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Graphs associated with nilpotent Lie algebras of maximal rank

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Abstract

In this paper, we use the graphs as a tool to study nilpotent Lie algebras. It implies to set up a link between graph theory and Lie theory. To do this, it is already known that every nilpotent Lie algebra of maximal rank is associated with a generalized Cartan matrix A and it is isomorphic to a quotient of the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra $\mathfrak{g}(A)$. Then, if A is affine, we can associate $\mathfrak{n}_+$ with a directed graph (from now on, we use the term digraph) and we can also associate a subgraph of this digraph with every isomorphism class of nilpotent Lie algebras of maximal rank and of type A. Finally, we show an algorithm which obtains these subgraphs and also groups them in isomorphism classes.

1. Introduction

Let $\mathfrak{L}$ be a finite-dimensional nilpotent Lie algebra, $\mathrm{Der}\,\mathfrak{L}$ its derivation algebra, $\mathrm{Aut}\,\mathfrak{L}$ its automorphism group.

Definition 1 *A* torus *on $\mathfrak{L}$ is a commutative subalgebra of* $\mathrm{Der}\mathfrak{L}$ *whose elements are semi-simple. A torus is said to be* maximal *if it is not contained in any other torus.*

Definition 2 *The* rank *of $\mathfrak{L}$ is the common dimension of all maximal tori on $\mathfrak{L}$. We say that $\mathfrak{L}$ is* of maximal rank *if its rank r is equal to the dimension ℓ of $\mathfrak{L}/[\mathfrak{L},\mathfrak{L}]$ (in general, the rank r is less than ℓ).*

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If $\mathfrak{L}$ is a nilpotent Lie algebra and T is a maximal torus on $\mathfrak{L}$, the elements of T can be diagonalized simultaneously, then $\mathfrak{L}$ is decomposed in a direct sum of root spaces for T:

$$\mathfrak{L} = \bigoplus_{\beta \in T^*} \mathfrak{L}^\beta ,$$

where T^* is the dual of the vector space T and

$$\mathfrak{L}^\beta = \{X \in \mathfrak{L} \,:\, tX = \beta(t)X, \, \forall t \in T\}.$$

Definition 3 *The set* $R(T) = \{\beta \in T^* \,:\, \mathfrak{L}^\beta \neq \{0\}\}$ *is called* root system *of* $\mathfrak{L}$ *associated with* T *(Favre calls root set to* $R(T)$ *in [1]).*

Definition 4 *A minimal system* $\{X_1, X_2, \ldots, X_\ell\}$ *of generators of* $\mathfrak{L}$ *verifying that for each* X_i *there exists* $\beta_i \in R(T)$ *such that* $X_i \in \mathfrak{L}^{\beta_i}$ *is called* T*-msg.*

We can obtain a T-msg for each torus T on a nilpotent Lie algebra $\mathfrak{L}$ (see 2.6 of [5]).

By 2.10 of [5], if $\mathfrak{L}$ is a nilpotent Lie algebra of maximal rank ℓ, T is a torus on $\mathfrak{L}$ and $\{X_1, X_2, \ldots, X_\ell\}$ is a T-msg with roots $\beta_1, \beta_2, \ldots, \beta_\ell$, then the set $\{\beta_1, \beta_2, \ldots, \beta_\ell\}$ is a basis of T^* and for each $\beta \in R(T)$ there exists $(d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$, non null, unique such that $\beta = \sum_{i=1}^{\ell} d_i \beta_i$ and we call $|\beta| = \sum_{i=1}^{\ell} d_i$ the *height* of β.

Definition 5 *We call* generalized Cartan matrix *a matrix* $A = (a_{ij})_{i,j=1}^{\ell}$ *with entries in* $\mathbb{Z}$ *satisfying:*

1. $a_{ij} = 2$, $\forall i = 1, \ldots, \ell$
2. $a_{ij} \leq 0$, $\forall i, j = 1, \ldots, \ell$, $i \neq j$
3. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, $\forall i, j = 1, \ldots, \ell$

The indecomposable generalized Cartan matrices are classified in 3 types: finite, affine and indefinite.

