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Hausdorff dimension of the graph of the Fractional Brownian Sheet

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Abstract

Let $\{B^{(\alpha)}(t)\}_{t\in\mathbb{R}^d}$ be the Fractional Brownian Sheet with multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, $0 < \alpha_i < 1$. In [14], Kamont has shown that, with probability 1, the box dimension of the graph of a trajectory of this Gaussian field, over a non-degenerate cube $Q \subset \mathbb{R}^d$ is equal to $d + 1 - \min(\alpha_1, \dots, \alpha_d)$. In this paper, we prove that this result remains true when the box dimension is replaced by the Hausdorff dimension or the packing dimension.

1. Introduction

The fractal dimensions of a subset E of $\mathbb{R}^{d+1}$ allow to account for its geometrical complexity. The most known are the Hausdorff dimension, the box dimension and the packing dimension. Two excellent references on them are [15] and [29]. Throughout this article these fractal dimensions will respectively be denoted by $\dim_H(E)$, $\dim_B(E)$ and $\dim_P(E)$. Recall that the inequalities

$$\dim_H(E) \le \dim_P(E) \le \dim_B(E) \le d + 1, \tag{1.1}$$

always hold. In this article we will mainly focus on the Hausdorff dimension so let us recall its definition. We refer to [15] and [29] for the definitions of the box dimension and of the packing dimension. For each $\alpha > 0$, the α-dimensional Hausdorff measure of E is defined by

$$\mathcal{H}^\alpha(E) = \lim_{\delta\to 0} \inf \Big\{ \sum_{i\in\mathbb{N}} (2r_i)^\alpha,\ E \subset \bigcup_{i\in\mathbb{N}} B(x_i, r_i),\ r_i \le \delta \Big\},$$

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where $B(x_i, r_i)$ denotes the open ball of radius r_i centered at x_i and

$$\dim_H(E) = \inf\{\alpha > 0,\, \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha > 0,\, \mathcal{H}^\alpha(E) = +\infty\}.$$

There has been considerable interest in the fractal dimensions of the images, graphs, level sets and multiple points of vector valued Gaussian fields (see for example [1, 6, 7, 12, 14, 10, 11, 24, 30, 31]). Let us recall an important theorem of Xiao [31] that allows to determine the Hausdorff dimensions of the image and of the graphs of a wide class of Gaussian fields, namely the β-index vector valued Gaussian fields. This Theorem is an extension of many other results in the literature (see for example [1, 8, 22]) and correct some of them. To state it we first need to recall the definition of the β-index Gaussian fields, which has been introduced by Adler (see for example [1]).

Definition 1.1 *Let $\{X(t)\}_{t\in\mathbb{R}^d} = \{(X_1(t), \dots, X_n(t))\}_{t\in\mathbb{R}^d}$ be an $\mathbb{R}^n$-valued and mean 0 Gaussian vector field defined on $\mathbb{R}^d$. We assume that the coordinate fields $X_1, \dots, X_n$ have stationary increments and for all $1 \le j \le n$ and $t \in \mathbb{R}^d$, we denote*

$$\sigma_j^2(t) = E(|X_j(t) - X_j(0)|^2). \tag{1.2}$$

If for each $j = 1, 2, \dots, n$ there exists a real $0 < \beta_j \le 1$ such that

$$\begin{aligned} \beta_j &= \sup\{\alpha > 0,\, \lim_{|t|\to 0} |t|^{-\alpha}\sigma_j(t) = 0\} \\ &= \inf\{\alpha > 0,\, \lim_{|t|\to 0} |t|^{-\alpha}\sigma_j(t) = +\infty\}, \end{aligned} \tag{1.3}$$

then the field $\{X(t)\}_{t\in\mathbb{R}^d}$ will be called a (d,n)-Gaussian field with index $\beta = (\beta_1, \dots, \beta_n)$.

The β-index Gaussian fields are extensions of the well known Levy's Fractional Brownian Field (LFBF) whose definition will be recalled below. Xiao has completely determined the Hausdorff dimensions of the images and the graphs of these fields. More precisely he proved that

Theorem 1.2 *[31] Let $\{X(t)\}_{t\in\mathbb{R}^d}$ be a (d,n)-Gaussian field of index β, with coordinate so arranged that they satisfy $0 = \beta_0 < \beta_1 \le \dots \le \beta_n \le 1$ and let $K \subset \mathbb{R}^d$ be a compact set. Suppose that there exists a constant $\epsilon > 0$ such that for all $(s,t) \in K \times K$,*

$$\det cov(X(t) - X(s)) \ge \epsilon \prod_{j=1}^{n} \sigma_j^2(t-s). \tag{1.4}$$

Then, with probability 1,

$$\dim_H X(K) = \min\Big\{n; \frac{\dim_H K + \sum_{i=1}^{j}(\beta_j - \beta_i)}{\beta_j}, 1 \le j \le n\Big\}, \tag{1.5}$$

and

$$\begin{aligned}\dim_H \Gamma(X,K) \\ = \min\Big\{\frac{\dim_H K + \sum_{i=1}^{j}(\beta_j - \beta_i)}{\beta_j}, 1 \le j \le n; \dim_H K + \sum_{i=1}^{n}(1-\alpha_i)\Big\},\end{aligned} \tag{1.6}$$

where $X(K)$ *is the random image* $X(K) = \{X(t),\, t \in K\}$ *and* $\Gamma(X,K)$ *is the random graph* $\Gamma(X,K) = \{(t, X(t)),\, t \in K\}$.

An important example of a Gaussian field, that does not fall into the class of β-index Gaussian fields, is the Fractional Brownian Sheet (FBS), which is an extension to the d dimensional parameter space $\mathbb{R}^d$ of the well known Fractional Brownian Motion (FBM). Recall that the standard Fractional Brownian Motion (FBM) of index $\beta \in]0,1[$ is the mean 0, real valued Gaussian process $\{B_\beta(t)\}_{t\in\mathbb{R}}$ defined on $\mathbb{R}$ and with the covariance kernel

$$E(B_\beta(t)B_\beta(s)) = \mathcal{K}_\beta(t,s) = \frac{1}{2}(|t|^{2\beta} + |s|^{2\beta} - |t-s|^{2\beta}), \tag{1.7}$$

for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$. This process was introduced in 1940 by Kolmogorov [13] and then made popular by Mandelbrot and Van Ness in 1968 [20]. Since then it turns out to be very useful in modeling. The monograph of Doukhan, Oppenheim and Taqqu offers a systematic treatment of FBM, as well as an overview of different areas of applications [9]. There are two possible extensions of the FBM to the d dimensional parameter space $\mathbb{R}^d$, the first one is the isotropic LFBF (see e.g. [27, 7]) and the second one is the FBS, also called the anisotropic Fractional Wiener Field (see e.g. [14, 2, 3]). Similarly to the FBM, the LFBF of index β is the mean 0, real valued, Gaussian field whose covariance kernel is given by the formula (1.7) but with the convention that the symbol $|.|$ stands for the euclidian norm on $\mathbb{R}^d$. While the FBS of multi-index $(\alpha_1,\dots,\alpha_d) \in]0,1[^d$ is the mean 0, real valued Gaussian field, whose covariance kernel is given for all $s = (s_1,\dots,s_d) \in \mathbb{R}^d$ and $t = (t_1,\dots,t_d) \in \mathbb{R}^d$ by the product

