|\Delta_k(h_{n+1} - h_n)\|_{L^2}^2 = \sum_{j=1}^{3}\int_{\mathbb{R}^2}\Delta_k J_j \Delta_k(h_{n+1} - h_n)dx.\end{aligned}$$

Below we only consider the case of $0 < s < 1$. Using Lemma 2.1, Lemma 2.2 and Lemma 2.3, and the fact that $\|u_n(t)\|_{H^{s+1}} \leq E_1$ and $\|h_n(t)\|_{H^{s+1}} \leq E_2$ when $t \leq T_1$, we have that

$$\begin{aligned}\partial_t\|u_{n+1} - u_n\|_{H^{s+1}}^2 &+ 2\nu\|\nabla(u_{n+1} - u_n)\|_{H^{s+1}}^2 \\ &\leq A_0(\|u_n - u_{n-1}\|_{H^{s+1}} + \|h_n - h_{n-1}\|_{H^{s+1}}) \\ &\qquad \times (\|u_{n+1} - u_n\|_{H^{s+1}} + \|\nabla(u_{n+1} - u_n)\|_{H^{s+1}}),\end{aligned}$$

and

$$\partial_t\|h_{n+1}-h_n\|^2_{H^{s+1}} \le A_0\Big(\|h_{n+1}-h_n\|^2_{H^{s+1}} + \big(\|u_n-u_{n-1}\|_{H^{s+1}} + \|\nabla(u_n-u_{n-1})\|_{H^{s+1}} + \|h_n-h_{n-1}\|_{H^{s+1}}\big)\|h_{n+1}-h_n\|_{H^{s+1}}\Big),$$

where A_0 is a constant, and $A_0 = O(E_1^4E_2^{-2})$. Using Cauchy-Schwarz inequality, we obtain

(3.11)
$$\partial_t\|u_{n+1}-u_n\|^2_{H^{s+1}} + \nu\|\nabla(u_{n+1}-u_n)\|^2_{H^{s+1}} \le \|u_{n+1}-u_n\|^2_{H^{s+1}} + \frac{A_0^2}{\nu}(\|u_n-u_{n-1}\|^2_{H^{s+1}} + \|h_n-h_{n-1}\|^2_{H^{s+1}}),$$

(3.12)
$$\partial_t\|h_{n+1}-h_n\|^2_{H^{s+1}} \le \widetilde{A}_0^2\|h_{n+1}-h_n\|^2_{H^{s+1}} + \frac{E_2^2}{4E_1^2}\|u_n-u_{n-1}\|^2_{H^{s+1}} + \frac{\nu E_2^2}{4E_1^2}\|\nabla(u_n-u_{n-1})\|^2_{H^{s+1}} + \|h_n-h_{n-1}\|^2_{H^{s+1}},$$

where $\widetilde{A}_0^2 = 4A_0^2(1+\frac{E_1^2}{\nu E_2^2})$.

We prove now that there exists a positive time T_2 ($\le T_1$) such that, for any n,

$$(C_n)\quad \begin{cases} \|u_n-u_{n-1}\|_{L^\infty([0,T_2],H^{s+1})} + \nu\|\nabla(u_n-u_{n-1})\|_{L^2([0,T_2],H^{s+1})} \le E_12^{-n}, \\ \|h_n-h_{n-1}\|_{L^\infty([0,T_2],H^{s+1})} \le E_22^{-n}. \end{cases}$$

We will prove (C_n) by induction on n. In fact, it is easy to see that (C_1) is valid if $T_2 \le T_1$.

We suppose now that (C_n) holds and prove that (C_{n+1}) is valid using the estimates (3.11) and (3.12). Taking integration from 0 to t on (3.11), we deduce

$$\|(u_{n+1}-u_n)(t)\|^2_{H^{s+1}} + \nu\int_0^t e^{t-t'}\|\nabla(u_{n+1}-u_n)(t')\|^2_{H^{s+1}}dt' \le e^t\|(u_{n+1}-u_n)(0)\|^2_{H^{s+1}} + te^t\frac{A_0^2}{\nu}(E_1^2+E_2^2)2^{-2n}.$$

If $T_2 = \min\{T_1, \nu(6A_0^2)^{-1}\}$ and $t \le T_2$, we have $e^t \le 3/2$, $te^t2\frac{A_0^2}{\nu} \le 3/2$, we also have

$$\|(u_{n+1}-u_n)(0)\|_{H^{s+1}} \le 2^{-(n+1)}\|\Delta_{n+1}u_0\|_{H^{s+2}} \le \frac{1}{2}E_12^{-(n+1)}.$$

Using (C_n), we obtain

$$\|(u_{n+1}-u_n)\|^2_{L^\infty([0,T_2],H^{s+1})} + \nu\|\nabla(u_{n+1}-u_n)\|^2_{L^2([0,t_2],H^{s+1})} \le E_1^22^{-2(n+1)}. \tag{3.13}$$

The same calculus for (3.12) yields

$$\|(h_{n+1}-h_n)(t)\|^2_{H^{s+1}} \le e^{\widetilde{A}_0^2 t}\|(h_{n+1}-h_n)(0)\|^2_{H^{s+1}} + 2te^{\widetilde{A}_0^2 t}E_2^2 2^{-2n}.$$

Finally if $T_2=\min\{T_1,\nu(6A_0^2)^{-1},\widetilde{A}_0^{-2}\}$ and $t\le T_2$, we obtain

$$\|(h_{n+1}-h_n)\|^2_{L^\infty([0,T_2],H^{s+1})} \le E_2^2 2^{-2(n+1)}. \tag{3.14}$$

Proposition 3.2 is proved now with $T_2=O(E_1^{-10}\nu^2E_2^6)$. ■

Regularity and uniqueness of solutions

From Proposition 3.2, we may conclude that the approximative sequence (u_n,h_n) of problems (3.4)-(3.6) is a Cauchy sequence in $\mathcal{X}([0,T_2],s{+}1,E_1,E_2)$ with $s>0$. So that the limit (u,h) is a solution of Cauchy problem (1.1)-(1.3). From Proposition 3.1, this sequence is bounded in $\mathcal{X}([0,T_1],s+2,E_1,E_2)$, so that it is also the Cauchy sequence in $\mathcal{X}([0,T_2],s'{+}2,E_1,E_2)$ for all $s'<s$ (by interpolation), and the limit is in $\mathcal{X}([0,T_2],s+2,E_1,E_2)$. So we have proved the existence of solution for Theorem 3.1.