In [5] a generalized Cartan matrix $A = (a_{i,j})_{i,j=1}^{\ell}$ is associated to every nilpotent Lie algebra $\mathfrak{L}$ of maximal rank ℓ and thus, one says that $\mathfrak{L}$ is *of type* A. This matrix is built considering maximal tori on $\mathfrak{L}$ and applying the nilpotency of $\mathfrak{L}$ on T-msg.

Then, by using generalized Cartan matrices we link the nilpotent Lie algebras of maximal rank with Kac-Moody algebras. Indeed, we have that every nilpotent Lie algebra $\mathfrak{L}$ of maximal rank and of type A is a quotient of the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra $\mathfrak{g}(A)$ associated with A.

For a general overview on Kac-Moody Lie algebras and Graph Theory the reader can see [3] and [4], respectively.

2. Associated graphs

Definition 6 *Let $\mathfrak{L}$ be a nilpotent Lie algebra of maximal rank ℓ, T a maximal torus on $\mathfrak{L}$, $R(T)$ the root system associated with T and $\{X_1, X_2, \ldots, X_\ell\}$ a T-msg with corresponding roots $\beta_1, \beta_2, \ldots, \beta_\ell$. We define the following digraph:*

- *the set of vertices $V(G_{\mathfrak{L}})$ is $R(T)$.*
- *we draw a directed edge from γ to μ if there exists β_i such that $\mu = \gamma + \beta_i$, where μ, $\gamma \in R(T)$.*

Theorem 7 *The digraph above mentioned is unique, up to isomorphism, for every nilpotent Lie algebra $\mathfrak{L}$ of maximal rank. We will denote it by $G_{\mathfrak{L}}$.*

Proof. Let $\mathfrak{L}$ a nilpotent Lie algebra of maximal rank ℓ.

1. If T and T' are two maximal tori, there exists $\theta \in \mathrm{Aut}\,(\mathfrak{L})$ such that $\theta T \theta^{-1} = T'$. Let $\{X_1, X_2, \ldots, X_\ell\}$ be a T-msg with corresponding roots $\beta_1, \beta_2, \ldots, \beta_\ell$. Then $\{\theta(X_1), \theta(X_2), \ldots, \theta(X_\ell)\}$ is a T'-msg with corresponding roots $\beta'_1, \beta'_2, \ldots, \beta'_\ell$ which are defining by
$$\beta'_i(t') = \beta_i(\theta^{-1} t' \theta),\ \forall t' \in T',\ i = 1, \ldots, \ell\,.$$
We have the following bijection from $R(T)$ on $R(T')$:
if $\beta \in R(T)$, then there exists an unique $(d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$ such that $\beta = \sum_{i=1}^{\ell} d_i \beta_i$ and we define $f(\beta) = \sum_{i=1}^{\ell} d_i \beta'_i$. Moreover, we have that $\theta(\mathfrak{L}^\beta) = \mathfrak{L}^{f(\beta)}$, $\forall \beta \in T^*$. Due to this bijection, the digraphs obtained from the torus T and the T-msg $\{X_1, X_2, \ldots, X_\ell\}$ and from the torus T' and the T'-msg $\{\theta(X_1), \theta(X_2), \ldots, \theta(X_\ell)\}$ are isomorphic.
2. Let T be a torus on $\mathfrak{L}$ and let $\{X_1, X_2, \ldots, X_\ell\}$ and $\{Y_1, Y_2, \ldots, Y_\ell\}$ be two T-msg with corresponding roots $\beta_1, \beta_2, \ldots, \beta_\ell$ and $\gamma_1, \gamma_2, \ldots, \gamma_\ell$, respectively. By 3.1 of [5], there exist a unique $\sigma \in \mathfrak{S}_\ell$ and $\lambda_1, \ldots, \lambda_\ell \in K - \{0\}$ such that $Y_i = \lambda_i X_{\sigma i}$ and $\gamma_i = \beta_{\sigma i}$, $i = 1, \ldots, \ell$. Then the digraphs obtained from T and the T-msg $\{X_1, X_2, \ldots, X_\ell\}$ and $\{Y_1, Y_2, \ldots, Y_\ell\}$, respectively, are isomorphic. ■