$$EB^{(\alpha)}(t)B^{(\alpha)}(s) = \prod_{i=1}^{d} \mathcal{K}_{\alpha_i}(t_i, s_i), \tag{1.8}$$

where for every $i = 1,\dots,d$, K_{α_i} is the covariance kernel of a standard 1-D Fractional Brownian Motion (FBM), of index α_i. The FBS seems to be an important Gaussian field. Indeed, it is a generalization of the well known Brownian Sheet (see e.g [1, chapter 8]) and there is more and more interest in it (see for instance [16, 5, 32]). Recall that the Brownian Sheet is an FBS whose indexes α_i are all equal to $1/2$. It arises in a natural fashion in a variety of statistical situations as a result of weak convergence arguments (see for example [23]). The FBS seems to share the same property. Indeed, it has been shown in [5] that it can be approximated by processes constructed from the Poisson process.

The box dimension of the graph of the FBS over a non degenerate cube Q of $\mathbb{R}^d$ was determined by Kamont in 1996 [14]. She showed that with probability 1,

$$\dim_B(\Gamma(B^{(\alpha)}, Q)) = d + 1 - \min(\alpha_1, \dots, \alpha_d). \tag{1.9}$$

Thus it follows from the inequality (1.1) that almost surely,

$$\begin{aligned} \dim_H(\Gamma(B^{(\alpha)}, Q)) &\leq \dim_P(\Gamma(B^{(\alpha)}, Q)) \\ &\leq \dim_B(\Gamma(B^{(\alpha)}, Q)) \\ &= d + 1 - \min(\alpha_1, \dots, \alpha_d). \end{aligned} \tag{1.10}$$

Up to now, no non trivial lower bound of $\dim_H(\Gamma(B^{(\alpha)}, Q))$ is known, mainly because of two difficulties. The first difficulty is that the FBS being not a β-index Gaussian field one cannot use Theorem 1.2 to determine the Hausdorff dimension of its graph. While the second difficulty, is that the techniques that have allowed to obtain the Hausdorff dimension of the graph of the Brownian Sheet (see for example [1] chapter 8) can hardly be extended to the FBS. The goal of our article is to show the following Theorem.

Theorem 1.3 *Let* $Q = \prod_{k=1}^d [a_k, b_k]$ *be a cube of* $\mathbb{R}^d$ *and let*

$$\Gamma(B^{(\alpha)}, Q) = \{(t, B^{(\alpha)}(t));\, t \in Q\}$$

be the random graph of the FBS over Q*. Then, with probability* 1*, the Hausdorff dimension, the packing dimension and the box dimension of its graph satisfy,*

$$\begin{aligned} \dim_H(\Gamma(B^{(\alpha)}, Q)) &= \dim_P(\Gamma(B^{(\alpha)}, Q)) \\ &= \dim_B(\Gamma(B^{(\alpha)}, Q)) = d + 1 - \min(\alpha_1, \dots, \alpha_d). \end{aligned} \tag{1.11}$$

It follows from relation (1.10) that to get Theorem 1.3, we only need to prove that the inequality

$$\dim_H(\Gamma(B^{(\alpha)}, Q)) \geq d + 1 - \min(\alpha_1, \dots, \alpha_d). \tag{1.12}$$

Thanks to the Frostman's Theorem (see e.g [1, 12, 15]) to obtain this inequality we only need to show that μ, the occupation measure of the field $t \mapsto (t, B^{(\alpha)}(t))$ has, with probability 1, a finite u-dimensional energy, for any $u \in]1, d+1-\min(\alpha_1, \dots, \alpha_d)[$. More precisely, for any Borel set $A \subset \mathbb{R}^{d+1}$, $\mu(A)$ is defined as the integral

$$\mu(A) = \int_Q \chi_{\{(t,B^{(\alpha)}(t)) \in A\}}\, dt,$$

where $\chi_{\{(t,B^{(\alpha)}(t)) \in A\}}$ denotes the characteristic function of the set $\{(t, B^{(\alpha)}(t));\ t \in \mathbb{R}^d\}$ and we need to prove that with probability 1 the integral

$$\int_{Q\times Q} |x-y|^{-u}\, \mu(dx)\mu(dy)$$

is finite. By a monotone class argument this is easily seen to be equivalent to $\int_{Q\times Q} (|s-t| + |B^{(\alpha)}(s) - B^{(\alpha)}(t)|)^{-u}\, dsdt < +\infty$, which follows from

$$\int_{Q\times Q} E[(|s-t| + |B^{(\alpha)}(s) - B^{(\alpha)}(t)|)^{-u}]\, dsdt < +\infty. \tag{1.13}$$

From now on, for the sake of simplicity, we will suppose that $Q = [1/2, 1]^d$. This is really not restrictive, Theorem 1.3 can be proved similarly when Q is any other cube of $\mathbb{R}^d$. The following Lemma allows to obtain an upper bound of the integral (1.13).

Lemma 1.4 *For all $(s,t) \in \mathbb{R}^d \times \mathbb{R}^d$, $s \neq t$ and for every real $u > 1$, we have*

$$E[(|s-t| + |B^{(\alpha)}(s) - B^{(\alpha)}(t)|)^{-u}] \le c|s-t|^{1-u}\sigma^{-1}(s,t), \tag{1.14}$$

where $\sigma^2(s,t) = E(|B^{(\alpha)}(s) - B^{(\alpha)}(t)|^2)$ and $c > 0$ is a constant.

Implicitly, this Lemma has often been used in the literature (see for example the proof of Theorem 16.4, page 244 of [15]) but for the sake of clarity and completeness we prefer to recall its proof, which is very short.

Proof. We have

$$\begin{aligned}
&E[(|s-t| + |\,B^{(\alpha)}(s) - B^{(\alpha)}(t)|)^{-u}] \\
&\qquad = \frac{1}{\sigma(s,t)\sqrt{2\pi}} \int_{\mathbb{R}} (|s-t| + |x|)^{-u} \exp\left(-\frac{x^2}{2\sigma^2(s,t)}\right) dx \\
&\qquad \le \frac{2}{\sigma(s,t)\sqrt{2\pi}} |s-t|^{-u} \int_0^{|s-t|} dx + \frac{2}{\sigma(s,t)\sqrt{2\pi}} \int_{|s-t|}^{+\infty} x^{-u}\, dx \\
&\qquad \le \frac{2}{\sigma(s,t)\sqrt{2\pi}} \left[|s-t|^{1-u} + \frac{|s-t|^{1-u}}{u-1} \right].
\end{aligned}$$

■

At last, using Lemma 1.4, one can easily see that to obtain (1.13), it is sufficient to prove the following Proposition.

