Transitive flows on manifolds

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Abstract

In this paper we characterize manifolds (topological or smooth, compact or not, with or without boundary) which admit flows having a dense orbit (such manifolds and flows are called transitive) thus fully answering some questions by Smith and Thomas. Namely, it is shown that a surface admits a transitive flow (which can be got smooth) if and only if it is connected and it is neither homeomorphic to the sphere nor the projective plane nor embeddable in the Klein bottle (or, alternatively, if it is connected and includes two orientable topological circles intersecting transversally at exactly one point). We also prove that any (connected) manifold with dimension at least 3 admits a transitive flow, which can be got smooth if the manifold admits a smooth structure.

In particular, this allows us to characterize ω-limit sets with non-empty interior for flows in a given n-manifold (as they can be described by the property of being the closure of its transitive n-submanifolds).

1. Introduction

A *flow* on a manifold M (topological or smooth, compact or not, with or without boundary) is a one-parameter family $\{\Phi^t\}_{t\in\mathbb{R}}$ of homeomorphisms of M such that $\Phi^t \circ \Phi^s = \Phi^{t+s}$ for any $t, s \in \mathbb{R}$ (we also assume that the map $\Phi : \mathbb{R} \times M \to M$ given by $\Phi(t,x) = \Phi^t(x)$, which we will identify with the flow in the sequel, is continuous). The *orbit* of a point $x \in M$ under a flow $\{\Phi^t\}_{t\in\mathbb{R}}$ is the set $\{\Phi^t(x) : t \in \mathbb{R}\}$. A flow on a manifold M is said to be *(topologically) transitive* if there is a point whose orbit under the flow is dense in M, and we say that a manifold M is *transitive* if it admits a transitive flow; if the manifold M admits a (fixed) smooth (C^∞) structure and Φ is smooth then we will refer to them as *smooth transitive*.

The problem of finding transitive flows on manifolds has a long tradition (cf. for instance the bibliography in [16]). Needless to say, the circle is the only transitive 1-manifold. That neither the sphere $\mathbb{S}^2$, the projective plane $\mathbb{P}^2$ nor the Klein bottle $\mathbb{B}^2$ are transitive was more or less known since the seminal work of Bendixson [2] a century ago in the sphere case (and probably in the projective plane case as well), and since at least 1969 [9] in the Klein bottle case. The existence of smooth transitive flows for all other closed surfaces is proved e.g. in [4], although probably it was known rather earlier. In the multidimensional case ($n \geq 3$) one should refer at least to Oxtoby and Ulam [12], who proved in 1941 that virtually all n-dimensional compact connected polyhedra admit (continuous) transitive flows, to a lesser known paper by Sidorov [15] (1968) where it is shown that any region in $\mathbb{R}^n$ is smooth transitive (with the standard differential structure) and to Anosov [1] (1974) whose more general result on ergodic flows implies in particular that all compact connected smooth n-manifolds (and also $\mathbb{R}^n$) are smooth transitive.

The task of classifying transitive manifolds was systematically undertaken by Smith and Thomas in 1988 in the papers [16] and [17]. Essentially, they characterize transitive (and, equivalently, smooth transitive) compact connected surfaces as those which cannot be embedded in $\mathbb{S}^2$, $\mathbb{P}^2$ or $\mathbb{B}^2$, and provide an alternative, simpler proof of the above-mentioned weaker version of Anosov's result: the problems of characterizing which non-compact surfaces and manifolds are transitive are posed as open there.

Up to our knowledge, the last steps in this direction where independently given by Benière in his Ph.D. Thesis [3] (1998) and the second author in his Master Thesis [18] (1999). In [3] it is proved that all connected orientable surfaces without boundary which are not embeddable in $\mathbb{S}^2$ are smooth minimal and hence smooth transitive. Alternatively, smooth transitive connected surfaces without boundary were characterized (under the additional assumption of finite genus) in [18] as those including two "crossing circles", that is, two orientable simple closed curves intersecting transversally at exactly one point.

In this paper we culminate the classification of transitive manifolds by proving first:

Theorem A. *Let S be a connected surface (resp. a connected orientable surface). Then the following statements are equivalent:*

(i) S is smooth transitive;

(ii) S is transitive;

(iii) S is not homeomorphic to $\mathbb{S}^2$, $\mathbb{P}^2$, nor to any surface in $\mathbb{B}^2$ (resp. is not homeomorphic to any surface in $\mathbb{S}^2$);

(iv) S has two crossing circles.

Remark 1.1. In the orientable case much of Theorem A follows of course from Beniére's result, but some rather strong tools are required (Falconer's results on Hausdorff measures and interval exchange transformations) which are unnecessary for our purposes. Thus we have decided to include a rather elementary and complete proof of Theorem A below.

We also prove:

Theorem B. *Let M be a connected n-manifold (resp. a connected smooth n-manifold), $n \geq 3$. Then it is transitive (resp. smooth transitive).*

Recall that if $x \in M$ and Φ is a flow on a n-manifold M then the ω-*limit set* of the orbit of x under the flow Φ is defined by

$$\omega_\Phi(x) = \{y \in X : \text{there is a sequence } (t_m)_m \to \infty \text{ with } (\Phi^{t_m}(x))_m \to y\}$$

(the α-*limit set* of the orbit of x, $\alpha_\Phi(x)$, is analogously defined just replacing ∞ by $-\infty$). If, moreover, $\omega_\Phi(x)$ has nonempty interior O, then O is connected and invariant for the flow Φ (in particular it includes the orbit of x) and has $\omega_\Phi(x)$ as its closure: see Lemma 2.2. Hence, Φ can be seen as a transitive flow on O, which is then a transitive n-submanifold of M. Conversely, let T be a transitive n-submanifold of M and let Φ be the corresponding transitive flow on T (then T is an ω-limit set for Φ by Lemma 2.4). As all maps Φ^t are homeomorphisms and, because of the theorem of invariance of domain [8, p. 475], any subset of M which is homeomorphic to an open set of $\mathbb{R}^n$ is open as well, the combinatorial boundary ∂T of T is invariant for Φ. Then we can see Φ as a flow on the open set $T \setminus \partial T$ and extend it via Lemma 2.3 to a flow on M still having T as one of its ω-limit sets. Thus, ω-limit sets (and, similarly, α-limit sets) with nonempty interior in M are exactly the closure of its transitive n-submanifolds and Theorems A and B allow us to characterize them.

2. Preliminaries

Let X be a metrizable space (much of the ensuing discussion also works in less restrictive settings, but this one is general enough for our purposes). We call a continuous map $\Phi : \Lambda \subset \mathbb{R} \times X \to X$ a *local flow* on X provided that the following properties hold:

(i) Λ is open in $\mathbb{R} \times X$; moreover, for any $x \in X$ the set of numbers t for which $\Phi(t, x)$ is defined is an open interval $I_x \ni 0$;

(ii) $\Phi(0, x) = x$ for any $x \in X$;

(iii) if $\Phi(t, x) = y$ then $I_y = \{s - t : s \in I_x\}$; moreover, $\Phi(u, y) = \Phi(u, \Phi(t, x)) = \Phi(u + t, x)$ for any $u \in I_y$.

If we are in the particular case $\Lambda = \mathbb{R} \times X$ then we get a *flow* on X, as we did in the Introduction in the case of manifolds. Notice that if X is compact then $I_x = \mathbb{R}$ for any $x \in X$ so any local flow is in fact a flow. We use the notation $\Phi(t,x) = \Phi^t(x) = \Phi_x(t)$ whenever it makes sense, being $\Phi_x(I_x)$ the *orbit* of a point x under the local flow Φ. We say that a point $x \in X$ is *singular* if its orbit consists of just the point x.

Of course we are primarily interested in manifolds, and local flows naturally arise there as associated to autonomous systems of differential equations (thus the need of introducing this otherwise artificial notion). More precisely, it is well known that if $\Phi : \Lambda \to M$ is a smooth local flow then there is a smooth vector field $F : M \to TM$ (its *associated vector field*) such that $\frac{\partial \Phi}{\partial t}(t,x) = F(\Phi(t,x))$ for any t and x and that, conversely, for any smooth vector field F on M there is a smooth local flow Φ on M (its *associated local flow*) such that $\frac{\partial \Phi}{\partial t}(t,x) = F(\Phi(t,x))$ for any t and x.

When we speak about an *n-manifold* (or sometimes just a *manifold*) we refer to a Hausdorff topological space with a countable base (usually denoted by M) which is locally homeomorphic to the closed ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ (hence it need not be compact and may have *(combinatorial) boundary points*, which are those having no neighbourhood homeomorphic to the *open* ball $\{x \in \mathbb{R}^n : \|x\| < 1\}$). As usual, if M is an n-manifold then we denote by ∂M the set of its boundary points. If $n = 2$ then M is called a *surface* and we will often use the symbol S to denote it; in the particular case when S is compact, connected and has empty boundary we call it a *closed surface*. As it is well known, all n-manifolds with $n \leq 3$ admit a smooth structure which is unique up to diffeomorphisms (cf. Theorem 2.1), but if $n > 3$ they may admit one, none or even infinitely many. Thus, when speaking about a *smooth* manifold in this setting, we will implicitly assume that we have fixed a smooth structure on it.

If X is metrizable and $A \subset X$ then Cl A, Int A, Bd A and diam A will denote the *closure*, the *interior*, the *boundary* of A and the *diameter* (for a fixed distance $d(\cdot, \cdot)$ in X) of A, and if f is a map defined on X, then $f|_A$ will denote its restriction to A. Notice that "boundary" refers now to *topological* boundary, as opposed to "combinatorial boundary" before (in the rest of the paper it should be clear what kind of "boundary" we are referring to in each case) We will use the symbol "$\cong$" to denote "homeomorphic to". A continuous map $f : X \longrightarrow Y$ is called an *embedding* if f maps X homeomorphically onto $f(X)$; in this situation we also say that X is *embeddable* or *can be embedded* in Y. If additionally X and Y are manifolds and f maps diffeomorphically X onto $f(X)$ then we say that f is a *smooth embedding*.