Example 1 We consider the following Lie algebra:
$$\mathfrak{M} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_9$$
with brackets
$$[e_1, e_2] = e_5, \quad [e_1, e_4] = e_6, \quad [e_2, e_3] = e_7,$$
$$[e_3, e_4] = e_8, \quad [e_2, e_8] = e_9, \quad [e_4, e_7] = -e_9.$$

This Lie algebra is nilpotent Lie algebra of maximal rank $\ell = 4$. A maximal torus on $\mathfrak{M}$ is:

$$T = \mathbb{C}t_1 \oplus \cdots \oplus \mathbb{C}t_4 \text{ with } t_i(e_j) = \delta_{ij}\, e_j,\ i, j = 1, \ldots, 4$$

The set $\{e_1, e_2, e_3, e_4\}$ is a T-msg with roots $\beta_1, \beta_2, \beta_3, \beta_4$. The root system associated with T is

$$R(T) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_1 + \beta_2, \beta_1 + \beta_4, \beta_2 + \beta_3, \beta_3 + \beta_4, \beta_2 + \beta_3 + \beta_4\}$$

Then we have the following root space decomposition:

$$\mathfrak{M} = \bigoplus_{\beta \in R(T)} \mathfrak{M}^\beta$$

where

$\mathfrak{M}^{\beta_i} = \mathbb{C}e_i$ for $i = 1, \ldots, 4$,
$\mathfrak{M}^{\beta_1+\beta_2} = \mathbb{C}e_5$, $\mathfrak{M}^{\beta_1+\beta_4} = \mathbb{C}e_6$, $\mathfrak{M}^{\beta_2+\beta_3} = \mathbb{C}e_7$, $\mathfrak{M}^{\beta_3+\beta_4} = \mathbb{C}e_8$,
$\mathfrak{M}^{\beta_2+\beta_3+\beta_4} = \mathbb{C}e_9$.

The corresponding associated digraph $G_{\mathfrak{M}}$ is:

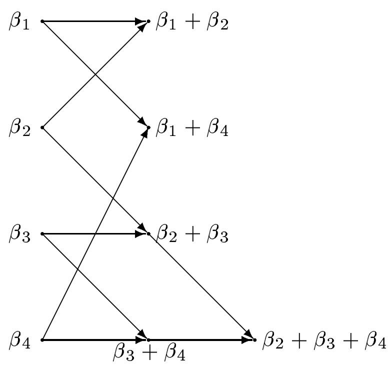


FIGURE 1: The digraph $G_{\mathfrak{M}}$.

and the generalized Cartan matrix associated with $\mathfrak{M}$ is:

$$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

which is the generalized Cartan matrix of affine type $A_3^{(1)}$ (see Chapter 4 of [3]).

Remark 1 *In the digraph $G_{\mathfrak{M}}$, we have drawn in each vertical line all the vertices corresponding with roots of $R(T)$ with the same height. The height is augmented from left to right in 1. If we have drawn a directed edge from γ to μ, then the height of μ is $|\mu| = |\gamma| + 1$. We will follow this indications to draw digraphs from now on.*

By the above remark, we draw the digraph $G_{\mathfrak{M}}$ in a more simplified form, substituting directed edges by edges (since each edge is directed from a root of height h to roots of height $h+1$):

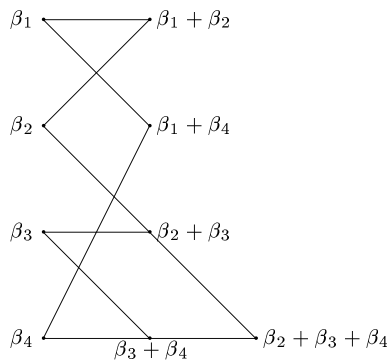


FIGURE 2: The digraph $G_{\mathfrak{M}}$ (simplified form).