Proposition 1.5 *For every $u \in]1, d+1-\min(\alpha_1, \dots, \alpha_d)[$, we have*

$$\sum_{l=0}^{\infty} 2^{l(u-1)} \int_{W_l} \sigma^{-1}(s,t)\, dsdt < \infty, \tag{1.15}$$

where for all integer $l \geq 0$, W_l is the domain defined as,

$$W_l = \{(s,t) \in Q \times Q;\ s_1 \neq t_1 \text{ and } 2^{-l-1} \leq |s-t| < 2^{-l}\}. \tag{1.16}$$

The remainder of our article is organized as follows. In section 2, we give a "wavelet decomposition" of the FBS, then in section 3, using this "nice" decomposition we prove Proposition 1.5.

2. "A wavelet" decomposition of the FBS

First, a word about notations.

- $\{2^{j/2}\psi(2^j x - k)\}_{(j,k)\in\mathbb{Z}\times\mathbb{Z}}$ will be a Lemarié-Meyer wavelet basis [17, 21]. Recall that such bases satisfy the following properties.

  (a) ψ and its Fourier transform $\hat{\psi}$ belong to the Schwartz class $S(\mathbb{R})$, namely the space of all infinitely differentiable functions $u(t)$ which satisfy, for all integers, m and n,

  $$\lim_{|t|\to\infty} t^m \left(\frac{d}{dt}\right)^n u(t) = 0.$$

  (b) The support of $\hat{\psi}$ is contained in the domain

  $$\left\{\xi;\ \frac{2\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}\right\}.$$

  (c) For all ξ, $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 1$.

- For every $H \in]0,1[$, ψ^H will be the function defined as,

  $$\psi^H(x) = \int_{\mathbb{R}} e^{ix\eta} \frac{\overline{\hat{\psi}(\eta)}}{|\eta|^{H+1/2}}\, d\eta. \tag{2.1}$$

  Thanks to properties (a) and (b) of the Lemarié-Meyer's wavelets, this definition makes sense and ψ^H belongs to the Schwartz class.

- For all $j=(j_1,\dots,j_d)\in\mathbb{Z}^d$, $k=(k_1,\dots,k_d)\in\mathbb{Z}^d$ and $t=(t_1,\dots,t_d)\in\mathbb{R}^d$, we set

$$\tilde{\psi}^{(\alpha)}_{j,k}(t)=\prod_{l=1}^{d}\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l), \tag{2.2}$$

where for every $l=1,\dots,d$

$$\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l)=\psi^{\alpha_l}(2^{j_l}t_l-k_l)-\psi^{\alpha_l}(-k_l). \tag{2.3}$$

Let us now state a fundamental result.

Proposition 2.1 *Let $\{\epsilon_{j,k}\}_{(j,k)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables. We will denote $(j,\alpha)=\sum_{l=1}^d j_l\alpha_l$ the scalar product of the vectors j and α. For every $t\in\mathbb{R}^d$, the series*

$$\tilde{B}^{(\alpha)}(t)=\sum_{(j,k)\in\mathbb{Z}^d\times\mathbb{Z}^d}2^{-(j,\alpha)}\epsilon_{j,k}\tilde{\psi}^{(\alpha)}_{j,k}(t), \tag{2.4}$$

is convergent in $L^2(\Omega)$, where Ω is the probability space. Moreover, up to a multiplicative constant, the field $\{\tilde{B}^{(\alpha)}(t)\}_{t\in\mathbb{R}^d}$ is an FBS with multi-index α.

Proof. Since the random variables $\epsilon_{j,k}$ are $\mathcal{N}(0,1)$ and independent, a necessary and sufficient condition for the series (2.4) to be convergent in $L^2(\Omega)$, is that

$$\sum_{(j,k)\in\mathbb{Z}^d\times\mathbb{Z}^d}2^{-2(j,\alpha)}|\tilde{\psi}^{(\alpha)}_{j,k}(t)|^2=\prod_{l=1}^{d}\left(\sum_{(j_l,k_l)\in\mathbb{Z}\times\mathbb{Z}}2^{-2j_l\cdot\alpha_l}|\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l)|^2\right)<\infty.$$

So, let us show that this last series is finite. First, we will prove that, for any $l=1,\dots,d$,

$$\sum_{(j_l,k_l)\in\mathbb{N}\times\mathbb{Z}}2^{-2j_l\alpha_l}|\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l)|^2<\infty. \tag{2.5}$$

As, the function ψ^{α_l} belongs to the Schwartz class, its derivative of any order n, satisfies

$$\left|\left(\frac{d}{dx}\right)^n\psi^{\alpha_l}(x)\right|\le c_1(2+|x|)^{-1}, \tag{2.6}$$

where $c_1>0$ is a constant, that depends on n.

Therefore, using (2.3) we get that

$$\begin{aligned}\sum_{(j_l,k_l)\in\mathbb{N}\times\mathbb{Z}} & 2^{-2j_l\alpha_l}\,|\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l)|^2 \\ &\le 2\sum_{(j_l,k_l)\in\mathbb{N}\times\mathbb{Z}} 2^{-2j_l\alpha_l}(|\psi^{\alpha_l}(2^{j_l}t_l-k_l)|^2+|\psi^{\alpha_l}(-k_l)|^2) \\ &\le c_2\sum_{(j_l,k_l)\in\mathbb{N}\times\mathbb{Z}} 2^{-2j_l\alpha_l}((2+|2^{j_l}t_l-k_l|)^{-2}+(2+|k_l|)^{-2}) \\ &\le c_3\sum_{j_l\in\mathbb{N}} 2^{-2j_l\alpha_l}<\infty,\end{aligned}$$

where $c_3 = 2c_2\sup_{x\in\mathbb{R}}\sum_{k_l\in\mathbb{Z}}(2+|x-k_l|)^{-2}<\infty$. Now, let us show that for any $l=1,\dots,d$

$$\sum_{(j_l,k_l)\in\mathbb{N}\times\mathbb{Z}} 2^{2j_l\alpha_l}|\tilde{\psi}^{\alpha_l}_{-j_l,k_l}(t_l)|^2<\infty. \tag{2.7}$$

It follows from (2.3) and the Mean Value Theorem, that for all $(j_l,k_l)\in\mathbb{N}\times\mathbb{Z}$,

$$|\tilde{\psi}^{\alpha_l}_{-j_l,k_l}(t_l)|\le 2^{-j_l}|t_l|M_{-j_l,k_l}(t_l),$$

where

$$M_{-j_l,k_l}(t_l)=\sup\left\{\left|\left(\frac{d}{dx}\right)\psi^{\alpha_l}(x)\right|,\quad x\in[-2^{-j_l}|t_l|-k_l,2^{-j_l}|t_l|-k_l]\right\}.$$