The proof of uniqueness of solution is similar to the proofs for the convergence of approximative sequence. In fact, we consider

$$\partial_t(u-v)-\nu\triangle(u-v) = G_1(u,h)-G_1(v,g), \tag{3.15}$$
$$\partial_t(h-g)-u\nabla(h-g) = (u-v)\nabla g+G_2(u,h)-G_2(v,g), \tag{3.16}$$

with initial data $u(x,0)=v(x,0)=u_0(x)\in H^{s+2}$ and $h(x,0)=g(x,0)=h_0(x)\in H^{s+2}$.

Following the proof of Proposition 3.2, we obtain that

$$\begin{aligned}\|(u-v)\|^2_{L^\infty([0,T_2],H^{s+1})}&+\nu\|\nabla(u-v)\|^2_{L^2([0,T_2],H^{s+1})}\\ &\le 2\|(u-v)(0)\|^2_{H^{s+1}}+\tfrac{1}{16}(\|u-v\|^2_{L^\infty([0,T_2],H^{s+1})}\\ &\quad+\|h-g\|^2_{L^\infty([0,T_2],H^{s+1})}+\nu\|u-v\|^2_{L^2([0,T_2],H^{s+1})}),\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\|(g-h)(t)\|^2_{H^{s+1}} \le{}& 2\|(h-g)(0)\|^2_{H^{s+1}}+\tfrac{1}{16}(\|u-v\|^2_{L^\infty([0,T_2],H^{s+1})}\\ &+\|h-g\|^2_{L^\infty([0,T_2],H^{s+1})}+\nu\|u-v\|^2_{L^2([0,T_2],H^{s+1})}).\end{aligned} \tag{3.18}$$

This gives the uniqueness of the solutions.

4. Global existence for small initial data

First, we prove *a priori estimates* for local solutions.

Theorem 4.1 (*a priori* estimate) *Suppose that the problem (1.1)-(1.3) has a solution* $(u,h)\in L^\infty([0,T],H^{s+1}),\nabla u\in L^2([0,T],H^{s+1}(\mathbb{R}^2))$ *for some* $T>0$ *with initial data* $u_0,h_0\in H^{s+1}(\mathbb{R}^2), s>0$*, and*

$$N(T)=\left(\|u\|^2_{L^\infty([0,T];H^{s+1})}+\|h\|^2_{L^\infty([0,T];H^{s+1})}+\nu\|\nabla u\|^2_{L^2([0,T];H^{s+1})}\right)^{1/2}\le E_0.$$

Then there exist positive constants ε *and* C_1 *with* $\varepsilon C_1\le E_0$*, which are independent of* T *such that, if* $N(T)\le\varepsilon$*, then*

$$N(T)\le C_1N(0). \tag{4.1}$$

A combination of local existence theorem 3.1 and above *a priori* estimate give the following theorem.

Theorem 4.2 *Suppose that* $u_0,h_0\in H^{s+2}(\mathbb{R}^2), s>0$*. Then there exists* $\varepsilon>0$ *such that if*

$$\|u_0\|_{H^{s+2}}+\|h_0\|_{H^{s+2}}\le\varepsilon,$$

then the Cauchy problem (3.1)-(3.3) has a unique global solution

$$(u,h)\in L^\infty([0,+\infty[,H^{s+2}(\mathbb{R}^2)),\quad \nabla u\in L^2([0,\infty[,H^{s+2}(\mathbb{R}^2)).$$

For the proof of this theorem we refer to Sundbye [11].

Remark. We get global solution with index $s+2$, since we have only local solution with index $s+2$ in Theorem 3.1. But we have proved the *a priori* estimate for small index $s+1$, so if we can get local solution for $s+1$, we will also get global solution for small index $s+1$.

We prove now Theorem 4.1. Let us linearize the equations (1.1) and (1.2) on $(h,u)=(1,0)$ as follows:

$$\begin{cases} u_t-\nu\triangle u+\nabla h=H_1,\\ h_t+\mathrm{div}u=H_2,\end{cases} \tag{4.2}$$

where

$$\begin{cases} H_1= & \frac{1}{1+h}\nabla h\nabla u-(u\cdot\nabla)u,\\ H_2= & -\mathrm{div}(hu),\end{cases}$$

In the following we will estimate (u,h) under the *a priori* assumption

$$\widetilde{N}(T)=\|h\|^2_{L^\infty([0,T],H^{s+1})}+\|u\|^2_{L^\infty([0,T],H^{s+1})}\le\delta_0\,, \tag{4.3}$$

where $s>0$ and $0<\delta_0<<1$.

Applying the operator Δ_k on (4.2), multiplying the first equation of (4.2) by $\Delta_k(u - \triangle u + \lambda\nabla h)$ and the second equation by $\Delta_k(h - \triangle h)$, summing them and integrating over $\mathbb{R}^2$ yields

$$
\begin{aligned}
\frac{1}{2}\partial_t(\|u_k\|^2_{H^1} + \|h_k\|^2_{H^1}) &+\nu\|\nabla u_k\|^2_{L^2} + \nu\|\triangle u_k\|^2_{L^2} + \lambda\|\nabla h_k\|^2_{L^2} \\
&= \int_{\mathbb{R}^2}(\Delta_k F_1 \Delta_k(u-\triangle u+\lambda\nabla h)+\Delta_k F_2\Delta_k(h-\triangle h))dx \\
&\quad -\lambda\int_{\mathbb{R}^2}(\partial_t u_k\nabla h_k - \nu\triangle u_k\nabla h_k)dx,
\end{aligned} \tag{4.4}
$$

where $0 < \lambda << 1, u_k = \triangle_k u, h_k = \triangle_k h$.