Then, let consider a generalized Cartan matrix A of affine type. Let Δ_+ be the set of positive roots, $\mathfrak{g}_\alpha$ the root subspace associated with $\alpha \in \Delta_+$ and $\alpha_1, \alpha_2 \ldots, \alpha_l$ the simple roots of the Kac-Moody algebra $\mathfrak{g}(A)$. If $\alpha \in \Delta_+$ there exist $d_1, \ldots, d_\ell$ such that $\alpha = \sum d_i \alpha_i$ and we call $|\alpha| = \sum d_i$ the *height* of α.

We have the following decomposition for the positive part $\mathfrak{n}_+$ of $\mathfrak{g}(A)$:

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

Definition 8 *We define the following digraph G_A, associated with $\mathfrak{n}_+$:*

- *the set of vertices $V(G_A)$ is $\{0\} \cup \Delta_+$.*
- *we draw a directed edge from γ to μ if there exists α_i such that $\mu = \gamma + \alpha_i$, where μ, $\gamma \in \{0\} \cup \Delta_+$.*

Due to the properties of the positive root system Δ_+ when A is affine, the following lemma is verified:

Lemma 9 *If A is a generalized Cartan matrix of affine type, then the digraph G_A is infinite, it has a countable infinite set $\{n\xi \,/\, n \geq 1\}$ of cut points and there are countably infinite many finite subgraphs G_n $(n \geq 0)$ of G_A such that:*

1. $V(G_{n-1}) \cap V(G_n) = \{n\xi\}$.
2. $V(G_n) = \{\alpha + n\xi \,/\, \alpha \in V(G_0)\}$.

Proof. If A is a generalized Cartan matrix of affine type, the positive root system of the Kac-Moody algebra $\mathfrak{g}(A)$ has the following structure:

there exist $r \in \{1, 2, 3\}$ and $\delta \in \Delta_+$ such that $\Delta_+ \cup \{0\} = \cup_{j \geq 0} \Delta_j$ where

$$\begin{aligned} \Delta_0 &= \{0\} \cup \{\gamma \in \Delta_+ \,/\, |\gamma| < |r\delta|\} \cup \{r\delta\}, \\ \Delta_j &= \{jr\delta + \gamma \,/\, \gamma \in \Delta_0\} \ \text{ if } j \geq 1. \end{aligned}$$

Since $V(G_A) = \Delta_+ \cup \{0\}$, we have that G_A is a infinite digraph. The vertices $n\xi = nr\delta$ are cut points in the digraph G_A because $nr\delta$ is the unique root of Δ_+ with height $nr|\delta|$. G_n is the subgraph of G_A whose set of vertices is Δ_n for $n \geq 0$ and, obviously, these subgraphs verify the properties 1 and 2. ■

As a consequence of this lemma and the results related to the root systems, see [3], the digraph G_A has the following structure:

$$G_0 \vee G_1 \vee \cdots \vee G_n \vee G_{n+1} \vee \cdots = \bigvee_{n \geq 0} G_n$$

where we identify the vertices $n\xi$ of G_{n-1} and G_n for all $n \geq 1$.

Example 2 We consider the following generalized Cartan matrix of affine type:

$$A_3^{(1)} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

The set Δ_+ of positive roots of the Kac-Moody algebra $\mathfrak{g}(A_3^{(1)})$ verifies that there exists $\delta \in \Delta_+$ $(r = 1)$ such that $\Delta_+ \cup \{0\} = \cup_{j \geq 0} \Delta_j$ where

$$\begin{aligned} \Delta_0 = \{\, & 0, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_0 + \alpha_1, \alpha_5 = \alpha_0 + \alpha_3, \alpha_6 = \alpha_1 + \alpha_2, \alpha_7 = \alpha_2 + \alpha_3, \\ & \alpha_8 = \alpha_0 + \alpha_1 + \alpha_2, \alpha_9 = \alpha_0 + \alpha_1 + \alpha_3, \alpha_{10} = \alpha_0 + \alpha_2 + \alpha_3, \\ & \alpha_{11} = \alpha_1 + \alpha_2 + \alpha_3, \delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \} \\ \Delta_j = \{\, & j\delta + \gamma \,/\, \gamma \in \Delta_0\} \ \text{ if } j \geq 1 \end{aligned}$$

with $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ the simple roots.