Consequently, we obtain that for some $\overline{J}_l$ big enough,

$$\sum_{j_l=\overline{J}_l}^{\infty}\sum_{k_l\in\mathbb{Z}}2^{2j_l\alpha_l}|\tilde{\psi}^{\alpha_l}_{-j_l,k_l}(t_l)|^2\le t_l^2c_4^2\sum_{j_l=\overline{J}_l}^{\infty}\sum_{k_l\in\mathbb{Z}}2^{-2(1-\alpha_l)j_l}(1+|k_l|)^{-2}<\infty.$$

At last, let us prove that the field $\{\tilde{B}^{(\alpha)}(t)\}_{t\in\mathbb{R}^d}$ is an FBS, up to a multiplicative constant, i.e. its covariance kernel satisfies the relation (1.8), up to a multiplicative constant. It follows from (2.1) and (2.3) that

$$2^{-j_l\alpha_l}\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l)=2^{-j_l\alpha_l}\int_{\mathbb{R}}\left(\frac{e^{i2^{j_l}t_l\eta}-1}{|\eta|^{\alpha_l+1/2}}\right)\overline{e^{ik_l\eta}\hat{\psi}(\eta)}\,d\eta.$$

Setting, $\xi_l=2^{j_l}\eta$, we get that

$$2^{-j_l\alpha_l}\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l)=2^{j_l/2}\int_{\mathbb{R}}\left(\frac{e^{it_l\xi_l}-1}{|\xi_l|^{\alpha_l+1/2}}\right)\overline{\mathcal{F}(\psi(2^{j_l}.+k_l))(\xi_l)}\,d\xi_l,$$

where $\mathcal{F}(\psi(2^{j_l}.+k_l))$ is the Fourier transform of the function $x\mapsto\psi(2^{j_l}x+k_l)$.

Then, since the sequence $\{2^{j_l/2}\mathcal{F}(\psi(2^{j_l}.+k_l));\ (j_l,k_l)\in\mathbb{Z}^2\}$ is, up to a multiplicative constant, an orthonormal basis for $L^2(\mathbb{R})$, it follows that

$$\sum_{(j_l,k_l)\in\mathbb{Z}\times\mathbb{Z}} 2^{-2j_l\alpha_l}\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l)\overline{\tilde{\psi}^{\alpha_l}_{j_l,k_l}(s_l)}=\int_{\mathbb{R}}\frac{(e^{it_l\xi_l}-1)(e^{-is_l\xi_l}-1)}{|\xi_l|^{2\alpha_l+1}}\,d\xi_l=c_4K_{\alpha_l}(t_l,s_l),$$

where $c_4>0$ is a constant and K_{α_l} is the covariance kernel of a standard, 1-D FBM, with index α_l (see (1.7)). Thus, we obtain that, for every $t=(t_1,\dots,t_d)\in\mathbb{R}^d$ and $s=(s_1,\dots,s_d)\in\mathbb{R}^d$,

$$E(\tilde{B}^{(\alpha)}(t)\tilde{B}^{(\alpha)}(s))=c_5\prod_{l=1}^{d}K_{\alpha_l}(t_l,s_l)=c_5\mathcal{K}_\alpha(t,s).$$

■

We will say, with some abuse since j is a multi-index, that the series (2.4) is a “wavelet decomposition” of the FBS.

Remark 2.2 *Observe that using techniques similar to that of [19] and [4], it has been shown in [3] that when $d=2$, the series (2.4) is almost surely, uniformly convergent in t, on every compact of $\mathbb{R}^2$.*

From now on, the field $\tilde{B}^{(\alpha)}$ will be identified with $B^{(\alpha)}$ the FBS with multi-index α.

3. Proof of Proposition 1.5

We can suppose without loss of generality that

$$\alpha_1=\min(\alpha_1,\dots,\alpha_d). \tag{3.1}$$

If $h=(h_1,h_2,\dots,h_d)\in\mathbb{R}^d$, we set $\hat{h}_1=(h_2,\dots,h_d)$ and for every $\hat{\jmath}_1,\hat{k}_1\in\mathbb{Z}^{d-1}$, $\tilde{\psi}^{\hat{\alpha}_1}_{\hat{\jmath}_1,\hat{k}_1}$ will denote the real valued function defined on $\mathbb{R}^{d-1}$, as

$$\tilde{\psi}^{\hat{\alpha}_1}_{\hat{\jmath}_1,\hat{k}_1}(\hat{t}_1)=\prod_{l=2}^{d}\tilde{\psi}^{\alpha_l}_{j_l,k_l}(t_l). \tag{3.2}$$

To obtain Proposition 1.5 we need the following Lemmas whose proofs will be given later.

Lemma 3.1 *There exists a constant $c_1>0$ such that the inequality,*

$$\left(\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}}|\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{\jmath}_1,\hat{k}_1}(\hat{s}_1)-\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{\jmath}_1,\hat{k}_1}(\hat{t}_1)|^2\right)^{1/2}\le c_1|\hat{s}_1-\hat{t}_1|, \tag{3.3}$$

holds for every $\hat{\jmath}_1\in\mathbb{N}^{d-1}$ and $\hat{s}_1,\hat{t}_1\in[0,1]^{d-1}$.

Lemma 3.2 *There exist an index $\hat{J}_1 \in \mathbb{N}^{d-1}$ and a constant $c_2 > 0$, such that, the inequality,*

$$\left(\sum_{\hat{k}_1 \in \mathbb{Z}^{d-1}} |\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{J}_1,\hat{k}_1}(\hat{s}_1)|^2 \right)^{1/2} \geq c_2, \tag{3.4}$$

holds for every $\hat{s}_1 \in [\frac{1}{2}, 1]^{d-1}$.

Lemma 3.3 *There exist two constants $c_3 > 0$ and $c_4 > 0$ satisfying the following property. For every $l \in \mathbb{N}$, $(s,t) \in W_l$, $s = (s_1, s_2, \ldots, s_d)$, $t = (t_1, t_2, \ldots t_d)$ and for some index $J_1 \in \mathbb{N}$, the inequalities,*

$$c_3 2^{-J_1 - 1} \leq |s_1 - t_1| < c_3 2^{-J_1} \tag{3.5}$$

and

$$\left(\sum_{k_1 \in \mathbb{Z}} |\tilde{\psi}^{\alpha_1}_{J_1,k_1}(s_1) - \tilde{\psi}^{\alpha_1}_{J_1,k_1}(t_1)|^2 \right)^{1/2} \geq c_4, \tag{3.6}$$

hold.

Lemma 3.4 *There exists a constant $c_5 > 0$, such that the inequality*

$$\left(\sum_{k_1 \in \mathbb{Z}} |\tilde{\psi}^{\alpha_1}_{j_1,k_1}(t_1)|^2 \right)^{1/2} \leq c_5 \tag{3.7}$$

holds, for every index $j_1 \in \mathbb{N}$ and real t_1.