We will denote by $\| \cdot \|$ the euclidean norm in $\mathbb{R}^n$, when $\mathbb{O}^n(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$, $\mathbb{D}^n(\rho) = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$, will be the corresponding open and closed balls of radius ρ. We will rewrite $\mathbb{O}^n := \mathbb{O}^n(1)$, $\mathbb{D}^n := \mathbb{D}^n(1)$ and put $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ as usual. Also, let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{P}^2 = \mathbb{S}^2/ \sim$, $\mathbb{M}^2 = (\mathbb{S}^1 \times (-1,1))/ \sim$ (with $x \sim y$ if $x = \pm y$ in both cases) and $\mathbb{B}^2 = (\mathbb{S}^1 \times [-1,1])/ \sim$ (here "$\sim$" identifies the points $(t,1)$ and $(-t,-1)$ for any $t \in \mathbb{S}^1$). We call an *open arc* (resp. an *arc*, a *circle*, a *disk*, a *closed disk*, a *sphere*, a *torus*, a *projective plane*, a *Klein bottle*, a *Möbius band*, an *annulus*, a *closed annulus*, an *n-cell* and an *n-sphere*) any space homeomorphic to an open interval (resp. a compact interval, $\mathbb{S}^1$, $\mathbb{O}^2$, $\mathbb{D}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$, $\mathbb{B}^2$, $\mathbb{M}^2$, $\mathbb{S}^1 \times (-1,1)$, $\mathbb{S}^1 \times [-1,1]$, $\mathbb{D}^n$ and $\mathbb{S}^n$). Notice that we are using "arc" and "circle" instead of the more usual, but longer terms, "simple curve" and "simple closed curve".

When denoting a circle by a greek letter, say α, we will use the symbol to simultaneously refer to the circle itself and a (fixed) homeomorphism mapping $\mathbb{S}^1$ onto the circle. For instance, if α and β are the components of the boundary of a closed annulus A and we say that they have *compatible orientations* then we mean that there is a homeomorphism $h : \mathbb{S}^1 \times [-1,1] \to A$ such that $h(t,-1) = \alpha(t)$ and $h(t,1) = \beta(t)$ (this amounts to say that both circles are either clockwise oriented or counterclockwise oriented by the homeomorphisms α and β after seeing A as a plane subset). Similarly, we say that two disjoint circles $\alpha, \beta \subset \mathbb{S}^2$ have *compatible orientations* if they have compatible orientations with regard to the closed annulus $\mathbb{S}^2 \setminus (D_\alpha \cup D_\beta)$, where D_α and D_β are the disjoint disks enclosed by α and β. If $\alpha \subset X$ is a circle then we will say that α is *orientable* (resp. *nonorientable*) if it admits a neighbourhood homeomorphic to the annulus $\mathbb{S}^1 \times (-1,1)$ (resp. the Möbius band $\mathbb{M}^2$) with the central circle being mapped by the homeomorphism onto α. It is well known that any circle in a surface S is either orientable or nonorientable (after we include among orientable circles those which are components of ∂S, as they admit a neighbourhood homeomorphic to a closed annulus). A surface admitting only orientable circles is called *orientable*; otherwise it is *nonorientable*.

And finally, we say that two circles $\alpha, \beta \in X$ *intersect transversally at x* if $x \in \alpha \cap \beta$ and there is a neighbourhood (a disk) D of x such that both $\gamma = \alpha \cap D$ and $\delta = \beta \cap D$ are open arcs, they decompose D into two disks D_1^γ, D_2^γ and D_1^δ, D_2^δ, and they are decomposed by x into two open arcs γ_1, γ_2 and δ_1, δ_2 with $\delta_i \subset D_i^\gamma$ and $\gamma_i \subset D_i^\delta$, $i = 1, 2$. If two orientable circles intersect (transversally) at exactly one point then we refer to them as two *crossing circles*.

We conclude this section stating and proving some previously cited results.

Theorem 2.1. *Any n-manifold ($n \leq 3$) admits a (unique up to diffeomorphisms) smooth structure. In particular, if M and N are homeomorphic n-manifolds ($n \leq 3$) then they are diffeomorphic.*

Proof. See for instance Example 3.1.6 from [19, p. 112] (the restrictions $\partial M = \partial N = \emptyset$ there are not essential). ∎

Lemma 2.2. *Let Φ be a flow on a metrizable space X and let $x \in X$. Assume that $O = \mathrm{Int}\,\omega_\Phi(x) \neq \emptyset$. Then O is invariant for Φ (in particular $\Phi_x(\mathbb{R}) \subset O$), O is connected and $\mathrm{Cl}\,O = \omega_\Phi(x)$.*

Proof. The orbit $\Phi_x(\mathbb{R})$ must intersect O at some point y. Since $\omega_\Phi(x)$ is invariant under Φ then $\Phi(\mathbb{R} \times O) \subset \omega_\Phi(x)$; in particular, $\Phi_x(\mathbb{R}) = \Phi_y(\mathbb{R}) \subset \omega_\Phi(x)$. Notice that $\Phi(\mathbb{R} \times O)$ is open (because it is the union of all sets $\Phi^t(O)$, which are open since Φ^t is a homeomorphism for every t), connected and dense in $\omega_\Phi(x)$ (these last two properties because it includes the orbit $\Phi_x(\mathbb{R})$). Observe that $\Phi(\mathbb{R} \times O) = O$ and then we are done. ∎

Lemma 2.3. *Let Y be a locally compact metrizable space and let $X \subset Y$ be open. Let $\Phi : \Lambda \to X$ be a local flow on X. Then there is a flow Ψ on Y for which all points from $Y \setminus X$ are singular and having the same orbits as those of Φ in X. Moreover, for any $x \in X$ the bijection $\theta_x : I_x \to \mathbb{R}$ satisfying $\Phi_x(t) = \Psi_x(\theta_x(t))$ for any t is increasing.*

Moreover, if Y is a smooth manifold and Φ is smooth on X, then the flow Ψ can be got smooth in Y.

Proof. We will first prove the lemma in its continuous version. To begin with, notice that it is not restrictive to assume that Y is connected (as if Z is an arbitrary component of Y then the open subset $X \cap Z$ of Z is a union of orbits of Φ, that is, the restriction of Φ to $\Lambda \cap (\mathbb{R} \times (X \cap Z))$ is a local flow on $X \cap Z$). Moreover, since Y is locally compact and metrizable we have then that Y has a countable base (see [11, pp. 257 (Theorem 41.4) and 261 (Exercise 10)]); hence, Y_∞ is metrizable as well [8, p. 43, Theorem 5].

Fix a distance d in Y_∞ for which there is some $y \in X$ such that $d(y, \mathrm{Bd}\,X) > 1$ (here $\mathrm{Bd}\,X$ is the boundary of X in Y_∞) and find positive numbers t_n, $n = 1, 2, \ldots$, small enough so that if $d(x, \mathrm{Bd}\,X) \geq 1/n$ then $d(\Phi(t, x), \mathrm{Bd}\,X) \geq 1/(n+1)$ for any $|t| \leq t_n$. Fix a continuous map $\Theta : X \to (0, \infty)$ satisfying $\Theta(x) \geq 1/t_n$ for any x with $d(x, \mathrm{Bd}\,X) \geq 1/(n+1)$. Now, for any $\theta \in \mathbb{R}$ and $x \in X$ there is exactly one number $t(\theta, x) := t \in I_x$ such that $\theta = \int_0^t \Theta(\Phi(s, x))\, ds$. It is simple to check that $t(\theta, x)$ is continuous and, then, that $\Psi(\theta, x) = \Phi(t(\theta, x), x)$ is a flow on X which can be continuously extended to the rest of Y_∞ (in particular to Y) by writing $\Psi(\theta, x) = x$ for any $\theta \in \mathbb{R}$ and any $x \in Y_\infty \setminus X$.

Now we generalize the ideas from [16, Lemma 2.1] to prove the smooth version of the lemma. See Y as an n-submanifold of $\mathbb{R}^m$ for $m = 2n$. We can suppose that Y is bounded in $\mathbb{R}^m$ and hence that $K = (\operatorname{Cl} Y) \setminus X$ is compact. Fix a collection $\{U_i\}_{i=1}^\infty$ of open bounded neighbourhoods of K in $\mathbb{R}^m$ such that $\operatorname{Cl} U_{i+1} \subset U_i$ for any i and $\bigcap_{i=1}^\infty U_i = K$. Find smooth maps $\lambda_i : \mathbb{R}^m \to [0,1]$ satisfying $\lambda_i \equiv 1$ on $\mathbb{R}^m \setminus U_i$ and $\lambda_i \equiv 0$ on $\operatorname{Cl} U_{i+1}$ and consider the (clearly well defined and smooth) vector fields $F_i : Y \to \mathbb{R}^m$ given by

$$F_i(x) = \begin{cases} \lambda_i(x)F(x) & \text{if } x \in X, \\ 0 & \text{otherwise,} \end{cases}$$

where $F : X \to \mathbb{R}^m$ is the smooth vector field associated to Φ.