High vertical frequencies estimates

Now we will give some estimates to the right hand of (4.4) for the case of high vertical frequencies. This means that for some M large enough, we study (4.4) for $k > M$. By lemma 2.4 we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2}\Delta_k(\tfrac{1}{1+h}\nabla h\nabla u)u_k dx\Big| \\
&\quad \le C_0 d_k^2 2^{-2ks}\|u\|_{H^s}(1+\|h\|_{H^{s+1}})(\|\triangle u\|^2_{H^s} + \|\nabla h\|^2_{H^s}),
\end{aligned} \tag{4.5}
$$

$$
\Big|\int_{\mathbb{R}^2}\Delta_k(\tfrac{1}{1+h}\nabla h\nabla u)\triangle u_k dx\Big| \le C_0 d_k^2 2^{-2ks}\|\nabla h\|_{H^s}(1+\|h\|_{H^{s+1}})\|\triangle u\|^2_{H^s}, \tag{4.6}
$$

and

$$
\begin{aligned}
&\lambda\Big|\int_{\mathbb{R}^2}\Delta_k(\tfrac{1}{1+h}\nabla h\nabla u)\nabla h_k dx\Big| \\
&\quad \le C_0\lambda d_k^2 2^{-2ks}\|\nabla h\|_{H^s}(1+\|h\|_{H^{s+1}})(\|\triangle u\|^2_{H^s} + \|\nabla h\|^2_{H^s}).
\end{aligned} \tag{4.7}
$$

We also have

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^2}\Delta_k((u\cdot\nabla)u)\triangle u_k dx\Big| &= \Big|\int_{\mathbb{R}^2}\Delta_k(T_u\nabla u + T_{\nabla u}u)\triangle u_k + R(u,\nabla u)\triangle u_k dx\Big| \\
\le \sum_{|q-k|\le N_1}&\left(\Big|\int_{\mathbb{R}^2}\Delta_k(S_q u\nabla u_q)\triangle u_k dx\Big| + \Big|\int_{\mathbb{R}^2}\Delta_k(S_q(\nabla u)u_q)\triangle u_k dx\Big|\right) \\
&+\sum_{q\ge k-N_2, j\in\{-1,0,1\}}\Big|\int_{\mathbb{R}^2}\Delta_k(u_q\nabla u_{q-j})\triangle u_k dx\Big| \\
\le C_0 d_k^2 2^{-2ks}&(\|u\|_{L^2}\|\triangle u\|^2_{H^s} + \|\nabla u\|_{L^2}\|\nabla u\|_{H^s}\|\triangle u\|_{H^s}) \\
\le C_0 d_k^2 2^{-2ks}&\|u\|_{H^{s+1}}\|\nabla u\|^2_{H^{s+1}}.
\end{aligned}
$$

Similarly,

$$\lambda\Big|\int_{\mathbb{R}^2}\Delta_k((u\cdot\nabla)u)\nabla h_k dx\Big|\leq C_0\lambda d_k^2 2^{-2ks}\|u\|_{H^{s+1}}(\|\nabla u\|^2_{H^{s+1}}+\|\nabla h\|^2_{H^s}).$$

Since $\|f_q\|_{H^s}\leq\|\nabla f_q\|_{H^s}$ for $q\geq 0$, we can obtain that

$$\Big|\int_{\mathbb{R}^2}\Delta_k((u\cdot\nabla)u)u_k dx\Big|\leq Cd_k^2 2^{-2ks}\|u\|_{H^s}\|\nabla u\|^2_{H^s},$$

and

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\mathrm{div}\Delta_k(uh)h_k dx\Big|=&\Big|\int_{\mathbb{R}^2}\Delta_k(uh)\nabla h_k dx\Big|\\ \leq& Cd_k^2 2^{-2ks}(\|u\|_{L^2}+\|h\|_{L^2})(\|\nabla u\|^2_{H^s}+\|\nabla h\|^2_{H^s}).\end{aligned}$$

Using lemma 2.5, we obtain

$$\Big|\int_{\mathbb{R}^2}\Delta_k(\mathrm{div}(hu)\triangle h_k dx\Big|\leq Cd_k^2 2^{-2ks}\|\nabla h\|_{H^s}(\|\nabla h\|^2_{H^s}+\|\nabla u\|^2_{H^{s+1}}).$$

It is easy to see that

$$\lambda\nu\Big|\int_{\mathbb{R}^2}\triangle u_k\nabla h_k dx\Big|\leq C\lambda\nu d_k^2 2^{-2ks}(\varepsilon^{-1}\|\triangle u\|^2_{H^s}+\varepsilon\|\nabla h\|^2_{H^s}).$$

Noting that

$$\int_{\mathbb{R}^2}(\partial_t u_k)(\nabla h_k)dx=\partial_t\int_{\mathbb{R}^2}u_k\nabla h_k dx-\int_{\mathbb{R}^2}u_k\partial_t(\nabla h_k)dx,$$

we have

$$\begin{aligned}\lambda\Big|\int_{\mathbb{R}^2}u_k\partial_t(\nabla h_k)dx\Big| \leq & C\lambda d_k^2 2^{-2ks}\Big(\|\nabla u\|^2_{H^s}\\ &+(\|h\|_{H^{s+1}}+\|u\|_{H^{s+1}})(\|\nabla u\|^2_{H^s}+\|\nabla h\|^2_{H^s})\Big),\end{aligned}$$

and

$$\begin{aligned}\lambda\Big|\int_0^t\partial_\tau\Big(\int_{\mathbb{R}^2}u_k\nabla h_k dx\Big)d\tau\Big| \leq & C\lambda d_k^2 2^{-2ks}\Big(\|u(t)\|_{H^s}\|\nabla h(t)\|_{H^s}\\ &+\|u(0)\|_{H^s}\|\nabla h(0)\|_{H^s}\Big).\end{aligned}$$

Multiplying inequality (4.4) by 2^{2ks} and integrating over $(0,t)$, we obtain

(4.8)

$$\begin{aligned}
&\| u_k(t)\|_{H^{s+1}}^2+\|h_k(t)\|_{H^{s+1}}^2+\int_0^t(\nu\|\nabla u_k(\tau)\|_{H^{s+1}}^2+\lambda\|\nabla h_k(\tau)\|_{H^s}^2)d\tau\\
&\quad\le Cd_k^2(\|u(0)\|_{H^{s+1}}^2+\|h(0)\|_{H^{s+1}}^2)+C\lambda\nu d_k^2\int(\varepsilon^{-1}\|\nabla u\|_{H^{s+1}}^2+\varepsilon\|\nabla h\|_{H^s}^2)d\tau\\
&\qquad+Cd_k^2\Big(\|h\|_{L^\infty([0,T],H^{s+1})}+\|h\|_{L^\infty([0,T],H^{s+1})}^2\\
&\qquad\quad+\|u\|_{L^\infty([0,T],H^{s+1})}+\|u\|_{L^\infty([0,T],H^{s+1})}^2\Big)\int_0^t(\|\nabla u\|_{H^{s+1}}^2+\|\nabla h\|_{H^s}^2)d\tau\\
&\qquad+C\lambda d_k^2(\|u(t)\|_{H^s}^2+\|h(t)\|_{H^{s+1}}^2).
\end{aligned}$$