Lemma 3.5 *There exists a constant $c_6 > 0$, such that the inequality*

$$\int_{W_l} |s_1 - t_1|^{-\alpha_1} \, ds_1 \ldots ds_d dt_1 \ldots dt_d \leq c_6 2^{-l(d-\alpha_1)}, \tag{3.8}$$

holds for every $l \in \mathbb{N}$.

Assuming Lemmas 3.1 to 3.5, Proposition 1.5 can be proved as follows.

Proof of Proposition 1.5 Let $l \geq 0$ be an arbitrary integer and $(s,t) \in W_l$, $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots, t_d)$. $J \in \mathbb{N} \times (\mathbb{Z}_-)^{d-1}$ will denote the index $J = (J_1, -\hat{J}_1)$, where $J_1 \in \mathbb{N}$ and $\hat{J}_1 \in \mathbb{N}^{d-1}$ have been introduced in Lemmas 3.2 and 3.3. Recall that we have set $\sigma^2(s,t) = E(|B^{(\alpha)}(s) - B^{(\alpha)}(t)|^2)$. It follows from the Relation (2.4) that

$$\begin{aligned} \sigma^2(s,t) &\geq \sum_{k \in \mathbb{Z}^d} 4^{-(J,\alpha)} \left| \tilde{\psi}^{(\alpha)}_{J,k}(s) - \tilde{\psi}^{(\alpha)}_{J,k}(t) \right|^2 \\ &= 4^{-J_1 \alpha_1 + (\hat{J}_1, \hat{\alpha}_1)} \sum_{k \in \mathbb{Z}^d} \left| \tilde{\psi}^{\alpha_1}_{J_1,k_1}(s_1) \tilde{\psi}^{\hat{\alpha}_1}_{-\hat{J}_1,\hat{k}_1}(\hat{s}_1) - \tilde{\psi}^{\alpha_1}_{J_1,k_1}(t_1) \tilde{\psi}^{\hat{\alpha}_1}_{-\hat{J}_1,\hat{k}_1}(\hat{t}_1) \right|^2. \end{aligned}$$

Thus, we have

$$\begin{aligned}\sigma(s,t) \quad \geq \quad & c_7 2^{-J_1\alpha_1}\Big(\sum_{k\in\mathbb{Z}^d}\big|\tilde{\psi}^{\alpha_1}_{J_1,k_1}(s_1)-\tilde{\psi}^{\alpha_1}_{J_1,k_1}(t_1)\big|^2\big|\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{J}_1,\hat{k}_1}(\hat{s}_1)\big|^2\Big)^{1/2}\\ & - c_7 2^{-J_1\alpha_1}\Big(\sum_{k\in\mathbb{Z}^d}\big|\tilde{\psi}^{\alpha_1}_{J_1,k_1}(t_1)\big|^2\big|\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{J}_1,\hat{k}_1}(\hat{s}_1)-\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{J}_1,\hat{k}_1}(\hat{t}_1)\big|^2\Big)^{1/2},\end{aligned}$$

where the constant $c_7 = 2^{(\hat{J}_1,\alpha_1)}$ does not depend on (s,t) (see Lemma 3.2). Then Lemmas 3.1 to 3.4 imply that

$$\sigma(s,t) \geq c_7 2^{-J_1\alpha_1}(c_4c_2 - c_5c_1|\hat{s}_1-\hat{t}_1|).$$

Moreover, it follows from (1.16) that $|\hat{s}_1-\hat{t}_1| \leq 2^{-l}$ and from (3.5) that $2^{-J_1\alpha_1} > c_3^{-\alpha_1}|s_1-t_1|^{\alpha_1}$. Thus, we obtain that for every $l\in\mathbb{N}$,

$$\sigma(s,t) \geq c_7c_3^{-\alpha_1}|s_1-t_1|^{\alpha_1}(c_4c_2-c_5c_1 2^{-l}).$$

Therefore, there exist a constant $c_8>0$ and an integer l_0, such that for every $l\geq l_0$ and $(s,t)\in W_l$, we have

$$\sigma(s,t) \geq c_8|s_1-t_1|^{\alpha_1}. \tag{3.9}$$

At last, let us show that for all $u\in]1,d+1-\alpha_1[$, the series

$$\sum_{l=l_0}^{\infty} 2^{l(u-1)}\int_{W_l}\sigma^{-1}(s,t)\,dsdt$$

is convergent. Using (3.9) and Lemma 3.5, it follows that

$$\begin{aligned}\sum_{l=l_0}^{\infty} 2^{l(u-1)}\int_{W_l}\sigma^{-1}(s,t)\,dsdt \quad & \leq \quad c_8^{-1}\sum_{l=l_0}^{\infty}2^{l(u-1)}\int_{W_l}|s_1-t_1|^{-\alpha_1}\,dsdt\\ & \leq \quad c_9\sum_{l=l_0}^{\infty}2^{-l(d+1-\alpha_1-u)} < \infty.\end{aligned}$$

■

Now, let us give the proofs of Lemmas 3.1 to 3.5.

Proof of Lemma 3.1. First, let us show that (3.3) holds for $\hat{\jmath}_1 = 0_{\mathbb{N}^{d-1}}$ and every $\hat{s}_1,\hat{t}_1\in[0,1]^{d-1}$, i.e.

$$\Big(\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}}|\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}(\hat{s}_1)-\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}(\hat{t}_1)|^2\Big)^{1/2} \leq c_1|\hat{s}_1-\hat{t}_1|. \tag{3.10}$$

We set for all $\hat{k}_1\in\mathbb{Z}^{d-1}$,

$$M_{\hat{k}_1} = \sup\Big\{\Big|\frac{\partial\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}}{\partial h_l}(\hat{h}_1)\Big|,\ \hat{h}_1\in]0,1[^{d-1},\ l=2,\dots,d\Big\}. \tag{3.11}$$

It follows from the Mean Value Theorem that

$$\Big(\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}} |\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}(\hat{s}_1)-\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}(\hat{t}_1)|^2\Big)^{1/2} \le \sqrt{d-1}\Big(\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}} M^2_{\hat{k}_1}\Big)^{1/2}|\hat{s}_1-\hat{t}_1|.$$

Thus, it is sufficient to show that

$$\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}} M^2_{\hat{k}_1}<\infty. \tag{3.12}$$

Recall that $\hat{k}_1=(k_2,\dots,k_d)$ and $\hat{h}_1=(h_2,\dots,h_d)$. It follows from (2.2) and (2.6) that for all $\hat{h}_1\in]0,1[^{d-1}$ and $l=2,\dots,d$,

$$\begin{aligned}\left|\frac{\partial\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}}{\partial h_l}(\hat{h}_1)\right| &= \left|\frac{d\psi^{\alpha_l}}{dh_l}(h_l-k_l)\right|\prod_{2\le n\le d, n\ne l}|\psi^{\alpha_n}(h_n-k_n)-\psi^{\alpha_n}(-k_n)|\\ &\le c\left(\frac{1}{2+|h_l-k_l|}\right)\prod_{2\le n\le d, n\ne l}\left(\frac{1}{2+|h_n-k_n|}+\frac{1}{2+|k_n|}\right)\\ &\le c'\prod_{2\le n\le d}(1+|k_n|)^{-1}.\end{aligned}$$