Take a countable atlas $\{\varphi_i\}_{i=1}^\infty$, $\varphi_i : V_i \subset \mathbb{R}^n \to Y$. It is not restrictive to assume that, if $\{\epsilon_i\}_{i=1}^\infty$ are appropriately chosen small positive numbers and we write $G_i(x) = \epsilon_i F_i(x)$, then, for any j, we have that the compositions $G_j \circ \varphi_i$, $i = 1, 2, \ldots, j$, have all its partial derivatives up to the order j uniformly bounded by $1/2^j$. Then it is clear that $G = \sum_{i=1}^\infty G_i$ is a smooth vector field in Y vanishing at $Y \setminus X$ and whose associated local flow Ψ has the same orbits, with the same orientations, as those of Φ. Moreover, we can assume that the choosing of the numbers ϵ_i guarantees that $\|G(x)\| < d(\mathbb{R}^m \setminus U_i, \operatorname{Cl} U_{i+1})$ (with $d(\cdot, \cdot)$ the euclidean distance) for any $x \in X \cap (U_i \setminus U_{i+1})$. Now, it is obvious that Ψ is in fact a flow and we have finished. ∎

Lemma 2.4. *Let Φ be a flow on a locally compact metrizable space X. Then the following statements are equivalent:*

(i) *there is a point $x \in X$ whose orbit is dense in X;*

(ii) *there is a G_δ dense set of points each of them having X as both its ω-limit and its α-limit set.*

Proof. We only have to prove (i)$\Rightarrow$(ii). As X is a Baire space [11, p. 299], we just have to find open dense sets $\{O_n\}_{n=1}^\infty$ in X such that $\alpha_\Phi(y) = \omega_\Phi(y) = X$ for any $y \in \bigcap_{n=1}^\infty O_n$.

Notice that X is connected and thus has a countable base $\{U_n\}_{n=1}^\infty$ of open sets. Take $x \in X$ with $\operatorname{Cl} \Phi_x(\mathbb{R}) = X$. Fix any pair U_m and U_n of these sets. Then there are points $a = a_{m,n}$ and $z = z_{m,n}$ in $\Phi_x(\mathbb{R}) \cap U_m$ and numbers $t_a < -n$ and $t_z > n$ with $\Phi_a(t_a), \Phi_z(t_z) \in U_n$ (and then $\Phi_u(t_a), \Phi_v(t_z) \in U_n$ for any $u \in W_a$ and $v \in W_z$ for some small neighbourhoods W_a and W_z of a and z). Now it suffices to take

$$O_n = \left(\bigcup_{m=1}^\infty W_{a_{m,n}} \right) \bigcap \left(\bigcup_{m=1}^\infty W_{z_{m,n}} \right). \qquad \blacksquare$$

3. Proof of Theorem A

A few words about the proof. After some preliminary work we will show that we can restrict ourselves to the empty boundary case (Lemma 3.4). Then, to prove (iv)⇒(i) (the implications (i)⇒(ii), (ii)⇒(iii) and (iii)⇒(iv) are easy or even trivial) we will proceed as follows. First we show that there is a dense region $O \subset S$, still having two crossing circles, which can be embedded in $\mathbb{T}^2$.

Next we prove that O is homeomorphic to $\mathbb{T}^2 \setminus A$, with A being a totally disconnected set which is included in an arc of constant irrational slope.

Finally we slightly alter the corresponding irrational flow in $\mathbb{T}^2$ so that A is exactly the set of critical points of the new flow (which, in particular, has a dense orbit in $\mathbb{T}^2 \setminus A$), apply Theorem 2.1 to carry this flow to a smooth transitive flow on O, and use Lemma 2.3 to get the desired transitive flow on S.

Before going into the details we need to recall some topological results on surfaces that will be used later. Throughout the section we will use liberally, without further reference, some intuitively obvious (but deep) results on the sphere. Among them: a circle decomposes the sphere into two disks, each of them having the circle as its boundary (the *Jordan curve theorem*); any homeomorphism between two circles in a sphere can be extended to a homeomorphism on the whole sphere (the *Schönflies theorem*); if we take off the interiors of two disjoint closed disks from the sphere then we get a closed annulus (the *two-dimensional annulus theorem*); no arc separates the sphere; and so on (a good general reference is [8, Chapter 10]). In fact, the previously stated property for any circle in a surface of being either orientable or nonorientable follows elementary (if rather tediously) from these results (plus of course the fact that a surface is locally euclidean).

Recall that two closed surfaces are homeomorphic if and only if they have the same genus and Euler characteristic. In what follows M_g and N_g will denote (appropriately fixed) orientable and nonrientable surfaces of genus g. Thus $M_0 \cong \mathbb{S}^2$, $M_1 \cong \mathbb{T}^2$, $N_1 \cong \mathbb{P}^2$ and $N_2 \cong \mathbb{B}^2$ but the corresponding equalities need not happen. Recall that $\chi(M_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$, where $\chi(S)$ denotes the Euler characteristic of S (that is, $\chi(S) = F + V - E$ with F, V and E being the number of faces, vertexes and edges of an arbitrary triangulation of S).

We next describe a typical procedure to generate all posible closed (and, indeed, non-closed) surfaces. Let $\{\alpha_i, \beta_i\}_{i=1}^{i_0}$, $\{\gamma_j\}_{j=1}^{j_0}$, $0 \le i_0, j_0 \le \infty$, be countable families (notice that they may be empty and/or infinite) of circles in $\mathbb{S}^2$. Assume additionally that the following properties hold ($\Sigma = \{\alpha_i, \beta_i\}_{i=1}^{i_0} \cup \{\gamma_j\}_{j=1}^{j_0}$):

(i) the circles $\sigma \in \Sigma$ are pairwise disjoint and enclose pairwise disjoint disks D_σ;

(ii) the circles α_i, β_i have compatible orientations for any i;

(iii) $d(D_\sigma, \bigcup_{\sigma' \neq \sigma} D_{\sigma'}) > 0$;

(iv) if $i_0 = \infty$ then $\lim_{i \to \infty} d(D_{\alpha_i}, D_{\beta_i}) = 0$;

(v) if $i_0 = \infty$ (resp. $j_0 = \infty$) then $\lim_{i \to \infty} \operatorname{diam} D_{\alpha_i} = \lim_{i \to \infty} \operatorname{diam} D_{\beta_i} = 0$ (resp. $\lim_{j \to \infty} \operatorname{diam} D_{\gamma_j} = 0$).

Define in $\mathbb{S}^2 \setminus \bigcup_{\sigma \in \Sigma} D_\sigma$ the following equivalence relation: $x \sim y$ if and only if either:

- $x = y$;

- $x = \alpha_i(t)$, $y = \beta_i(t)$ (or $y = \alpha_i(t)$, $x = \beta_i(t)$) for some $t \in \mathbb{S}^1$ and some i;

- $x = \gamma_j(t)$, $y = \gamma_j(-t)$ for some $t \in \mathbb{S}^1$ and some j.

We call the compact, connected (indeed even metrizable by the Nagata-Smirnov theorem) quotient space $\mathbb{S}^2_\Sigma := \mathbb{S}^2 / \sim$ a *pseudosurface*. Notice if $[A] = [A]_\Sigma$ denotes the set of equivalence classes under $\sim$ including points from a set $A \subset \mathbb{S}^2$, then $[\sigma]$ is a circle in $\mathbb{S}^2_\Sigma$ for any $\sigma \in \Sigma$, which is orientable in the case $\sigma = \alpha_i$ or $\sigma = \beta_i$, and nonorientable in the case $\sigma = \gamma_j$. It turns out that if $i(\Sigma) := i_0$ and $j(\Sigma) := j_0$ are both finite then $\mathbb{S}^2_\Sigma$ is a closed surface, with $\mathbb{S}^2_\Sigma \cong M_{i(\Sigma)}$ in the case $j(\Sigma) = 0$ and $\mathbb{S}^2_\Sigma \cong N_{2i(\Sigma)+j(\Sigma)}$ if $j(\Sigma) > 0$.

A nice thing about pseudosurfaces is that non-closed surfaces can also be embedded into them [14]:

Theorem 3.1. *Let S be a connected surface, $\partial S = 0$. Then there is a embedding $e : S \to \mathbb{S}^2_\Sigma$ for some collection of circles Σ such that $\mathbb{S}^2_\Sigma \setminus e(S)$ is totally disconnected (that is, each of its components consists of one point) and intersects none of the circles $[\sigma]$, $\sigma \in \Sigma$. Moreover, for any $\epsilon > 0$ the maximum distance from the points of the circles $[\sigma]$ to $\mathbb{S}^2_\Sigma \setminus e(S)$ is less than ϵ (except at most for finite number of circles).*

The next result is essentially proved e.g. in [6, Lemma 2.4]. We will only need to use it in some particular cases, but for completeness we state it in its full generality. Recall that a circle $C \subset X$ is *null homotopic* if there is a continuous map $H : [0,1] \times [0,1] \to X$ satisfying $H([0,1] \times \{0\}) = C$ and $H(0,s) = H(1,s) = H(t,1) = x$ for some $x \in C$ and every $t, s \in [0,1]$.

Lemma 3.2. *Let S be a closed surface, let $\alpha \subset S$ be a circle and let g be the genus of S.*

(i) *If α is nonorientable (thus $S \cong N_g$) then either $S \setminus \alpha \cong M_{(g-1)/2} \setminus D$ or $S \setminus \alpha \cong N_{g-1} \setminus D$ for any closed disk D in the corresponding surface.*

(ii) *If α is orientable and $S \setminus \alpha$ is connected then $S \setminus \alpha \cong M_{g-1} \setminus (D_1 \cup D_2)$ (if $S \cong M_g$), and $S \setminus \alpha \cong M_{(g-2)/2} \setminus (D_1 \cup D_2)$ or $S \setminus \alpha \cong N_{g-2} \setminus (D_1 \cup D_2)$ (if $S \cong N_g$) for any pairwise disjoint closed disks D_1 and D_2 in the corresponding surface.*

(iii) *If α is orientable and non-null homotopic, and $S \setminus \alpha = O_1 \cup O_2$ for some pairwise disjoint open sets O_1 and O_2, then there are positive integers g_1, g_2 such that $g_1 + g_2 = g$ with $O_i \cong M_{g_i} \setminus D_i$, $i = 1, 2$ (if $S \cong M_g$), and such that $2g_1 + g_2 = g$ with $O_1 \cong M_{g_1} \setminus D_1$, $O_2 \cong N_{g_2} \setminus D_2$, or $g_1 + g_2 = g$ with $O_i \cong N_{g_i} \setminus D_i$, $i = 1, 2$ (if $S \cong N_g$), for any closed disks D_i, $i = 1, 2$, in the corresponding surfaces.*

Moreover, if in cases (ii) and (iii) $\alpha \subset U \subset S$ is a closed annulus which is decomposed by α into two closed annuli U_i, $i = 1, 2$, then the restriction of the corresponding homeomorphism f with domain $S \setminus \alpha$ to $U_i \setminus \alpha$ can be extended (after renaming if necessary the closed disks D_i) to a homeomorphism $f_i : U_i \to f(U_i \setminus \alpha) \cup \mathrm{Bd}\, D_i$, $i = 1, 2$.