Low vertical frequencies estimates

Now we will consider the low vertical frequencies: denoting $S_M=\sum_{k<M}\Delta_k$, applying the operator S_M on (4.2), multiplying the first equation of (4.2) by $S_k(u+\lambda\nabla h)$and the second equation by S_kh, summing them and integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}
&\frac{1}{2}\partial_t(\|S_Mu\|_{L^2}^2+\|S_Mh\|_{L^2}^2)+\nu\|\nabla S_Mu\|_{L^2}^2+\lambda\|\nabla S_Mh\|_{L^2}^2\\
&\qquad=\int_{\mathbb{R}^2}\big(S_M(F_1)S_M(u+\lambda\nabla h)+S_M(F_2)S_Mh\big)dx\\
&\qquad\quad-\lambda\int_{\mathbb{R}^2}(\partial_tS_Mu\nabla S_Mh-\nu\triangle S_Mu\nabla S_Mh)dx,
\end{aligned}\tag{4.9}$$

where $0<\lambda<<1$. As in the proof of (4.8), we will give some estimates to the right hand of (4.9). It is easy to see that

$$\Big|\int_{\mathbb{R}^2}S_M\Big(\frac{\nabla h}{1+h}\nabla u\Big)S_Mudx\Big|\le C\Big\|\frac{1}{1+h}\Big\|_{L^\infty}\|\nabla h\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2},$$

and

$$\begin{aligned}
&\lambda\Big|\int_{\mathbb{R}^2}S_M\Big(\frac{\nabla h}{1+h}\nabla u-(u\cdot\nabla)u\Big)S_M(\nabla h)dx\Big|\\
&\qquad\le C\lambda\Big(\Big\|\frac{1}{1+h}\Big\|_{L^\infty}\|\nabla h\|_{L^2}^2\|\nabla u\|_{L^\infty}+\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla h\|_{L^2}\Big).
\end{aligned}$$

Using the estimates (2.2), we have

$$\Big|\int_{\mathbb{R}^2}S_M((u\cdot\nabla)u)S_Mudx\Big|\le C\|u\|_{\dot H^{1/2}}^2\|\nabla u\|_{L^2}\le C\|u\|_{L^2}\|\nabla u\|_{L^2}^2,$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^2} S_M(\operatorname{div}(uh))S_M h dx\right| &\le C\|u\|_{\dot H^{1/2}}\|h\|_{\dot H^{1/2}}\|\nabla h\|_{L^2}\\ &\le C(\|u\|_{L^2}+\|h\|_{L^2})(\|\nabla u\|_{L^2}^2+\|\nabla h\|_{L^2}^2).\end{aligned}$$

Note that $\|S_M\triangle u\|_{L^2}\le 2^M\|S_M\nabla u\|_{L^2}$, which implies

$$\left|\lambda\nu\int_{\mathbb{R}^2} S_M(\triangle u)S_M(\nabla h)dx\right|\le C\lambda\nu(\varepsilon^{-1}\|\nabla u\|_{L^2}^2+\varepsilon\|\nabla h\|_{L^2}^2).$$

As in the above proofs, we have

$$\begin{aligned}\lambda\left|\int_{\mathbb{R}^2} S_M u S_M(\partial_t(\nabla h))dx\right| &\le C\lambda(\|h\|_{L^2}+\|u\|_{L^2})(\|\nabla u\|_{L^2}^2+\|\nabla h\|_{L^2}^2)\\ &\quad +C\lambda(\varepsilon^{-1}\|\nabla u\|_{L^2}^2+\varepsilon\|\nabla h\|_{L^2}^2),\end{aligned}$$

and

$$\lambda\left|\int_0^t\partial_t\int_{\mathbb{R}^2} S_M u S_M(\nabla h)dxd\tau\right|\le C\lambda(\|u\|_{L^2}\|\nabla h\|_{L^2}+\|u(0)\|_{L^2}\|\nabla h(0)\|_{L^2}).$$

Integrating both sides of (4.9) over $(0,t)$, and adding the above estimates to the right hand of (4.9), we have

$$\begin{aligned}&(\|S_M u\|_{L^2}^2+\|S_M h\|_{L^2}^2)+\int_0^t(\nu\|\nabla S_M u\|_{L^2}^2+\lambda\|\nabla S_M h\|_{L^2}^2)d\tau\\ &\quad\le C(\|u\|_{L^\infty([0,T],H^{s+1})}+\|h\|_{L^\infty([0,T],H^{s+1})})\int_0^t(\|\nabla u\|_{H^s}^2+\|\nabla h\|_{H^s}^2)d\tau\\ &\qquad + C\lambda(\|u(t)\|_{L^2}^2+\|\nabla h(t)\|_{L^2}^2)+C(\|u(0)\|_{L^2}^2+\|\nabla h(0)\|_{L^2}^2)\\ &\qquad + C\lambda\nu\int_0^t(\varepsilon^{-1}\|\nabla u\|_{L^2}^2+\varepsilon\|\nabla h\|_{L^2}^2)d\tau.\end{aligned}$$