Thus, we have

$$\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}} M^2_{\hat{k}_1}\le c''\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}}\prod_{2\le n\le d}(1+|k_n|)^{-2}\le c''\prod_{2\le n\le d}\left(\sum_{k_n\in\mathbb{Z}}(1+|k_n|)^{-2}\right)<\infty$$

and we obtain (3.10). At last, let us show that (3.3) holds for every $\hat{\mathrm{j}}_1=(j_2,\dots,j_d)\in\mathbb{N}^{d-1}$. For all, $\hat{h}_1=(h_2,\dots,h_d)\in\mathbb{R}^d$, we set $2^{-\hat{\mathrm{j}}_1}.\hat{h}_1=(2^{-j_2}h_2,\dots,2^{-j_d}h_d)$. Since, $\hat{s}_1$ and $\hat{t}_1$ belong to $[0,1]^{d-1}$, it is clear that $2^{-\hat{\mathrm{j}}_1}.\hat{s}_1$ and $2^{-\hat{\mathrm{j}}_1}.\hat{t}_1$ belong to $[0,1]^{d-1}$. Thus, using (3.10), we obtain that

$$\begin{aligned}&\Big(\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}}\left|\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{\mathrm{j}}_1,\hat{k}_1}(\hat{s}_1)-\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{\mathrm{j}}_1,\hat{k}_1}(\hat{t}_1)\right|^2\Big)^{1/2}\\ &=\Big(\sum_{\hat{k}_1\in\mathbb{Z}^{d-1}}\left|\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}(2^{-\hat{\mathrm{j}}_1}.\hat{s}_1)-\tilde{\psi}^{\hat{\alpha}_1}_{0,\hat{k}_1}(2^{-\hat{\mathrm{j}}_1}.\hat{t}_1)\right|^2\Big)^{1/2}\le c_1|2^{-\hat{\mathrm{j}}_1}.(\hat{s}_1-\hat{t}_1)|\le c_1|\hat{s}_1-\hat{t}_1|.\end{aligned}$$

■

We need the following result to prove Lemma 3.2.

Remark 3.6 *For every $H\in]0,1[$, let ψ^H be the function defined in (2.1). Its derivative satisfies*

$$\sum_{k\in\mathbb{Z}}\left|\frac{d\psi^H}{dx}(k)\right|^2>0.$$

Proof of Remark 3.6 For all $x \in \mathbb{R}$, we have $\frac{d\psi^H}{dx}(x) = i \int_{\mathbb{R}} e^{ix\eta} \overline{\frac{\eta\hat{\psi}(\eta)}{|\eta|^{H+1/2}}}\, d\eta$. Thus, for every $k \in \mathbb{Z}$,

$$\frac{d\psi^H}{dx}(k) = i \int_0^{2\pi} e^{ik\eta} \left(\overline{\sum_{n\in\mathbb{Z}} \frac{(\eta + 2\pi n)\hat{\psi}(\eta + 2\pi n)}{|\eta + 2\pi n|^{H+1/2}}} \right) d\eta. \tag{3.13}$$

Suppose now, *ad absurdum*, that for every $k \in \mathbb{Z}$, $\frac{d\psi^H}{dx}(k) = 0$. Then, it follows from (3.13) that all the Fourier coefficients, of the 2π-periodic C^∞ function

$$\eta \longmapsto \sum_{n\in\mathbb{Z}} \frac{(\eta + 2\pi n)\hat{\psi}(\eta + 2\pi n)}{|\eta + 2\pi n|^{H+1/2}}$$

vanish, and consequently, this function vanishes as well. Thus, we would have for all $\eta \in \mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} \frac{(\eta + 2\pi n)\hat{\psi}(\eta + 2\pi n)}{|\eta + 2\pi n|^{H+1/2}} = 0.$$

By setting, in this last equality, $\eta = 4\pi/3$, we would obtain by property (b) of the Lemarié-Meyer wavelets that $\hat{\psi}(4\pi/3) = 0$ but this would contradict their property (c). ■

Proof of Lemma 3.2 It follows from (3.2) that, for every $\hat{s}_1 \in [\frac{1}{2}, 1]^{d-1}$ and $\hat{J}_1 = (j_2, \ldots, j_d) \in \mathbb{N}^{d-1}$, we have

$$\left(\sum_{\hat{k}_1 \in \mathbb{Z}^{d-1}} |\tilde{\psi}^{\hat{\alpha}_1}_{-\hat{J}_1, \hat{k}_1}(\hat{s}_1)|^2 \right)^{1/2} = \prod_{m=2}^{d} \left(\sum_{k_m \in \mathbb{Z}} |\tilde{\psi}^{\alpha_m}_{-j_m, k_m}(s_m)|^2 \right)^{1/2}.$$

Thus, it is sufficient to show that for every $m = 2, \ldots, d$, there exists a constant $c_m > 0$, such that the inequality,

$$\left(\sum_{k_m \in \mathbb{Z}} |\tilde{\psi}^{\alpha_m}_{-j_m, k_m}(s_m)|^2 \right)^{1/2} \geq c_m, \tag{3.14}$$

holds, for all $s_m \in [\frac{1}{2}, 1]$. One can apply Taylor's formula to order 2 and get

$$\begin{aligned} \tilde{\psi}^{\alpha_m}_{-j_m, k_m}(s_m) &= \psi^{\alpha_m}(2^{-j_m} s_m - k_m) - \psi^{\alpha_m}(-k_m) \\ &= (2^{-j_m} s_m) \frac{d\psi^{\alpha_m}}{dx}(-k_m) + \frac{(2^{-j_m} s_m)^2}{2} \frac{d^2\psi^{\alpha_m}}{dx^2}(-k_m + e_{-j_m, k_m}(s_m) s_m), \end{aligned}$$

where $e_{-j_m, k_m}(s_m) \in]0, 1[$.