Remark 3.3. To complete Lemma 3.2 let us emphasize that if α is a null homotopic circle in S then there is a closed disk $D \in S$ with $\mathrm{Bd}\, D = \alpha$ (in particular, any null homotopic circle is orientable).

After this preparatory work we are ready to prove Theorem A. Let us first emphasize that:

Lemma 3.4. *Theorem A holds for every surface if and only if it holds for all surfaces with empty boundary.*

Proof. It suffices to prove that, under the assumption that Theorem A holds true in the empty boundary case, a surface S satisfies (i) (resp. (ii), (iii) or (iv)) in the statement of Theorem A if and only if $R = S \setminus \partial S$ satisfies (i) (resp. (ii), (iii) or (iv)). Indeed, for properties (i) and (ii) our claim follows from Lemma 2.3, and for property (iv) from the fact that if S has two crossing circles then they can be easily modified to get two crossing circles in R (using for instance that, according to the *collaring theorem* [5, pp. 113–114] , there is an embedding $e : \partial S \times [0, 1) \to S$ with $e(x, 0) = x$ for any $x \in \partial S$).

We must be a bit more careful regarding property (iii). Clearly, it suffices to show that if R is embeddable in $\mathbb{B}^2$ (resp. in $\mathbb{S}^2$) then S is embeddable in $\mathbb{B}^2$ (resp. in $\mathbb{S}^2$) as well.

To do this we use again the collaring theorem to embed S in a surface with empty boundary T (which is orientable if S is orientable) so that $T \backslash S \cong \partial S \times (0,1)$. Now it is simple to check that if α, β are two crossing circles in T then they both must intersect R and hence can be modified to get two crossing circles in R. Since R cannot have two crossing circles (recall that we are assuming that Theorem A holds true in the empty boundary case), T cannot have them either and we can apply Theorem A to the surface without boundary T to conclude that it is embeddable in $\mathbb{B}^2$ (resp. in $\mathbb{S}^2$). Hence S is embeddable in $\mathbb{B}^2$ (resp. in $\mathbb{S}^2$) as we desired to show. ∎

Thus we will prove Theorem A under the additional assumption $\partial S = \emptyset$. The statement (i)$\Rightarrow$(ii) is trivial, and (ii)$\Rightarrow$(iii) follows from the fact that all ω-limit sets for flows in $\mathbb{S}^2$, $\mathbb{P}^2$ or $\mathbb{B}^2$ have empty interior (see e.g. [17]).

We next prove (iii)$\Rightarrow$(iv) by showing that if S has no crossing circles then it is homeomorphic to either $\mathbb{S}^2$, $\mathbb{P}^2$ or a region in $\mathbb{B}^2$ (or, if S is orientable, to a region in $\mathbb{S}^2$).

Consider the embedding $e : S \rightarrow \mathbb{S}^2_\Sigma$ from Theorem 3.1. We claim that $i(\Sigma) = 0$ and $j(\Sigma) \leq 2$.

Assume $i(\Sigma) \neq 0$. Then for some totally disconnected closed set $C \subset \mathbb{S}^2 \setminus (\bigcup_{\sigma \in \Sigma} \mathrm{Cl}\, D_\sigma)$ we have that $S \cong \mathbb{S}^2_\Sigma \setminus [C]_\Sigma$. Moreover, since C is totally disconnected and then cannot separate $\mathbb{S}^2$ (use e.g. [8, p. 189 and p. 539, Th. 5]), there is an arc $\delta \subset \mathbb{S}^2$ having each of its endpoints in α_1 and β_1, respectively, and intersecting no other points from $C \cup \bigcup_{\sigma \in \Sigma} \mathrm{Cl}\, D_\sigma$. Since α_1 and β_1 have compatible orientations, $[\delta]_\Sigma$ and $[\alpha_1]_\Sigma = [\beta_1]_\Sigma$ are crossing circles in $\mathbb{S}^2_\Sigma \setminus [C]_\Sigma$. Thus S has two crossing circles, a contradiction.

Now we show $j(\Sigma) \leq 2$ (notice that if additionally S is orientable then we automatically get $j(\Sigma) = 0$ and S is embeddable in $\mathbb{S}^2$ as required). Again we argue by contradiction to find in S three pairwise disjoint nonorientable circles, say δ, ϕ_1 and ϕ_2 (see Figure 1). Starting from an orientable circle ζ enclosing δ it is easy to find two nonorientable circles δ_1 and δ_2 close to δ (hence disjoint from ϕ_1 and ϕ_2) intersecting transversally at exactly one point $x \in \delta$. Further, there are pairwise disjoint closed disks K_1 and K_2 in S such that $K_l \cap \delta_l = \mu_l$ and $K_l \cap \phi_l = \rho_l$ are arcs in $\mathrm{Bd}\, K_l$, $l = 1, 2$. Say $\mathrm{Bd}\, K_l = \mu_l \cup \tau_{l,1} \cup \rho_l \cup \tau_{l,2}$ for appropriate (minimal) arcs $\tau_{l,1}, \tau_{l,2}$. Then $(\delta_l \setminus \mu_l) \cup \tau_{l,1} \cup (\phi_l \setminus \rho_l) \cup \tau_{l,2}$, $l = 1, 2$, are orientable circles intersecting transversally at x, a contradiction.

Thus, $i(\Sigma) = 0$ and $j(\Sigma) \leq 2$. This means that S is embeddable in $\mathbb{S}^2$, $\mathbb{P}^2$ or $\mathbb{B}^2$. Moreover, if it is not homeomorphic to any of the first two surfaces then, as any proper region of $\mathbb{S}^2$ or $\mathbb{P}^2$ is embeddable in $\mathbb{B}^2$ (cf. Lemma 3.2(i)), it would be embeddable in $\mathbb{B}^2$ as well. The statement (iii)$\Rightarrow$(iv) is proved.

It only rests to show (iv)$\Rightarrow$(i). As indicated at the beginning of the section, the proof consists of three steps.
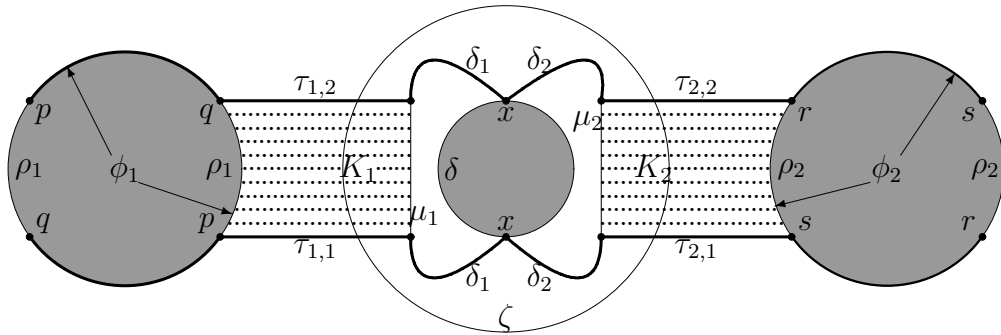
Figure 1: Proof of $j(\Sigma) \leq 2$. Here δ, ϕ_1 and ϕ_2 are nonorientable Jordan curves, when each point is identified with its opposite as marked for x, p, q, r and s.

STEP 1: *Finding the region O.*

As in the proof of (iii)⇒(iv) we retort to the embedding $e : S \rightarrow \mathbb{S}^2_\Sigma$ when, according to the reasoning there, we must have either $i(\Sigma) \neq 0$ or $j(\Sigma) \geq 3$. If we are in the case $i(\Sigma) \neq 0$ then it suffices to take off from S all the circles which are mapped by e onto the circles $[\alpha_i] = [\beta_i]$ ($i \neq 1$) and $[\gamma_j]$ to get the desired set O (notice that with the notation we used there, $e^{-1}([\delta])$ and $e^{-1}([\alpha_1])$, are crossing circles in O; moreover, O is trivially embeddable in $\mathbb{S}^2_{\Sigma'}$, $\Sigma' = \{\alpha_1, \beta_1\}$, which is homeomorphic to $\mathbb{T}^2$.

The case $j(\Sigma) \geq 3$ requires more work. We already know how to proceed to find a dense region $O' \subset S$ embeddable in N_3 and having two crossing circles. We next show how to take off a nonorientable circle from O' so than the resultant region O (which is still dense in S) still has two crossing circles but can now be embedded in $\mathbb{T}^2$.