We obtain the following digraph associated with $A_3^{(1)}$:

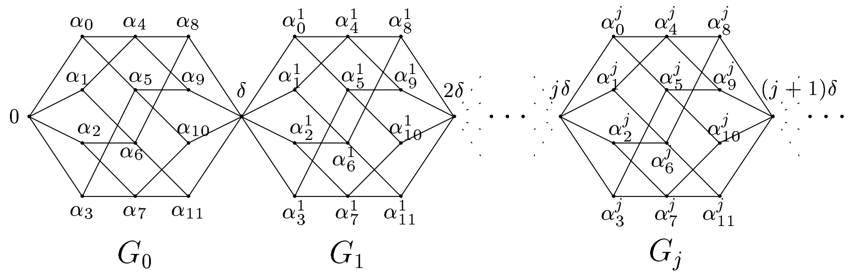


Figure 3: The digraph $G_{A_3^{(1)}}$.

Remark 2 *In drawing the above digraph we have followed the same indications as in figure 2. Moreover we have used the notation α_i^j for the root $j\delta + \alpha_i \in \Delta_j$, where $\alpha_i \in \Delta_0$.*

Theorem 10 *If $\mathfrak{L}$ is a nilpotent Lie algebra of maximal rank and of type A, then the corresponding associated digraph $G_{\mathfrak{L}}$ is isomorphic to a subgraph $G'_{\mathfrak{L}}$ of G_A.*

Proof. Let $A = (a_{ij})_{i,j=1}^{\ell}$ be a generalized Cartan matrix of affine type, $\mathfrak{L}$ be a nilpotent Lie algebra of maximal rank and of type A and $\mathfrak{n}_+$ be the positive part of the Kac-Moody algebra associated with A and we define the following ideal $\mathfrak{n}_{++}$ of $\mathfrak{n}_+$:

$$\mathfrak{n}_+ = \Big(\bigoplus_{\substack{1 \le i \ne j \le \ell \\ 0 \le k \le -a_{ji}}} \mathfrak{g}_{\alpha_i + k\alpha_j} \Big) \oplus \mathfrak{n}_{++}.$$

By 5.10 of [5], there exists an homogeneous ideal $\mathfrak{a}$ of $\mathfrak{n}_+$ included in $\mathfrak{n}_{++}$ such that $\mathfrak{L}$ and $\mathfrak{n}_+/\mathfrak{a}$ are isomorphic.

$\mathfrak{n}_+/\mathfrak{a}$ is a nilpotent Lie algebra of maximal rank and the digraph associated with this algebra, $G_{\mathfrak{n}_+/\mathfrak{a}}$, can be obtained, by 4.9 of [5], from the torus $T = \oplus_{i=1}^{\ell} K t_i$, where $t_i \in \mathrm{Der}(\mathfrak{n}_+/\mathfrak{a})$ is defined by $t_i E_j = \delta_{ij} E_j$, $i = 1, \ldots, \ell$, with $E_1, \ldots, E_\ell$ the Chevalley generators of $\mathfrak{n}_+$. Furthermore $\{E_1, E_2, \ldots, E_\ell\}$ is a T-msg whose corresponding roots can be identified to $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. Then the set $\{\alpha_1, \ldots, \alpha_\ell\}$ is a basis for T^* and $R(T) \subset \Delta_+$.