Thus, it follows that

$$\Big(\sum_{k_m\in\mathbb{Z}}\big|\tilde{\psi}^{\alpha_m}_{-j_m,k_m}(s_m)\big|^2\Big)^{1/2} \geq 2^{-j_m}s_m\Big(\sum_{k_m\in\mathbb{Z}}\Big|\frac{d\psi^{\alpha_m}}{dx}(-k_m)\Big|^2\Big)^{1/2} - \frac{(2^{-j_m}s_m)^2}{2}\Big(\sum_{k_m\in\mathbb{Z}}\Big|\frac{d^2\psi^{\alpha_m}}{dx^2}(-k_m+e_{-j_m,k_m}(s_m)s_m)\Big|^2\Big)^{1/2}.$$

Using (2.6) and since $s_m\in[\frac{1}{2},1]$, we obtain that

$$\Big(\sum_{k_m\in\mathbb{Z}}\big|\tilde{\psi}^{\alpha_m}_{-j_m,k_m}(s_m)\big|^2\Big)^{1/2} \geq \frac{2^{-j_m}}{2}\Big(\sum_{k_m\in\mathbb{Z}}\Big|\frac{d\psi^{\alpha_m}}{dx}(k_m)\Big|^2\Big)^{1/2} - 2^{-2j_m}\Big(\sum_{k_m\in\mathbb{Z}}(1+|k_m|)^{-2}\Big)^{1/2}.$$

Then it follows from Remark 3.6 and this last inequality that there exists an index $j_m\in\mathbb{N}$, such that for every $s_m\in[\frac{1}{2},1]$, one has

$$\Big(\sum_{k_m\in\mathbb{Z}}|\tilde{\psi}^{\alpha_m}_{-j_m,k_m}(s_m)|^2\Big)^{1/2}\geq c_m,$$

where $c_m=\frac{2^{-j_m}}{4}\Big(\sum_{k_m\in\mathbb{Z}}\big|\frac{d\psi^{\alpha_m}}{dx}(-k_m)\big|^2\Big)^{1/2}>0.$ ■

To prove Lemma 3.3, we need the following result of [25] and [26] whose proof is given for completeness.

Remark 3.7 *[25, 26] For every $H\in]0,1[$, let ψ^H be the function defined in (2.1). There exist a compact set $A\subset\mathbb{R}$ and a constant $c_7>0$, satisfying the following properties:*

(i) $\cup_{k\in\mathbb{Z}}(A+k)=\mathbb{R}$, where the set $(A+k)=\{x+k,\, x\in A\}$,

(ii) for every $x\in A$, $|\psi^H(x)|\geq c_7$.

Proof of Remark 3.7 First, similarly to the proof of Remark 3.6, one can show that for every real x, there exists an integer k, such that $\psi^H(x-k)\neq 0$. Then, using the continuity and the 1-periodicity of the function, $g(x)=\sum_{k\in\mathbb{Z}}|\psi^H(x-k)|^2$ this implies that for some constant $c'>0$, the inequality,

$$\sum_{k\in\mathbb{Z}}|\psi^H(x-k)|^2\geq c', \tag{3.15}$$

holds for every real x. Now, the function ψ^H being in the Schwartz class, there exists $c_0>0$, a constant such that for all x,

$$|\psi^H(x)|\leq c_0(2+|x|)^{-1}. \tag{3.16}$$

Let $K_0 > 0$ be an integer such that

$$c_0^2 \sum_{|k|>K_0} (1+|k|)^{-2} \leq c'/2. \tag{3.17}$$

At last, for all $x \in [0,1]$, we have

$$\sum_{|k|\leq K_0} |\psi^H(x-k)|^2 \geq c'/2. \tag{3.18}$$

Indeed, it follows from (3.15) and (3.17) that

$$\begin{aligned}\sum_{|k|\leq K_0} |\psi^H(x-k)|^2 &\geq c' - c_0^2 \sum_{|k|>K_0} (2+|x-k|)^{-2} \\ &\geq c' - c_0^2 \sum_{|k|>K_0} (2+|k|-|x||)^{-2} \geq c' - c_0^2 \sum_{|k|>K_0} (1+|k|)^{-2} \geq \ c'/2.\end{aligned}$$

Therefore, the constant c_7 and the compact set A can be defined as $c_7 = \left(\frac{c'}{2(2K_0+1)}\right)^{1/2}$ and $A = \{x \in \mathbb{R}, |\psi^H(x)| \geq c_7\}$. ■

Proof of Lemma 3.3 Let A and c_7 be the compact set and the constant, that have been introduced in Remark 3.7. Here, we suppose that $H = \alpha_1$. Since A is bounded, there exists a real $c_8 > 0$, such that for every $x \in A$, one has $|x| \leq c_8$. Moreover, as the function ψ^{α_1} is in the Schwartz class, for some constant $c_9 > 0$, the inequality

$$|\psi^{\alpha_1}(y)| \leq c_9(1+c_8+|y|)^{-1}, \tag{3.19}$$

holds, for every y. Now, let c_3 be a constant satisfying

$$c_9(1+2^{-1}c_3)^{-1} \leq 2^{-1}c_7. \tag{3.20}$$

For all $l \in \mathbb{N}$ and $(s,t) \in W_l$, $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots t_d)$, there exists a unique index $J_1 \in \mathbb{N}$, such that

$$c_3 2^{-J_1-1} \leq |s_1 - t_1| < c_3 2^{-J_1}. \tag{$*$}$$

It follows from Remark 3.7 (i) that for some $K_1 \in \mathbb{Z}$ one has $2^{J_1}s_1 - K_1 \in A$, i.e.

$$|\psi^{\alpha_1}(2^{J_1}s_1 - K_1)| \geq c_7. \tag{3.21}$$

At last, (2.3), (3.21), (3.19), $(*)$ and (3.20) imply that

$$\begin{aligned}|\tilde{\psi}^{\alpha_1}_{J_1,K_1}(s_1) - \tilde{\psi}^{\alpha_1}_{J_1,K_1}(t_1)| &= |\psi^{\alpha_1}(2^{J_1}s_1 - K_1) - \psi^{\alpha_1}(2^{J_1}t_1 - K_1)| \\ &\geq |\psi^{\alpha_1}(2^{J_1}s_1 - K_1)| - |\psi^{\alpha_1}(2^{J_1}t_1 - K_1)| \\ &\geq c_7 - c_9(1+c_8+|2^{J_1}t_1 - K_1|)^{-1} \\ &\geq c_7 - c_9(1+c_8+2^{J_1}|s_1-t_1| - |2^{J_1}s_1 - K_1|)^{-1} \\ &\geq c_7 - c_9(1+c_8+2^{-1}c_3 - c_8)^{-1} \geq 2^{-1}c_7.\end{aligned}$$

Therefore, we obtain that

$$\left(\sum_{k_1\in\mathbb{Z}}\left|\tilde{\psi}^{\alpha_1}_{J_1,k_1}(s_1)-\tilde{\psi}^{\alpha_1}_{J_1,k_1}(t_1)\right|^2\right)^{1/2}\geq c_4,$$

where $c_4=2^{-1}c_7$. ■

Proof of Lemma 3.4 It follows from (2.3) and (2.6) that

$$\left(\sum_{k_1\in\mathbb{Z}}|\tilde{\psi}^{\alpha_1}_{j_1,k_1}(t_1)|^2\right)^{1/2}\leq\left(\sum_{k_1\in\mathbb{Z}}|\psi^{\alpha_1}(2^{j_1}t_1-k_1)|^2\right)^{1/2}+\left(\sum_{k_1\in\mathbb{Z}}|\psi^{\alpha_1}(-k_1)|^2\right)^{1/2}$$
$$\leq 2c_0\sup_{x\in\mathbb{R}}\left\{\left(\sum_{k_1\in\mathbb{Z}}(2+|x-k_1|)^{-2}\right)^{1/2}\right\}<\infty.$$