First we need to find an appropriate description for N_3. Let $M_2 \subset \mathbb{R}^3$ be chosen in such a way that $x \in M_2$ implies $-x \in M_2$ as well. Let "$\approx$" denote the equivalence relation in M_2 identifying x and $-x$ for all $x \in M_2$. Then $N_3 \cong M_2/ \approx$ and we can assume in fact $O' \subset N_3 = M_2/ \approx$. [Indeed it is clear that $M_2/ \approx$ is a closed surface. To prove that it is nonorientable let $\pi : M_2 \rightarrow M_2/ \approx$ denote the quotient map, fix points $x, -x \in M_2$ and construct an arc $\alpha \subset W$ having x and $-x$ as its endpoints. It is not restrictive to assume that α contains no other pair of symmetrical points (otherwise take an open arc $\beta \subset \alpha$ small enough having no such pairs, include it in a maximal open arc $\gamma \subset \alpha$ with the same property and replace α by the closure of γ). Clearly, there is a homeomorphism $h : [-1, 1] \times (-1, 1) \rightarrow U \subset M_2$ such that $h([-1, 1] \times \{0\}) = \alpha$ and $h(-1, t) = -h(1, -t)$ for any $t \in [-1, 1]$. Then $\pi(\alpha)$ is a circle in $M_2/ \approx$ having $\pi(U)$ (a Möbius band) as a neighbourhood and $\pi(\alpha)$ is nonorientable. Finally, $\chi(M_2/ \approx) = \frac{\chi(M_2)}{2} = -1 = \chi(N_3)$ is easy to prove by constructing a triangulation of M_2 such that if K is any of its faces, edges or vertexes then $-K = \{-x : x \in K\}$ is also a face, an edge or a vertex of the triangulation.]

It turns out that $\pi^{-1}(O')$ is connected. This follows from the fact that if $V \subset N_g$ is an arbitrary region and U is one of the components of $\pi^{-1}(V)$ then $\pi|_U$ is surjective, and hence we have that either $\pi^{-1}(V)$ is connected or $\pi^{-1}(V)$ consists of exactly two components, homeomorphic to V, and symmetrical each other: in the particular case $O' = V$, the nonorientability implies that the first possibility holds. [To prove the surjectivity of $\pi|_U$ argue by contradiction to find a boundary point b of $\pi(U)$ in V (because V is connected). Notice that, since $\pi(U)$ is open (because π is an open map), $b \notin \pi(U)$. Find a sequence $(y_n)_n$ of points of $\pi(V)$ converging to b and take points x_n in U with $\pi(x_n) = y_n$; it is not restrictive to assume that $(x_n)_n$ converges, say to a. Since $\pi(a) = b$, $a \notin U$. Now it suffices to take a very small connected neighbourhood U_a of a and note that $\pi(U \cup U_a) \subset V$, contradicting the definition of U.]

Now it is easy to prove that if $\sigma, \tau \subset O'$ are crossing circles then $\pi^{-1}(\sigma) = \sigma_1 \cup \sigma_2$ and $\pi^{-1}(\tau) = \tau_1 \cup \tau_2$, where σ_l and τ_l are crossing circles, $l = 1, 2$, and $(\sigma_1 \cup \tau_1) \cap (\sigma_2 \cup \tau_2) = \emptyset$. Clearly, $\pi^{-1}(O') \setminus (\sigma_1 \cup \tau_1 \cup \sigma_2 \cup \tau_2)$ is then connected, so we can construct in it a (minimal) arc δ connecting two symmetrical points $x, -x \in O'$. Now $\alpha = \pi(\delta)$ is a nonorientable circle and $\sigma, \tau \subset O := O' \setminus \alpha$. According to Lemma 3.2(i), O is embeddable either in $\mathbb{B}^2$ or in $\mathbb{T}^2$. The existence in O of crossing circles excludes the first possibility.

STEP 2: *Constructing the homeomorphism to* $\mathbb{T}^2 \setminus A$.

Apply Theorem 3.1 to the surface O to find the corresponding embedding $e'' : O \to \mathbb{S}^2_{\Sigma''}$. Since O is orientable we have $j(\Sigma'') = 0$, since O has two crossing circles we have $i(\Sigma'') \geq 1$, and since O is embeddable in the torus we have $i(\Sigma'') \leq 1$ (otherwise O would have two pairs of pairwise disjoint crossing circles, and since each of these circles is orientable and non-null homotopic —cf. Remark 3.3— we would apply Lemma 3.2(ii) to arrive a contradiction). Thus $\mathbb{S}^2_{\Sigma''} \cong \mathbb{T}^2$ and we can assume $O \subset \mathbb{T}^2$ and that $C = \mathbb{T}^2 \setminus O$ is totally disconnected.

Denote by α and β the crossing circles in O. Fix an annular neighbourhood $B \subset O$ of β intersecting α at exactly a small open arc γ. Use Lemma 3.2(ii) to find a homeomorphism $f : \mathbb{T}^2 \setminus \alpha \to \mathbb{S}^1 \times (-1, 1)$ such that, if U is a closed annulus which is decomposed by α into two closed annuli U_l, $l = -1, 1$, then the restriction of f to $U_l \setminus \alpha$ can be extended to a homeomorphism $f_l : U_l \to f(U_l \setminus \alpha) \cup (\mathbb{S}^1 \times \{l\})$, $l = -1, 1$. Notice that the circles $\mathbb{S}^1 \times \{l\} = f_l \circ \alpha =: \alpha_l$ have compatible orientations in $\mathbb{S}^1 \times [-1, 1]$ (because the disk $f(B \setminus \gamma)$ connects them).

Take into account that, because of [8, p. 539, Th. 5], any totally disconnected compact subset of $\mathbb{S}^2$ is included into some arc, that is, a homeomorphic set to a compact interval. Further, for any two given arcs in $\mathbb{S}^2$ there is a homeomorphism of the sphere mapping one of them onto the

other one [8, p. 535, Th. 1]. Combining these facts it is easy to construct a circle μ including $f(C)$ and separating $\mathbb{S}^1 \times [-1, 1]$ into two closed annuli: let us denote them by P_{-1} and P_1. We can assume that α_l and μ have compatible orientations in P_l, $l = -1, 1$. Then we can construct a homeomorphism p from $\mathbb{S}^1 \times [-1, 1]$ into itself such that $p(t, l) = \alpha_l(t)$, $l = -1, 1$, and $p(t, 0) = \mu(t)$ for any $t \in \mathbb{S}^1$.

Fix now in $\mathbb{S}^1 \times (-1, 1)$ an arc δ of constant (say ρ) irrational slope, include it in a circle ϕ separating $\mathbb{S}^1 \times [-1, 1]$ into closed annuli Q_l, $l = -1, 1$, and, as before, find a homeomorphism q from $\mathbb{S}^1 \times [-1, 1]$ into itself leaving its boundary invariant and mapping $\mathbb{S}^1 \times \{0\}$ onto ϕ. Clearly, it is not restrictive to assume that $r = q \circ p^{-1}$ maps $f(C)$ into δ. Notice also that $(r \circ f_l \circ \alpha)(t) = (r \circ \alpha_l)(t) = (t, l)$ for any $t \in \mathbb{S}^1$ and $l = -1, 1$. Let $A = (r \circ f)(C)$. Then it is clear that O is homeomorphic to $\mathbb{T}^2 \setminus A$ (now we are seeing $\mathbb{S}^1 \times (-1, 1)$ as a subset of $\mathbb{T}^2$).

STEP 3: *Finishing the proof.*

Consider the irrational flow of slope ρ in $\mathbb{T}^2$, take a diffeomorphic copy $M_1 \subset \mathbb{R}^3$ of $\mathbb{T}^2$ and carry this flow to a smooth flow in M_1 with associated vector field $F : M_1 \to \mathbb{R}^3$. Multiply F by a smooth nonnegative function $\lambda : M_1 \to [0, \infty)$ such that the resulting vector field has an associated flow which, after being carried to $\mathbb{T}^2$, has A as its set of singular points. Use Theorem 2.1 to construct a smooth transitive flow on O and extend it to a smooth flow Φ on the whole S by means of Lemma 2.3. This concludes the proof of (iii)$\Rightarrow$(iv) and Theorem A.

4. Proof of Theorem B

The idea of the proof is fairly simple. The "bricks" of our construction are compact cylinders $C \subset M$ ("blocks") with pairwise disjoint interiors, which will be useful to parametrize in order to fix their tops $U(C)$, their bottoms $R(C)$, and their open fibres ("vertical" curves in $\operatorname{Int} C$ connecting one point from $U(C)$ with another one from $R(C)$). Starting from a "basement" B, we successively add "flats" (blocks whose bottom is included in the top of a previous flat or the basement) and "stairs" (blocks whose bottom is also included in the top of some flat or the basement and now with its top included in the bottom of the basement) to get an "infinite tower" dense in M. This will be done in such a way that the open fibres of the blocks are (not necessarily maximal) orbits of an appropriate flow and, moreover, one of the orbits of this flow (a countable union of open fibres from all the blocks and points from the corresponding tops and bottoms) is dense in M.

Throughout the proof we will assume, without loss of generality (cf. Lemma 2.3), that M has empty boundary.

Recall that a set $C \subset M$ is called an *n-cell* if it is homeomorphic to the closed ball $\mathbb{D}^n$. If additionally $\theta : [-1,1] \times \mathbb{D}^{n-1} \to C$ is a homeomorphism then we will call the pair (C, θ) a *parametrized n-cell*. We will write $R(C) = R(C, \theta) = \theta(\{-1\} \times \mathbb{D}^{n-1})$ and $U(C) = U(C, \theta) = \theta(\{1\} \times \mathbb{D}^{n-1})$, and the arcs $\theta([-1,1] \times \{z\})$ ($z \in \mathbb{D}^{n-1}$) will be called the *closed fibres* of C (with respect to the *parametrization θ*). If in these last definitions we replace $\mathbb{D}^{n-1}$ by $\mathbb{O}^{n-1}$ and $[-1,1]$ by $(-1,1)$ then we get $R_O(C) = R_O(C, \theta)$, $U_O(C) = U_O(C, \theta)$ and the *open fibres* of C (with respect to the *parametrization θ*).