Therefore we have that $G_{\mathfrak{n}_+/\mathfrak{a}}$ is a subgraph of G_A and, since $\mathfrak{L}$ and $\mathfrak{n}_+/\mathfrak{a}$ are isomorphic, $G_{\mathfrak{L}}$ and $G'_{\mathfrak{L}} = G_{\mathfrak{n}_+/\mathfrak{a}}$ are isomorphic. ■

This digraph $G'_{\mathfrak{L}}$ is a subgraph of G_A which is obtained from the nilpotent Lie algebra $\mathfrak{n}_+/\mathfrak{a}$, where $\mathfrak{a}$ is an homogeneous ideal of $\mathfrak{n}_+$. Thus we have that $\mathfrak{a} = \oplus_{\alpha \in \Delta_+} \mathfrak{a}_\alpha$ where $\mathfrak{a}_\alpha = \mathfrak{a} \cap \mathfrak{g}_\alpha$. Then there exists $j \ge 0$ such that $jr\delta \notin \{\alpha \in \Delta_+ \,/\, \mathfrak{a}_\alpha \ne \emptyset\}$ and $nr\delta \in \{\alpha \in \Delta_+ \,/\, \mathfrak{a}_\alpha \ne \emptyset\}$, $\forall n \ge j+1$.

Therefore $G'_{\mathfrak{L}}$ verifies that there exists $j \geq 0$ such that $j\xi \in V(G'_{\mathfrak{L}})$ and $n\xi \notin V(G'_{\mathfrak{L}}), \forall n \geq j+1$ and thus, we can consider that $G'_{\mathfrak{L}}$ is a subgraph of the finite digraph

$$G_0 \vee G_1 \vee \cdots \vee G_j = \bigvee_{n=0}^{j} G_n \,.$$

Let the digraph $\tilde{G}_{\mathfrak{L},j}$ be the subgraph of $\bigvee_{n=0}^{j} G_n$ whose set of vertices $V(\tilde{G}_{\mathfrak{L},j})$ is $V(G_A) - V(G_{\mathfrak{L}})$ and whose edges are all edges in $\bigvee_{n=0}^{j} G_n$ which connect two vertices in $V(\tilde{G}_{\mathfrak{L},j})$. This digraph $\tilde{G}_{\mathfrak{L},j}$ is a subgraph of the digraph G_j which verifies the following properties:

1. $(j+1)\delta \in V(\tilde{G}_{\mathfrak{L},j})$ and $\delta \notin V(\tilde{G}_{\mathfrak{L},j})$
2. if $\alpha \in V(\tilde{G}_{\mathfrak{L},j})$ and $\alpha+\alpha_i \in V(G_j)$, then $\alpha+\alpha_i \in V(\tilde{G}_{\mathfrak{L},j})$ and the edge from α to $\alpha+\alpha_i$ belongs to $\tilde{G}_{\mathfrak{L},j}$.

Theorem 11 *To classify nilpotent Lie algebras of maximal rank and of type A it is necessary to compute (up to isomorphism), as a first step, all the subgraphs of G_j verifying the properties* 1 *and* 2 *above mentioned, for each $j \geq 0$.*

Proof. Let $j \geq 0$ and G be a subgraph of G_j verifying the properties:

1. $(j+1)\delta \in V(G)$ and $\delta \notin V(G)$
2. if $\alpha \in V(G)$ and $\alpha+\alpha_i \in V(G_j)$, then $\alpha+\alpha_i \in V(G)$ and the edge from α to $\alpha+\alpha_i$ belongs to G.

Then there exists a nilpotent Lie algebra $\mathfrak{L}$ such that the digraph

$$G_0 \vee G_1 \vee \cdots \vee G_{j-1} \vee \tilde{G},$$

where $\tilde{G}$ is the subgraph of G_j whose set of vertices $V(\tilde{G})$ is $V(G_j) - V(G)$ and whose edges are all edges in G_j which connect two vertices in $V(\tilde{G})$, is isomorphic to $G_{\mathfrak{L}}$.