Since $\|S_M f\|_{H^s}\le 2^{Ms}\|S_M f\|_{L^2}$, we can write

$$\begin{aligned}&(\|S_M u\|_{H^{s+1}}^2+\|S_M h\|_{H^{s+1}}^2)+\int_0^t(\|\nabla S_M u\|_{H^{s+1}}^2+\lambda\|\nabla S_M h\|_{H^s}^2)d\tau\\ &\quad\le C(\|u\|_{L^\infty([0,T],H^{s+1})}+\|h\|_{L^\infty([0,T],H^{s+1})})\int_0^t(\|\nabla u\|_{H^{s+1}}^2+\|\nabla h\|_{H^s}^2)d\tau\\ &\qquad + C\lambda(\|u(t)\|_{H^s}^2+\|\nabla h(t)\|_{H^s}^2)+C(\|u(0)\|_{H^s}^2+\|\nabla h(0)\|_{H^s}^2)\\ &\qquad + \lambda\int_0^t(\varepsilon^{-1}\|\nabla u\|_{H^s}^2+\varepsilon\|\nabla h\|_{H^s}^2)d\tau.\end{aligned}$$

The inequality above, together with estimate (4.8), yield:

$$
\begin{aligned}
&\|u(t)\|_{H^{s+1}}^2+\|h(t)\|_{H^{s+1}}^2+\int_0^t(\|\nabla u(\tau)\|_{H^{s+1}}^2+\lambda\|\nabla h(\tau)\|_{H^s}^2)d\tau\\
&\quad\le C(\|h\|_{L^\infty([0,T],H^{s+1})}+\|u\|_{L^\infty([0,T],H^{s+1})}+\|h\|_{L^\infty([0,T],H^{s+1})}^2+\|u\|_{L^\infty([0,T],H^{s+1})}^2)\\
&\qquad\times\int_0^t(\|\nabla u\|_{H^{s+1}}^2+\|\nabla h\|_{H^s}^2)d\tau+C(\|u(0)\|_{H^{s+1}}^2+\|h(0)\|_{H^{s+1}}^2)\\
&\qquad+C\lambda(\|u(t)\|_{H^{s+1}}^2+\|h(t)\|_{H^{s+1}}^2)+C\lambda\nu\int_0^t(\varepsilon^{-1}\|\nabla u\|_{H^{s+1}}^2+\varepsilon\|\nabla h\|_{H^s}^2)d\tau.
\end{aligned}
$$

Taking ε and λ small enough, such that $C\varepsilon=\frac{1}{2}$ and $C\lambda\varepsilon^{-1}=\frac{1}{4}$, we get

$$
\begin{aligned}
&\|u(t)\|_{H^{s+1}}^2+\|h(t)\|_{H^{s+1}}^2+\int_0^t(\nu\|\nabla u(\tau)\|_{H^{s+1}}^2+\lambda\|\nabla h(\tau)\|_{H^s}^2)d\tau\\
&\quad\le C(\|h\|_{L^\infty([0,T],H^{s+1})}+\|h\|_{L^\infty([0,T],H^{s+1})}^2+\|u\|_{L^\infty([0,T],H^{s+1})}+\|u\|_{L^\infty([0,T],H^{s+1})}^2)\\
&\qquad\times\int_0^t(\nu\|\nabla u\|_{H^{s+1}}^2+\lambda\|\nabla h\|_{H^s}^2)d\tau+C(\|u(0)\|_{H^{s+1}}^2+\|h(0)\|_{H^{s+1}}^2).
\end{aligned}
$$

For fixed λ, taking δ_0 small enough, such that $C\delta_0<\frac{1}{5}\min\{1,\lambda\}$, we obtain

$$
\begin{aligned}
\|u(t)\|_{H^{s+1}}^2+\|h(t)\|_{H^{s+1}}^2+\int_0^t(\nu\|\nabla u(\tau)\|_{H^{s+1}}^2+\lambda\|\nabla h(\tau)\|_{H^s}^2)d\tau&\\
\le C(\|u(0)\|_{H^{s+1}}^2+\|h(0)\|_{H^{s+1}}^2)&.
\end{aligned}
$$

This proves Theorem 4.1.

5. Losing energy estimates

We prove now the losing energy estimates of section 2. The proofs of this section are technical.

Lemma 5.1 *Let $\tau>1$ and $-1\le k<+\infty$, then there exists $C>0$ such that for all $v,\nabla v,g,\nabla g\in H^\tau$, we have*

$$
\left|\int_{\mathbb{R}^2}\Delta_k(v\nabla g)\Delta_k g dx\right|\le Cd_k^2 2^{-2k\tau}\|v\|_{H^{\tau+1}}\|g\|_{H^\tau}^2,
$$

with $\{d_k\}\in\ell^2$ and $\|\{d_k\}\|_{\ell^2}\le 1$.

Proof. Using the paraproduct calculus, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2}\Delta_k(v\nabla g)\Delta_k g dx \;=\;& \int_{\mathbb{R}^2}\Delta_k(T_{\nabla g}v)\Delta_k g dx+\int_{\mathbb{R}^2}\Delta_k(T_v\nabla g)\Delta_k g dx\\
&+\int_{\mathbb{R}^2}\Delta_k R(v,\nabla g)\Delta_k g dx=I_1+I_2+I_3.
\end{aligned}
$$

Then there exists $N_1 > 0$ such that for any fixed $M > N_1$ and $k > M$,

$$
\begin{aligned}
|I_1| &\leq \sum_{|q-k|\leq N_1} \|S_q(\nabla g)\|_{L^\infty} \|\Delta_q v\|_{L^2} \|\Delta_k g\|_{L^2} \\
&\leq \sum_{|q-k|\leq N_1} \|S_q g\|_{L^\infty} \|\Delta_q(\nabla v)\|_{L^2} \|\Delta_k g\|_{L^2} \leq C d_k^2 2^{-2k\tau} \|g\|^2_{H^\tau} \|v\|_{H^{\tau+1}}.
\end{aligned}
$$

Here we have used Sobolev inequality for $\|g\|_{L^\infty}$ since $\tau > 1$. For $k \leq M$, using that $\sum_{|q-k|\leq N_1} \|S_q(\nabla g)\|_{L^\infty} \leq C 2^M \|g\|_{L^\infty}$, we get the same results.