Let $\{(B, \beta)\} \cup \{(F_i, \phi_i)\}_{i=1}^r \cup \{(S_j, \sigma_j)\}_{j=1}^s$ be a finite collection of parametrized n-cells in M. We call this collection (which, if no confusion arises, we will identify with the union set $T = B \cup \bigcup_{i=1}^r F_i \cup \bigcup_{j=1}^s S_j$) a *tower* with *basement* $B = (B, \beta)$, *flats* $F_i = (F_i, \phi_i)$ and *stairs* $S_j = (S_j, \sigma_j)$ (we will generically refer to B, F_i and S_j as the *blocks* of T), provided that, for any block C, there is a nonnegative integer $l(C)$ (the *level* of C) such that the following properties hold:

(i) $l(B) = 0$ and $l(C) > 0$ for any other block C;

(ii) if F is a flat of level l then there is a flat F^* of level $l-1$ such that $F \cap F^* = R(F) \subset U_O(F^*)$ (if $l = 1$ then F^* means the basement B); moreover, $F \cap C = \emptyset$ for any other block C with $l(C) \leq l$;

(iii) if S is a stair of level l then $S \cap B = U(S) \subset R_O(B)$ and there is a flat F^* of level $l-1$ such that $S \cap F^* = R(S) \subset U_O(F^*)$ (if $l = 1$ then we mean $F^* = B$ and $S \cap F^* = R(S) \cup U(S)$ with $U(S) \subset R_O(B)$ and $R(S) \subset U_O(B)$); moreover, $S \cap C = \emptyset$ for any other block C.

Assume additionally that there is a number $0 < \epsilon < 1$ such that for any block (C, θ) of T there is a continuous (or, if M is smooth, a smooth) embedding extending θ, $e(\theta) : (-1-\epsilon, 1+\epsilon) \times \mathbb{O}^{n-1}(1+\epsilon) \to M$, and put $e(C) := e(\theta)((-1-\epsilon, 1+\epsilon) \times \mathbb{O}^{n-1}(1+\epsilon))$. Moreover, assume that:

(iv) if C and C' are disjoint blocks then $e(C)$ and $e(C')$ are disjoint as well;

(v) if $F = (F, \phi)$ and $F^* = (F^*, \phi^*)$ are as in (ii) then $e(\phi)(\{-1\} \times \mathbb{O}^{n-1}(1+\epsilon)) \subset U_O(F^*)$ and $e(\phi)(t, z) = e(\phi^*)(t+2, z^*)$ for any $t \in (-1-\epsilon, -1+\epsilon)$ whenever z and z^* are such that $e(\phi)(-1, z) = e(\phi^*)(1, z^*)$;

(vi) if $S = (S, \sigma)$ and $F^* = (F^*, \phi^*)$ are as in (iii) then $e(\sigma)(\{-1\} \times \mathbb{O}^{n-1}(1+\epsilon)) \subset U_O(F^*)$ and $e(\sigma)(t, z) = e(\phi^*)(t+2, z^*)$ for any $t \in (-1-\epsilon, -1+\epsilon)$ whenever z and z^* are such that $e(\sigma)(-1, z) = e(\phi^*)(1, z^*)$; also, $e(\sigma)(\{1\} \times \mathbb{O}^{n-1}(1+\epsilon)) \subset R_O(B)$ and $e(\sigma)(t, w) = e(\beta)(t-2, w^*)$ for any $t \in (1-\epsilon, 1+\epsilon)$ whenever w and w^* are such that $e(\sigma)(1, w) = e(\beta)(-1, w^*)$.

Then we will call T a *regularizable tower* and also write $e(T) = e(B) \cup \bigcup_{i=1}^{r} e(F_i) \cup \bigcup_{j=1}^{s} e(S_j)$. The *fibres* of T are the maximal path-connected subsets of Int T consisting of a countable union of open or closed fibres of any of the blocks which the towers T_m are made of. See Figure 2.
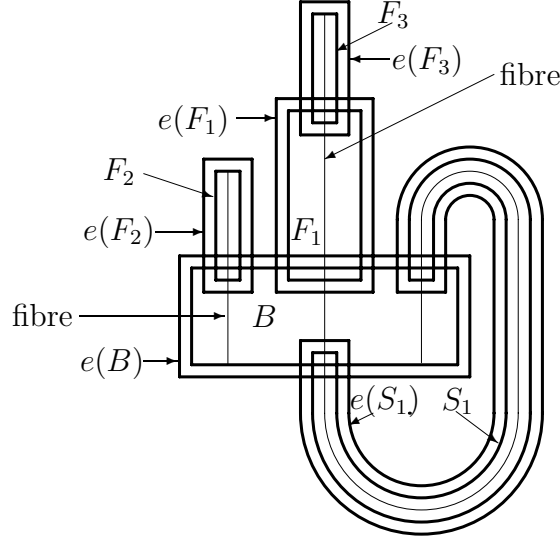

Figure 2: Regularizable tower with basement B, flats F_1, F_2 and F_3 and stair S_1 for which $l(B) = 0$, $l(F_1) = l(F_2) = l(S_1) = 1$ and $l(F_3) = 2$.

Finally, we call T an *infinite regularizable tower* if $T = \bigcup_{m=0}^{\infty} T_m$, where all T_m are regularizable towers such that $T_0 = \{(B, \beta)\}$ and, for any $m \geq 1$, there is a block (C_m, θ_m) of T_m such that $T_m = T_{m-1} \cup \{(C_m, \theta_m)\}$. The *fibres* of T are similarly defined as before.

We intend to construct an infinite regularizable tower T dense in M containing a dense fibre. The hardest part of the process consists of describing how to add to a given regularizable tower a new block containing (in an appropriate place) a prescribed point outside the initial tower. Next proposition pursues this aim:

Proposition 4.1. *Let $T \subset M$ be a regularizable tower (we keep the notation above). Let $x \in U_O(F)$ for some flat F of T of level $l \geq 0$ (if $l = 0$ then we mean $F = B$) and assume that $x \notin C$ for any other block C of T.*

(a) *Let $y \in M \setminus T$. Then there is a regularizable tower $T^* = T \cup \{(F^*, \phi^*)\}$ for which F^* is a flat of level $l + 1$ and such that $x = \phi^*(-1, \mathbf{0})$, $y = \phi^*(1, \mathbf{0})$.*

(b) *Let $y \in R_O(B)$. Then there is a regularizable tower $T^* = T \cup \{(S^*, \sigma^*)\}$ for which S^* is a stair of level $l + 1$ and such that $x = \sigma^*(-1, \mathbf{0})$, $y = \sigma^*(1, \mathbf{0})$.*

We will divide the proof into two steps. First we will assume that M is just a topological manifold; then we will deal with the smooth case.

Proof of Proposition 4.1 (the continuous case). It is easy to show that $e(T) \setminus T$ is connected (here we need of course $n \geq 3$), and since $e(T)$ is an open neighbourhood of T in M we get that $M \setminus T$ is connected as well. Then we can use that T is regularizable to easily construct an arc A having x and y as its endpoints so that $A \setminus \{x, y\} \subset M \setminus T$. Due to the compactness of A, there is a number $\delta > 0$ small enough such that if $A' \subset A$ is an arc whose diameter is less than δ then there is an open set W in M, homeomorphic to $\mathbb{R}^n$, such that $A' \subset W$.

We will begin by assuming that diam $A < \delta$, fixing an open set W homeomorphic to $\mathbb{R}^n$ including A, and proving Part (a) of the proposition.

Construct small disjoint n-cells C_x and C_y in W containing respectively x and y in their boundaries B_x and B_y. Indeed, we will take $C_x = e(\phi)([1, 1 + \mu] \times D)$ for a small number $\mu < \epsilon$ and a small closed ball $D \subset \mathbb{O}^{n-1}$ and assume that $e(\phi)(1, a) = x$ for the centre a of D. Notice that we can also assume that both B_x and B_y are *regularizable* n-spheres, that is, there are open neighbourhoods O_x and O_y of B_x and B_y and homeomorphisms h_x, h_y from $(-1, 1) \times \mathbb{S}^n$ onto O_x and O_y such that $h_x(\{0\} \times \mathbb{S}^n) = B_x$, $h_y(\{0\} \times \mathbb{S}^n) = B_y$. If $e(\phi)(1 + \mu, a) = x^*$, we can finally suppose that there is an arc $A^* \subset W$ connecting x^* and y and containing no other point from $C_x \cup C_y \cup T$.

Let W_∞ denote the one-point compactification of W. As both B_x and B_y are regularizable, the annulus theorem [10, 7, 13] guarantees the existence of a homeomorphism $h : [-1, 1] \times \mathbb{S}^n \to W_\infty \setminus \text{Int}(C_x \cup C_y)$ mapping $\{-1\} \times \mathbb{S}^n$ onto B_x, and $\{1\} \times \mathbb{S}^n$ onto B_y. It is then rutinary to construct closed $(n-1)$-cells K_{x^*} and K_y in B_x and B_y (with indeed $K_{x^*} = e(\phi)(\{1 + \mu\} \times E)$ for some smaller closed ball $E \subset D$ also centred in a), and a parametrized n-cell (F^+, ϕ^+) in $W \setminus \text{Int}(C_x \cup C_y)$, very close to A^* (hence not intersecting T), satisfying $F^+ \cap B_x = R(F^+) = K_{x^*}$, $F^+ \cap B_y = U(F^+) = K_y$, and such that $\phi^+(-1, \mathbf{0}) = x^*$, $\phi^+(1, \mathbf{0}) = y$.

The next step is to parametrize $F^{++} = F^+ \cup e(\phi)([1, 1 + \mu] \times E)$ with a map ϕ^{++} such that $R(F^{++}) = K_x := e(\phi)(\{1\} \times E)$ (and still $U(F^{++}) = K_y$) and with the additional property that if $u \in K_{x^*}$ and $e(\phi)(1 + \mu, z) = u$ for some $z \in E$ then there is a $z^{++} \in \mathbb{D}^{n-1}$ such that $\phi^{++}(t, z^{++}) = e(\phi)(2 + t, z)$ for any $t \in [-1, -1 + \mu]$. Certainly we can also assume $\phi^{++}(-1, \mathbf{0}) = x$, $\phi^{++}(1, \mathbf{0}) = y$.