■

Proof of Lemma 3.5 For all $l\in\mathbb{N}$, we set

$$G_l=\{(x,y)\in[0,1]^2;\ |x-y|\leq 2^{-l}\}. \tag{3.22}$$

Thus, it follows from (1.16) that for every $(s,t)\in\mathbb{R}^d\times\mathbb{R}^d$, $s=(s_1,\ldots,s_d)$, $t=(t_1,\ldots,t_d)$, we have

$$\chi_{W_l}(s,t)\leq\prod_{n=1}^{d}\chi_{G_l}(s_n,t_n), \tag{3.23}$$

where χ_{W_l} (resp. χ_{G_l}) denotes the characteristic function of W_l (resp. G_l). Consequently, we obtain that,

$$\int_{W_l}|s_1-t_1|^{-\alpha_1}\,dsdt\leq\left(\int_{G_l}|s_1-t_1|^{-\alpha_1}\,ds_1dt_1\right)\left(\int_{G_l}dxdy\right)^{d-1}. \tag{3.24}$$

Now, using Fubini's Theorem, we get that

$$\int_{G_l}|s_1-t_1|^{-\alpha_1}\,ds_1dt_1\leq c2^{-l(1-\alpha_1)} \tag{3.25}$$

$$\int_{G_l}ds_1dt_1\leq c'2^{-l}. \tag{3.26}$$

At last, Lemma 3.5 follows from (3.24), (3.25) and (3.26). ■

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References

[1] Adler, R. J.: *The geometry of random fields*. Wiley, New York, 1981.

[2] Ayache, A., Léger, S. and Pontier, M.: Drap Brownien Fractionnaire. *Potential Anal.* **17** (2002), 31–43.

[3] Ayache, A., Léger, S. and Pontier, M.: Les ondelettes à la conquête du drap brownien fractionnaire. *C. R. Math. Acad. Sci. Paris* **335** (2002), 1063–1068.

[4] Ayache, A. and Taqqu, M.,S.: Rate optimality of wavelet series approximations of fractional Brownian motion. *J. Fourier Anal. Appl.* **9** (2003), 451–471.

[5] Bardina, X., Jolis, M. and Tudor, C. A.: Weak convergence to the fractional brownian sheet. *Prepublicacions UAB* **6**, 2002.

[6] Berman, S. M.: Gaussian sample functions: uniform dimension and Hölder conditions nowhere. *Nagoya Math. J.* **46** (1972), 63–86.

[7] Ciesielski, Z. and Kamont, A.: Levy's fractional Brownian random field and function spaces. *Acta Sci. Math. (Szeged)* **60** (1995), 99–118.

[8] Cuzick, J.: Some local properties of Gaussian vector fields. *Ann. Probab.* **6** (1978), 984–994.

[9] Doukhan, P., Oppenheim, G. and Taqqu, M. S. (eds.): *Theory and applications of long-range dependence*. Birkhäuser, Boston, 2003.

[10] Geman, D. and Horowitz, J.: Occupation densities. *Ann. Probab.* **8** (1980), 1–67.

[11] Goldman, A.: Points multiples des trajectoires de processus Gaussiens. *Z. Wahrsch. Verw. Gebiete* **57** (1981), 481–494.

[12] Kahane, J. P.: *Some random series of functions*, 2nd. ed. Cambridge Studies in Advanced Mathematics **5**. Cambridge Univ. Press, Cambridge, 1985.

[13] Kolmogorov, A.: Wienersche Spiralen und einige andere interessante Kurven in Hilbertsche Raum. *C. R. (Dokl.) Acad. Sci. URSS* **26** (1940), 115–118.

[14] Kamont,A.: On the fractional anisotropic Wiener field. *Probab. Math. Statist.* **16** (1996), 85–98.

[15] Falconer, K.: *Fractal geometry. Mathematical foundations and applications*. John Wiley and Sons, Chichester, 1990.

[16] Kühn, T. and Linde, W.: Optimal series representation of fractional Brownian sheets. *Bernoulli* **8** (2002), 669–696.

[17] Lemarié, P.-G. and Meyer, Y.: Ondelettes et bases hilbertiennes. *Rev. Mat. Iberoamericana* **2** (1986), 1–18.

[18] Lindstrom, T.: Fractional Brownian fields as integrals of white noise. *Bull. London Math. Soc.* **25** (1993), 83–88.

[19] Meyer, Y., Sellan, F. and Taqqu, M. S.: Wavelets, generalized white noise and fractional integration: the synthesis of fractional brownian motion. *J. Fourier Anal. Appl.* **5** (1999), 465–494.

[20] Mandelbrot, B. and van Ness, J.: Fractional Brownian motion, fractional noises and applications. *Siam Rev.* **10** (1968), 422–437.

[21] Meyer, Y.: *Ondelettes et opérateurs I: ondelettes*. Actualités Mathématiques. Herman, Paris, 1990.

[22] Nolan, J. P.: Path properties of index-β stable fields. *Ann. Probab.* **16** (1988), 1596–1607.

[23] Pyke, R.: Partial sums of matrix arrays and Brownian Sheets. In *Stochastic analysis (a tribute to the memory of Rollo Davidson)*, 331–348. Wiley, London, 1973.

[24] Pitt, L. D.: Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** (1978), 309–330.

[25] Roueff, F.: *Dimension de Hausdorff du graphe d'une fonction continue: une étude analytique et statistique*. Ph. D. Thesis, Telecom, Paris, 2000.

[26] Roueff, F.: Almost sure Hausdorff dimensions of graphs of random wavelet series. *J. Fourier Anal. Appl.* **9** (2003), 237–260.

[27] Samorodnitsky, G. and Taqqu, M. S.: *Stable non-Gaussian random processes. Stochastic models with infinite variance*. Chapman & Hall, New York, 1994.

[28] Testard, F.: Polarité, points multiples et géométrie de certains processus Gaussiens. PhD. Thesis, University of Orsay, 1987.

[29] Tricot, C.: *Courbes et Dimension Fractale*. Springer-Verlag, Paris, 1993.

[30] Trotter, H. F.: A property of Brownian motion paths. *Illinois J. Math.* **2** (1958), 425–433.

[31] Xiao, Y.: Dimension results for Gaussian vector fields and index-α stable fields. *Ann. Probab.* **23** (1995), 273–271.

[32] Xiao, Y. and Zhang, T.: Local times of fractional Brownian sheets. *Probab. Theory Related Fields* **124** (2002), 204–226.

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