For the term I_2, in order to pass the operator ∇ from g to v, let us rewrite

$$
\begin{aligned}
I_2 &= \sum_{|q-k|\leq N_1} \int_{\mathbb{R}^2} \Delta_k(S_q v \Delta_q(\nabla g)) \Delta_k g dx \\
&= \sum_{|q-k|\leq N_1} \Big(\int_{\mathbb{R}^2} [\Delta_k, S_q v] \Delta_q(\nabla g) \Delta_k g dx + \int_{\mathbb{R}^2} (S_q - S_k) v \Delta_k \Delta_q(\nabla g) \Delta_k g dx \\
&\qquad + \int_{\mathbb{R}^2} S_k v \Delta_k(\nabla g) \Delta_k g dx \Big).
\end{aligned}
$$

Note that the operators Δ_k are convolution operators in $\mathbb{R}^2$, therefore

$$
[\Delta_k, S_q v] \Delta_q(\nabla g) = 2^{2k} \int_{\mathbb{R}^2} (S_q v(x) - S_q v(y)) f(2^k(x-y)) \Delta_q(\nabla g)(y) dy,
$$

where $f(x) = (\mathcal{F}^{-1}\varphi)(x)$. Using the fact that $|q-k| \leq N_1$ and Hausdorff-Young inequality, we have

$$
\begin{aligned}
&\sum_{|q-k|\leq N_1} \|[\Delta_k, S_q v] \Delta_q(\nabla g)\|_{L^2} \\
&\qquad \leq C \sum_{|q-k|\leq N_1} 2^{2k} \|\nabla(S_q v)\|_{L^\infty} 2^{-k} \|(2^k \cdot) f(2^k \cdot)\|_{L^1} \|\Delta_q(\nabla g)\|_{L^2} \\
&\qquad \leq C d_k 2^{-k\tau} \|\nabla v\|_{L^\infty} \|g\|_{H^\tau},
\end{aligned}
$$

A similar computation for other terms yields :

$$
|I_2| \leq C d_k^2 2^{-2k\tau} \|\nabla v\|_{H^\tau} \|g\|^2_{H^\tau}.
$$

Finally, for I_3, there exists $N_1 > 0$ such that

$$
\begin{aligned}
|I_3| &\leq \sum_{q\geq k-N_2, j\in\{-1,0,1\}} \Big| \int_{\mathbb{R}^2} \Delta_k(\Delta_q v \Delta_{q-j}(\nabla g)) \Delta_k g dx \Big| \\
&\leq C \sum_{q\geq k-N_2} \|\Delta_q v\|_{L^2} \|\Delta_{q-j}(\nabla g)\|_{L^\infty} \|\Delta_k g\|_{L^2} \\
&\leq C d_k 2^{-2k\tau} \Big(\sum_{q\geq k-N_2} d_q 2^{-(q-k)\tau} \Big) \|v\|_{H^{\tau+1}} \|g\|^2_{H^\tau}.
\end{aligned}
$$

Denote $d'_k = (\sum_{q\geq k-N_2} d_q 2^{-(q-k)\tau})$, then $\{d'_k\} \in l^2$ since $q > k$ and $\tau > 1$. For the sake of convenience, below we also denote d'_k by d_k. Thus

$$|I_3| \leq C d_k^2 2^{-2k\tau} \|v\|_{H^{\tau+1}} \|g\|_{H^\tau}^2.$$

Lemma 5.1 is proved. ∎

Lemma 5.2 *(a) Let $\tau > 2$ and $-1 \leq k < +\infty$, then there exists $C > 0$ such that for all $f, v, g, u, \nabla u \in H^\tau$, we have*

$$\Big|\int_{\mathbb{R}^2} \Delta_k\Big(\frac{\nabla f}{1+g}\nabla v\Big)\Delta_k u dx\Big| \leq C d_k^2 2^{-2k\tau} H_0(g)\|f\|_{H^\tau}\|v\|_{H^\tau}(1+\|g\|_{H^\tau})\|u\|_{H^{\tau+1}},$$

with $\{d_k\} \in \ell^2$, where

$$H_0(g) = 1 + \|(1+g)^{-1}\|_{L^\infty} + B_0(\|g\|_{L^\infty}),$$

and B_0 is the function give in (2.1).

(b) Let $1 < \tau < 2$ and $-1 \leq k < +\infty$, then there exists $C > 0$ such that for all $f, g, u, \nabla u, v, \nabla v \in H^\tau$, we have

$$\Big|\int_{\mathbb{R}^2} \Delta_k\Big(\frac{\nabla f}{1+g}\nabla v\Big)\Delta_k u dx\Big| \leq C d_k^2 2^{-2k\tau} H_0(g)\|f\|_{H^\tau}(1+\|g\|_{H^\tau}) U_\tau(u,v),$$

with $\{d_k\} \in \ell^2$, where

$$U_\tau(u,v) =: \|\nabla v\|_{L^\infty}\|u\|_{H^{\tau+1}} + \|\nabla v\|_{H^\tau}(\|\nabla u\|_{H^1} + \|u\|_{H^\tau}).$$

Proof. (a) As in the proof of Lemma 5.1, first we have

$$\int_{\mathbb{R}^2}\Delta_k\Big(\frac{\nabla f}{1+g}\nabla v\Big)\Delta_k u dx = \int_{\mathbb{R}^2}\Delta_k\Big(T_{\frac{\nabla f}{1+g}}\nabla v + T_{\nabla v}\frac{\nabla f}{1+g} + R\Big(\frac{\nabla f}{1+g}, \nabla v\Big)\Big)\Delta_k u dx.$$

For $k > M$, it is easy to obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\Delta_k(T_{\frac{\nabla f}{1+g}}\nabla v)\Delta_k u dx\Big| &\leq \sum{}_{|q-k|\leq N_1}\Big\|\frac{\nabla f}{1+g}\Big\|_{L^\infty}\|\Delta_q(\nabla v)\|_{L^2}\|\Delta_k u\|_{L^2} \\ &\leq \sum{}_{|q-k|\leq N_1}\Big\|\frac{\nabla f}{1+g}\Big\|_{L^\infty} 2^q\|\Delta_q v\|_{L^2} 2^{-k}\|\Delta_k\nabla u\|_{L^2} \\ &\leq \ C d_k^2 2^{-2k\tau}\Big\|\frac{1}{1+g}\Big\|_{L^\infty}\|f\|_{H^\tau}\|v\|_{H^\tau}\|\nabla u\|_{H^\tau}.\end{aligned}$$