Finally we just write $F^* = \phi^{++}([-1, 1] \times \mathbb{D}^{n-1}(\nu))$ for some $\nu < 1$ and define $\phi^*(t, z) = \phi^{++}(t, z/\nu)$. Obviously $T^* = T \cup \{(F^*, \phi^*)\}$ is regularizable (notice that we can use the same extensions for the blocks from T just restricting them for an appropriate $\epsilon^+ < \epsilon$).

This ends the proof of Part (a) of the proposition under the additional assumption that $\operatorname{diam} A < \delta$. The proof of Part (b) has no substantial novelties; one just has to proceed similarly with both x and y, using the basement B for y as we previously did with F and x.

In the general case, choose points $x = x_0, x_1, \ldots, x_k = y$ decomposing A into arcs $A_1, \ldots, A_k$ with diameters less than δ. Reason as in the proof of Part (a) above to construct a parametrized n-cell (F_1^*, ϕ_1^*) with $x_0 = \phi_1^*(-1, \mathbf{0})$, $x_1 = \phi_1^*(1, \mathbf{0})$ and such that $T_1 = T \cup \{(F_1^*, \phi_1^*)\}$ is a regularizable tower for which F_1^* is a flat of level $l + 1$. Notice that the construction allows us to additionally assume that $T_1 \cap (A_3 \cup \cdots \cup A_k) = \emptyset$. Further (replacing if necessary A_2 for a very close arc with the same endpoints and not intersecting the closed fibre in F_1^* containing both x_0 and x_1, and then taking a thinner F_1^*), it is not restrictive to suppose that $(A_2 \setminus \{x_1\}) \cap T_1 = \emptyset$. Thus we proceed similarly as before to extend T_1 via a flat (F_2^*, ϕ_2^*) of level $l + 2$ with $x_1 = \phi_2^*(-1, \mathbf{0})$, $x_2 = \phi_2^*(1, \mathbf{0})$ to a regularizable tower $T_2 = T_1 \cup \{(F_2^*, \phi_2^*)\}$.

Reasoning in this way we can construct in M a regularizable tower $T \cup \{(F_i^*, \phi_i^*)\}_{i=1}^k$ (if we are proving Part (a)) or $T \cup \{(F_i^*, \phi_i^*)\}_{i=1}^{k-1} \cup \{(S^*, \sigma^*)\}$ (if we are proving Part (b)) such that $x_{i-1} = \phi_i^*(-1, \mathbf{0})$, $x_i = \phi_i^*(1, \mathbf{0})$ for any i (when we mean $\phi_k^* = \sigma^*$ in Part (b)). Now, if we take a very small ν and define $E_1 = \mathbb{D}^{n-1}(\nu)$ and, inductively, E_i for any $1 < i \leq k$ by the property $\phi_i^*(\{-1\} \times E_i) = \phi_{i-1}^*(\{1\} \times E_{i-1})$, it is clear that $\bigcup_{i=1}^k \phi_i^*(I \times E_i)$ is an appropriate choice for respectively F^* or S^* (the corresponding ϕ^* or σ^* can be easily defined). ∎

Proof of Proposition 4.1 (the smooth case). The basic ideas are analogous to those previously used (and we will maintain the notation above); only, in the part of the proof when $\operatorname{diam} A < \delta$ we have to proceed more carefully in order to guarantee differentiability. We next show how to do it (for Part (a) as before).

Notice first that if $W \subset M$ is the homeomorphic set to $\mathbb{R}^n$ such that $A \subset W$ we can assume, without loss of generality, that it is diffeomorphic to $\mathbb{R}^n$ as well.

Recall that $\lambda := e(\theta) : (1 - \epsilon, 1 + \epsilon) \times \mathbb{O}^{n-1}(1 + \epsilon) \to M$ is now a smooth embedding and take as before $C_x = \lambda([1, 1 + \mu] \times D)$ for a small number $\mu < \epsilon$ and a small closed ball $D \subset \mathbb{O}^{n-1}$ with $\lambda(1, a) = x$ for the centre a of D. Just to manage a symmetrical notation and stress the similarity of the argument for Part (b), we can assume that there is a smooth embedding $\gamma : (1 - \epsilon, 1 + \epsilon) \times \mathbb{O}^{n-1}(1 + \epsilon) \to M$ such that $C_y := \gamma([-1 - \mu, -1] \times K)$ for a small closed ball $K \subset \mathbb{O}^{n-1}$ whose centre b satisfies $\gamma(-1, b) = y$. We will assume that the radius of both D and K is μ, that $\lambda([1 - \mu, 1 + \mu] \times D)$ and

$\gamma([-1-\mu,-1+\mu]\times K)$ are disjoint and included in W and that $\gamma([-1-\mu,-1+\mu]\times K)$ does not intersect T. In what follows we will identify W to $\mathbb{R}^n$ (thus $T' := T \cap W$ will be seen as a subset of $\mathbb{R}^n$); as the ensuing argument will make clear, it is not restrictive to do it so.

Consider the curves $\alpha : [1-\mu,1+\mu] \to \mathbb{R}^n$ and $\eta : [-1-\mu,-1+\mu] \to \mathbb{R}^n$ respectively defined by $\alpha(t) = \lambda(t,a)$ and $\eta(t) = \gamma(t,b)$. Notice that, since λ and γ are diffeomorphisms, we have $\alpha'(t) \neq \mathbf{0}$ and $\eta'(t) \neq \mathbf{0}$ for any t. Now it is simple to connect them in a smooth way, that is, there is an injective smooth map $\kappa : [-1-\mu,1+\mu] \to \mathbb{R}^n$ satisfying $\kappa'(t) \neq \mathbf{0}$ for any t, $\kappa(t) = \alpha(t+2)$ for any $t \in [-1-\mu,-1+\mu]$, and $\kappa(t) = \eta(t-2)$ for any $t \in [1-\mu,1+\mu]$. We can additionally assume that $\kappa([-1+\mu,1-\mu]) \subset \mathbb{R}^n\setminus T'$ and that

(4.1)
$$\kappa([-1+\mu,1-\mu]) \cap \lambda([1-\mu,1+\mu/2]\times D) = \emptyset,$$
$$\kappa([-1+\mu,1-\mu]) \cap \gamma([-1-\mu/2,-1+\mu]\times K) = \emptyset.$$

From [5, p. 69, Proposition 1.2] $\kappa'(t)/\|\kappa'(t)\|$ cannot map the interval $[-1-\mu,1+\mu]$ onto $\mathbb{S}^n$, that is, there is a point $p \in \mathbb{S}^n$ such that $\kappa'(t)/\|\kappa'(t)\| \neq p$ for any t. Fix a diffeomorphism $g : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1} \setminus \{p\}$, and define $G_r : \mathbb{S}^{n-1} \setminus \{p\} \to \mathbb{R}^n$, $1 \leq r \leq n-1$, by

$$G_r(q) = \frac{\partial g}{\partial z_r}(g^{-1}(q)),$$

where z_r denotes the r-th component in $\mathbb{R}^{n-1}$. As all vectors $G_r(\kappa'(t)/\|\kappa'(t)\|)$ are ortogonal to $\kappa'(t)$, $\{\kappa'(t), G_1(\kappa'(t)/\|\kappa'(t)\|), \ldots, G_{n-1}(\kappa'(t)/\|\kappa'(t)\|)\}$ is a base for any t and we can assume, provided that $\mu^* < \mu$ is small enough, that $\xi : (-1-\mu,1+\mu) \times \mathbb{O}^{n-1}(\mu^*) \to \mathbb{R}^n$ given by

$$\xi(t,z_1,\ldots,z_{n-1}) = \kappa(t) + z_1 G_1(\kappa'(t)/\|\kappa'(t)\|) + \cdots + z_{n-1}G_{n-1}(\kappa'(t)/\|\kappa'(t)\|)$$

is an embedding and
(4.2)
$$\xi([-1+\mu/4,1-\mu/4]\times \mathbb{D}^{n-1}(\mu^*/2)) \cap \lambda([1-\mu,1+\mu/8]\times D) = \emptyset,$$
$$\xi([-1+\mu/4,1-\mu/4]\times \mathbb{D}^{n-1}(\mu^*/2)) \cap \gamma([-1-\mu/8,-1+\mu]\times K) = \emptyset.$$

Let $\rho_{c,d,s} : \mathbb{R}^n \to \mathbb{R}$, $d > c > 0$, $s > 0$, be a nonnegative smooth map satisfying $\rho_{c,d,s}(t,z) = 1$ for any $(t,z) \in [-c,c] \times \mathbb{D}^{n-1}(s/2)$ and vanishing exactly at $\mathbb{R}^n \setminus (-d,d) \times \mathbb{O}^{n-1}(s)$, and construct in $\mathbb{R}^n$ the vector fields defined by

$$X(\lambda(t,z)) = \rho_{\mu/4,\mu/2,\mu}(t-1,z-a)\frac{\partial\lambda}{\partial t}(t,z),$$

$$Y(\gamma(t,z)) = \rho_{\mu/4,\mu/2,\mu}(t+1,z-b)\frac{\partial\gamma}{\partial t}(t,z),$$

$$Z(\xi(t,z)) = \rho_{1-\mu/2,1-\mu/4,\mu^*/2}(t,z)\frac{\partial\xi}{\partial t}(t,z),$$

whenever the expressions above make sense and zero elsewhere. Clearly, they all are well defined and smooth. Moreover, let $H = X + Y + Z$. As H vanishes outside a bounded subset of $\mathbb{R}^n$, its associated flow Φ is well defined as well. Notice that, because of (4.1), the arc $\kappa([-1 - \mu, 1 + \mu])$ is included in one of the orbits of Φ, and, because of (4.2), we have

$$
(4.3) \qquad
\begin{aligned}
\Phi(t, \lambda(1, z)) &= \lambda(t + 1, z), \quad |t| \leq \mu/8, \ z \in D, \\
\Phi(t, \gamma(-1, z)) &= \gamma(t - 1, z), \quad |t| \leq \mu/8, \ z \in K.
\end{aligned}
$$