For $k \leq M$, we have

$$\Big|\int_{\mathbb{R}^2}\Delta_k(T_{\frac{\nabla f}{1+g}}\nabla u)\Delta_k u dx\Big| \leq C d_k^2 2^{-2k\tau}\Big\|\frac{1}{1+g}\Big\|_{L^\infty}\|f\|_{H^\tau}\|v\|_{H^\tau}\|u\|_{H^\tau}.$$

For the second term, we rewrite

$$\Delta_q\Big(\frac{\nabla f}{1+g}\Big)=\Delta_q(\nabla f)-\Delta_q\Big(T_{\frac{g}{1+g}}\nabla f+T_{\nabla f}\frac{g}{1+g}+R\Big(\nabla f,\frac{g}{1+g}\Big)\Big),$$

thus we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\Delta_k\Big(T_{\nabla v}\frac{\nabla f}{1+g}\Big)\Delta_k u dx\Big|\\ &\qquad\le Cd_k^2 2^{-2k\tau}H_0(g)\|\nabla v\|_{H^\tau}\|f\|_{H^\tau}(1+\|g\|_{H^\tau})(\|\nabla u\|_{H^\tau}+\|u\|_{H^\tau}),\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\Delta_k\Big(R(\nabla v,\frac{\nabla f}{1+g}\Big)\Delta_k u dx\Big|\\ &\qquad\le\sum_{q\ge k-N_2,j\in\{-1,0,1\}}\Big|\int_{\mathbb{R}^2}\Delta_k(\Delta_q(\frac{\nabla f}{1+g})\Delta_{q-j}(\nabla v))\Delta_k u dx\Big|\\ &\qquad\le Cd_k^2 2^{-2k\tau}H_0(g)\|\nabla v\|_{H^\tau}\|f\|_{H^\tau}\|u\|_{H^{\tau+1}}.\end{aligned}$$

Therefore, part (a) is proved.

(b) Let us write

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\Delta_k\Big(\frac{\nabla f}{1+g}\nabla v\Big)\Delta_k u dx\Big|&\le\Big|\int_{\mathbb{R}^2}\Delta_k(T_{\frac{\nabla v}{1+g}}\nabla f)\Delta_k u dx\Big|\\ &+\Big|\int_{\mathbb{R}^2}\Delta_k\Big(T_{\nabla f}\frac{\nabla v}{1+g}\Big)\Delta_k u dx\Big|+\Big|\int_{\mathbb{R}^2}\Delta_k R\Big(\frac{\nabla v}{1+g},\nabla f\Big)\Delta_k u dx\Big|.\end{aligned}$$

The estimates for the first term and the third term are easy, so we discuss only the second term for which we consider two cases:

1) $k>M$. We have that

$$\Big|\int_{\mathbb{R}^2}\Delta_k\Big(T_{\nabla f}\frac{\nabla v}{1+g}\Big)\Delta_k u dx\Big|\le\sum_{|q-k|\le N_1}\|S_q(\nabla f)\|_{L^\infty}\Big\|\Delta_q\Big(\frac{\nabla v}{1+g}\Big)\Big\|_{L^2}\|\Delta_k u\|_{L^2}.$$

Since $1<\tau<2$, using the Sobolev's inequality, we obtain

$$\|S_q(\nabla f)\|_{L^\infty}\le\sum_{p\le q+2}2^{2p}\|\Delta_p f\|_{L^2}\le C2^{-q(\tau-2)}\|f\|_{H^\tau},$$

and

$$\|\frac{\nabla v}{1+g}\|_{H^\tau}\le\|\nabla v\|_{H^\tau}\Big(1+\|\frac{g}{1+g}\|_{H^\tau}\Big)\le\|\nabla v\|_{H^\tau}\Big(1+\Big\|\Big(\frac{1}{1+g}\Big)^2\Big\|_{L^\infty}\|g\|_{H^\tau}\Big),$$

which implies

$$\Big|\int_{\mathbb{R}^2}\Delta_k\Big(T_{\nabla f}\frac{\nabla v}{1+g}\Big)\Delta_k u dx\Big|\leq Cd_k^2 2^{-2k\tau}H_0(g)(1+\|g\|_{H^\tau})\|f\|_{H^\tau}\|\nabla v\|_{H^\tau}\|\nabla u\|_{H^1}.$$

2) $k\leq M$. It is easy to see that

$$\Big|\int_{\mathbb{R}^2}\Delta_k\Big(T_{\nabla f}\frac{\nabla v}{1+g}\Big)\Delta_k u dx\Big|\leq Cd_k^2 2^{-2k\tau}H_0(g)(1+\|g\|_{H^\tau})\|f\|_{H^\tau}\|\nabla v\|_{H^\tau}\|u\|_{H^\tau}.$$

And the lemma is proved. ■

Lemma 5.3 *(a) Let $\tau>2$ and $-1\leq k<+\infty$, then there exists $C>0$ such that for all $f,v,u,\nabla u,g_1,g_2\in H^\tau$, we have*

$$\Big|\int_{\mathbb{R}^2}\Delta_k\Big(\frac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f\nabla v\Big)\Delta_k u dx\Big|$$
$$\leq Cd_k^2 2^{-2k\tau}H_1(g_1,g_2)\|f\|_{H^\tau}\|v\|_{H^\tau}\|g_1-g_2\|_{H^\tau}(\|\nabla u\|_{H^\tau}+\|u\|_{H^\tau}),$$

with $\{d_k\}\in\ell^2$, and

$$\begin{aligned}H_1(g_1,g_2)= &\ (1+\|(1+g_1)^{-1}\|_{L^\infty}^2\|g_1\|_{H^\tau})(1+\|(1+g_2)^{-1}\|_{L^\infty}^2\|g_2\|_{H^\tau})\\ &+\|(1+g_1)^{-1}\|_{L^\infty}^2\|(1+g_2)^{-1}\|_{L^\infty}^2.\end{aligned}$$

(b) Let $1<\tau<2$ and $-1\leq k<+\infty$, then there exists $C>0$ such that for all $f,v,g_1,g_2,u,\nabla u,v,\nabla v\in H^\tau$,we have

$$\Big|\int_{\mathbb{R}^2}\Delta_k\Big(\frac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f\nabla v\Big)\Delta_k u dx\Big|$$
$$\leq Cd_k^2 2^{-2k\tau}H_1(g_1,g_2)\|f\|_{H^\tau}\|g_1-g_2\|_{H^\tau}U_\tau(u,v),$$

with $\{d_k\}\in\ell^2$, and $U_\tau(u,v)$ as in Lemma 5.2 (b).

The proof of this lemma is similar to Lemma 5.2. Let us remark that if $F_j=\frac{1}{1+g_j},\bar F_j=\frac{g_j}{1+g_j}(j=1,2)$, then we have

$$F=\frac{g_1-g_2}{(1+g_1)(1+g_2)}=(g_1-g_2)F_1F_2=(g_1-g_2)(1-\bar F_1-\bar F_2+\bar F_1\bar F_2)$$

with the following estimates

$$\begin{aligned}\|F\|_{L^\infty}&\leq C\|g_1-g_2\|_{L^\infty}\|F_1\|_{L^\infty}\|F_2\|_{L^\infty},\\ \|\Delta_q F\|_{L^2}&\leq Cd_q^2 2^{-2q\tau}\|g_1-g_2\|_{H^\tau}(1+\|F_1\|_{L^\infty}^2\|g_1\|_{H^\tau})(1+\|F_2\|_{L^\infty}^2\|g_2\|_{H^\tau}).\end{aligned}$$

In the following we will consider the losing energy estimate for the case of high vertical frequencies, i. e., $k>M$. Here, we assume that $M>N_1+N_2$.

Lemma 5.4 *Let $\tau > 0$ and $M \le k < \infty$, then there exists $C > 0$ such that for all $g, u, v, \nabla g, \nabla u \in H^\tau$, we have*

$$\Big| \int_{\mathbb{R}^2} \Delta_k \Big(\frac{1}{1+h} \nabla h \nabla u \Big) \Delta_k v dx \Big| \le C d_k^2 2^{-2k\tau} H_0(h)(1 + \|h\|_{H^{\tau+1}}) \|\triangle u\|_{H^\tau} \|\nabla h\|_{H^\tau} \|v\|_{H^\tau},$$

with $\{d_k\} \in l^2$.

The proof of this lemma is similar to the proof of lemma 5.2 and the following lemma.

Lemma 5.5 *Let $\tau > 0$ and $M \le k < \infty$, then there exists $C > 0$ such that for all $g \in H^{\tau+1}$ and $u \in H^{\tau+2}$, we have*

$$\Big| \int_{\mathbb{R}^2} \Delta_k (\mathrm{div}(hu)) \Delta(\Delta_k h) dx \Big| \le C d_k^2 2^{-2k\tau} \|\nabla h\|_{H^\tau} (\|\nabla h\|_{H^\tau}^2 + \|\nabla u\|_{H^{\tau+1}}^2),$$

with $\{d_k\} \in l^2$.

Proof. First, let us observe that

$$\Big| \int_{\mathbb{R}^2} \Delta_k (\mathrm{div}(hu)) \Delta(\Delta_k h) dx \Big| \le \Big| \int_{\mathbb{R}^2} \Delta_k (\nabla h \mathrm{div} u) \Delta_k (\nabla h) dx \Big| + \Big| \int_{\mathbb{R}^2} \Delta_k (h \nabla (\mathrm{div} u) \Delta_k (\nabla h) dx \Big| + \Big| \int_{\mathbb{R}^2} \Delta_k (u \nabla h) \Delta_k (\triangle h) dx \Big|.$$

It is easy to estimate the first and the second terms by

$$C d_k^2 2^{-2k\tau} (\|\nabla u\|_{L^\infty} \|\nabla h\|_{H^\tau}^2 + \|\nabla h\|_{L^2} \|\triangle u\|_{H^\tau} \|\nabla h\|_{H^\tau}),$$

while for the third term, since we cannot control $\triangle h$, it is convenient to write

$$\begin{aligned}
\int_{\mathbb{R}^2} \Delta_k (T_u \nabla h) \Delta_k (\triangle h) dx &= \sum_{|q-k| \le N_1} \int_{\mathbb{R}^2} \Delta_k (S_q u \nabla h_q) \Delta_k (\triangle h) dx \\
&= \sum_{|q-k| \le N_1} \int_{\mathbb{R}^2} ((S_q u \Delta_k \Delta_q (\nabla h)) + [\Delta_k, S_q u] \Delta_q (\nabla h)) \Delta_k (\triangle h) dx \\
&= \sum_{|q-k| \le N_1} \int_{\mathbb{R}^2} ((S_q - S_k) u \Delta_k \Delta_q (\nabla h) + [\Delta_k, S_q u] \Delta_q (\nabla h)) \Delta_k (\triangle h) dx \\
&\quad + \int_{\mathbb{R}^2} S_k u \nabla (\Delta_k h) \triangle (\Delta_k h) dx = K_1 + K_2 + K_3.
\end{aligned}$$

Since $(S_q - S_k)u = -\sum_{q\le p\le k-1}\Delta_p u$,

$$\begin{aligned} K_1 &\le C\sum_p \|\Delta_p u\|_{L^\infty}\|\nabla\Delta_k h\|_{L^2}\|\triangle\Delta_k h\|_{L^2} \\ &\le Cd_k^2 2^{-2k\tau}\|\triangle u\|_{H_\tau}\|\nabla h\|_{L^2}\|\nabla h\|_{H^\tau}. \end{aligned}$$

As in the proof of lemma 5.1, we have

$$K_2 \le Cd_k^2 2^{-2k\tau}\|\nabla u\|_{L^\infty}\|\nabla h\|^2_{H^\tau}.$$

Then, using the following computation

$$\begin{aligned} &\int_{\mathbb{R}^2}(S_k u\nabla(\Delta_k h))\triangle(\Delta_k h)dx \\ &\qquad = \int_{\mathbb{R}^2}\Big(\frac{1}{2}\mathrm{div}(S_k u)|\nabla(\Delta_k h)|^2 - \sum_{i,j}\partial_j(S_k u^i)\partial_i(\Delta_k h)\partial_j(\Delta_k h)\Big)dx, \end{aligned}$$

we get immediately that

$$K_3 \le Cd_k^2 2^{-2k\tau}\|\nabla u\|_{L^\infty}\|\nabla h\|^2_{H^\tau}\,,$$

and this proves the Lemma. ■

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