Say $\Phi(t_x, x) = y$ (thus $\Phi_x([0, t_x]) = \kappa([-1, 1])$) and define $\varphi : \mathbb{R} \times \operatorname{Int} D \to \mathbb{R}^n$ by $\varphi(t, z) = \Phi(t, \lambda(1, z))$. We claim that $d\varphi(t, z)$ is regular (its determinant does not vanish) for any (t, z). Indeed

$$
\frac{\partial \varphi}{\partial z_r}(t, z) = d\Phi_t(\lambda(1, z)) \frac{\partial \lambda}{\partial z_r}(1, z),
$$

$1 \leq r \leq n - 1$, and

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t}(t, z) &= \lim_{h \to 0} \frac{\Phi_t(\Phi(h, \lambda(1, z))) - \Phi_t(\Phi(0, \lambda(1, z)))}{h} \\
&= \frac{\partial}{\partial h} \Phi_t(\Phi(h, \lambda(1, z))) \bigg|_{h=0} \\
&= d\Phi_t(\lambda(1, z)) \frac{\partial \varphi}{\partial t}(0, z) = d\Phi_t(\lambda(1, z)) \frac{\partial \Phi}{\partial t}(0, \lambda(1, z)) \\
&= d\Phi_t(\lambda(1, z)) H(\lambda(1, z)) = d\Phi_t(\lambda(1, z)) \frac{\partial \lambda}{\partial t}(1, z).
\end{aligned}
$$

The claim follows from the fact that both λ and Φ_t are diffeomorphisms.

Let $\mu^{**} > 0$ be so small a number that $\gamma^{-1} \circ \varphi$ is well defined on $(t_x - \mu^{**}, t_x + \mu^{**}) \times O^{**}$, with O^{**} the open disk of radius μ^{**} centred in a, and let τ denote the first component of γ^{-1}. Then

$$
\begin{aligned}
\frac{\partial(\tau \circ \varphi)}{\partial t}(t_x, a) &= \nabla\tau(t_x, a) \cdot \frac{\partial \varphi}{\partial t}(t_x, a) \\
&= \nabla\tau(t_x, a) \cdot H(\Phi(t_x, \lambda(1, a))) \\
&= \nabla\tau(y) \cdot H(y) \\
&= \nabla\tau(y) \cdot \frac{\partial \gamma}{\partial t}(-1, b) \\
&= 1
\end{aligned}
$$

and (provided that μ^{**} is small enough) we can apply the implicit function theorem to find a smooth map $t = t(z)$ satisfying $t(a) = t_x$ and $\varphi(t(z), z) \in \gamma(-1, K)$ for any $z \in O^{**}$.

Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be nondecreasing smooth maps satisfying, respectively:

$$f_1(t) = t, \; t \in \left(-\infty, \frac{\mu}{8}\right]; \; f_1(t) = \frac{\mu}{4}, \; t \in \left[\frac{\mu}{4}, \infty\right); \; f_1'(t) > 0, \; t \in \left(-\infty, \frac{\mu}{4}\right);$$

$$f_2(t) = 0, \; t \in (-\infty, 0]; \quad f_2(t) = 2 - \frac{\mu}{2}, \; t \in [1, \infty); \quad f_2'(t) > 0, \; t \in (0, 1),$$

and define

$$f(t, z) = f_1(t) + f_2\left(\frac{t - \mu/8}{t(z) - \mu/4}\right) + \mu/4 - f_1(t(z) - t).$$

As $\varphi(t, z)$ is a local diffeomorphism, it is rutinary to check (again if μ^{**} is sufficiently small) that the map $\varphi^* : (-\mu/8, 2 + \mu/8) \times O^{**} \to \mathbb{R}^n$ given by

$$\varphi^*(f(t, z), z) = \varphi(t, z)$$

is a well defined diffeomorphism, satisfying for any $z \in O^{**}$

$$\varphi^*(2, z) \in \gamma(\{-1\} \times K)$$

and (due to (4.3))

$$\varphi^*(t, z) = \lambda(t - 1, z), \quad |t| \leq \mu/8,$$
$$\varphi^*(t + 2, z) = \gamma(t - 1, w) \quad \text{(if } z \text{ and } w \text{ are such that } \varphi(2, z) = \gamma(-1, w)),$$
$$|t| \leq \mu/8.$$

Now we define $\phi^*(t, z) = \varphi^*(t + 1, 2(z - a)/\mu^{**})$ and $F^* = \phi^*([-1, 1], \mathbb{D}^{n-1})$ and we are done.　∎

Proof of Theorem B. Using Proposition 4.1 we can construct an infinite regularizable tower T in M with the additional property that one of its fibres, ϱ^*, is dense in M. More precisely, fix a dense sequence of points $\{u_m\}_{m=0}^{\infty}$ in M. Then it suffices to construct inductively (see the next paragraph) the corresponding regularizable towers T_m with common basement B and union the desired T so that all fibres of all T_m are open arcs (thus their closures are arcs), and there are fibres ϱ_m of T_m such that, for any m,

(*) neither of the endpoints of $\mathrm{Cl}\, \varrho_m$ belong to the intersection of two blocks of T_m,

(**) $\varrho_m \subset \varrho_{m+1}$,

(***) $d(\varrho_m, u_m) < 1/2^m$,

because then $\varrho^* := \bigcup_{m=0}^{\infty} \varrho_m$ would do the job.

It is worth emphasizing that if α is an arbitrary fibre of some T_m then one of the endpoints belongs to $R_O(B)$ and the other one belongs to $U_O(F)$ for some block (a flat or the basement) F of T_m, and neither of them belong to another $R_O(C)$ or $U_O(C)$, with C any block of T_m. Thus (*) means that if $z_m \in R_O(B)$ and $x_m \in U_O(F_m)$ are the endpoints of $\mathrm{Cl}\,\varrho_m$ then they do not belong to another $R(C)$ or $U(C)$.

The choosing of T_0 is trivial, as it is that of T_{m+1} in the case when $u_{m+1} \notin T_m$: just use Proposition 4.1(a) to add a flat (F,ϕ) to T_m so that $\phi(-1,\mathbf{0}) = x_m$ and $\phi(1,\mathbf{0}) = u_{m+1}$, when $\varrho_{m+1} := \varrho_m \cup \phi([-1,1) \times \{\mathbf{0}\})$. If $u_{m+1} \in T_m$ then we find a fibre α of T_m very close to u_{m+1} with endpoints $z \in R_O(B)$ and $x \in U_O(F)$ so that neither of them belong to any other $R(C)$ or $U(C)$. The way to do this is first choosing a fibre α_1 of T_m just close enough to u_{m+1}, then (if necessary) a fibre α_2 with its endpoint in $R_O(B)$ very close to that of α_1 so that it belongs to $U(S)$ for no stair S of T_m, and finally (if necessary) the fibre α with both of its endpoints very close to those of α_2 and satisfying the required property. Then we apply Proposition 4.1(b) to add a stair (S,σ) to T_m satisfying $\sigma(-1,\mathbf{0}) = x_m$ and $\sigma(1,\mathbf{0}) = z$, being $\varrho_{m+1} := \varrho_m \cup \alpha \cup \sigma([-1,1] \times \{\mathbf{0}\})$. Observe that we must be careful in choosing S thin enough so that all fibres of T_m starting from $U(S)$ end in $U_O(F)$, and all fibres of T_m ending at $R(S)$ start from points of $R_O(B)$ outside $U(S)$. In this way, we guarantee that all fibres from T_{m+1} are open arcs.

Notice that if ϱ is a fibre of T and $u \in \varrho$ then there are an open interval $I_u \ni 0$ and a bijection $\varrho_u : I_u \to \varrho$ (both unambiguously defined) such that $\varrho_u(0) = u$ and having the following property: for any $v \in \varrho \cap e(C)$ for some block (C,ϕ) in T, $v = \varrho_u(t^*) = e(\phi)(t^{**}, z)$, there is a small number $\epsilon_v > 0$ such that $\varrho_u(t+t^*) = e(\phi)(t+t^{**}, z)$ for all $|t| < \epsilon_v$. It is not difficult to realize that $\Phi(t,u) := \varrho_u(t)$ is a well defined local flow on M (which is differentiable in the smooth case as, once we see M as embedded in $\mathbb{R}^{2n}$ and put with the notation above $F(v) := \frac{\partial e(\phi)}{\partial t}(t^{**}, z)$, it is clear that the map $F : O \to \mathbb{R}^{2n}$ so constructed is a well defined smooth vector field on M whose associated local flow is precisely Φ). Apply Lemma 2.3 to finish the proof. ∎

Remark 4.2. It must be stressed that, in a sense, the proof of Theorem B can be shortened as one could dispose of Part (b) of Proposition 4.1. Indeed, using just Part (a) of the proposition it is not hard to prove that $\mathbb{R}^n$ can be densely embedded in M, and now we only need to use a transitive flow in $\mathbb{R}^n$ as those of [15], [1] or [16].

Yet we think that our approach is preferable. On the one hand, it provides a self-contained proof of the transitivity of $\mathbb{R}^n$ which is rather simpler than those of [1] or [16]. Moreover, Part (b) generalizes (and thus brings

into attention to the western readership) the beautiful ideas in Sidorov's paper [15]. Finally, when constructing a transitive flow one should ideally tried to devise it with a number of singular points as small as possible, as then one could have a better chance to derive a minimal flow from it: Part (b), up to some extent, "reduces" this number. In this context it is worth recalling that the very important problem of proving or disproving that $\mathbb{S}^3$ admits a minimal flow still remains open.

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