The second step to classify nilpotent Lie algebras of maximal rank and of type A will be to obtain all nilpotent Lie algebras of maximal rank (up to isomorphism) whose digraph associated is $G_0 \vee G_1 \vee \cdots \vee G_{j-1} \vee \tilde{G}$, for each subgraph $\tilde{G}$ of G_j obtained in the first step, for each $j \geq 0$. ■

This last step is not dealt in this paper due to reasons of length. For a more general overview on this result the reader can see [2].

Moreover, as $V(G_j) = \{\alpha + j\xi \,/\, \alpha \in V(G_0)\}$, we have a bijection between the set of subgraphs of G_0 and the set of subgraphs of G_j, for $j \geq 1$. So, it is sufficient to obtain, up to isomorphism, all the subgraphs of G_0 which verify properties 1 and 2 (see [2]).

Finally, since that the digraph G_0 has a great number of vertices and edges (which increase with the generalized Cartan matrix order), we have designed an algorithm which allows, in the first place, to obtain all the subgraphs of G_0 verifying these properties and secondly, to group them in isomorphism classes.

The main steps of this algorithm, described in a short way, are the following:

Algorithm

Input:

- The digraph G_0 such that

$$G = \bigvee_{n \geq 0} G_n$$

  is the digraph associated with the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra $\mathfrak{g}(A)$. $V(G_0) = \{0, v_1, \ldots, v_p, v_{p+1} = \delta\}$

- The automorphism group of the matrix A, $\mathrm{Aut}\,(A)$.

Output: For each isomorphism class of subgraphs of G_0 which verify the properties 1 and 2, we give a representative.

Method

Step 1: We calculate the subgraphs of G_0 which verify the properties 1 and 2. (If I is a subgraph of G_0 which verifies the property 2, then I is generated by $v_{i_1}, \ldots, v_{i_k} \in V(I)$ and , by the property 1, $v_{i_j} \neq 0$, $j = 1, \ldots, k$.)

Step 1.1: We obtain the list of the subgraphs generated by one vertex $v \neq 0$.

Step 1.1.1: For each vertex v we determine the subgraph generated by v, $\langle v \rangle$.

Step 1.2: We obtain the list of the ideals generated by two or more vertices non nulls.

Step 1.2.1: For each ideal $\langle v_{i_1}, \ldots, v_{i_k} \rangle$ obtained in the preceding iteration (considering the step 1.1.1 as the first iteration) and for each nonzero vertex v_j of G_0, we determine if $j > i_k$ and if $\langle v_{i_1}, \ldots, v_{i_k}, v_j \rangle$ is a new subgraph with $k+1$ generators and in this case we add this subgraph to the list of subgraphs generated by $k + 1$ roots.

Step 2: We determine the action of Aut (A) on G_0. (The set of vertices $V(G_A)$ is $\{0\} \cup \Delta_+$ and G_0 is a subgraph of G_A.)

Step 3: We calculate the isomorphism classes of subgraphs of G_0 which verify 1 and 2. (We use Aut (A), since I and I' are isomorphic subgraphs of G_0 which verify 1 and 2 if and only if it exists an automorphism $\sigma \in$ Aut (A) such that $\sigma(I) = I'$.)

Step 3.1: We obtain a representative of each isomorphism class, by recurrence on the number k of vertices generating the subgraph.

We are now dealing with the computational complexity of algorithm and its implementation for doing specific computations with nilpotent Lie algebras.

Example 3 We consider the generalized Cartan matrix of affine type $A_3^{(1)}$ again. The digraph G_0 such that $G_{A_3^{(1)}} = \bigvee_{n\geq 0} G_n$ is:

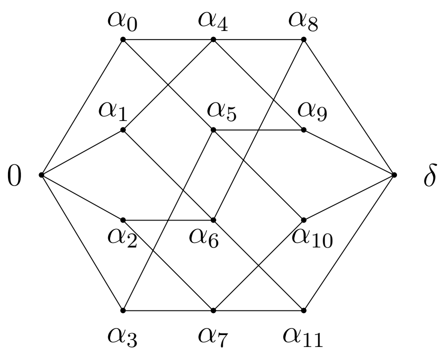


Figure 4: The digraph G_0 corresponding to the Kac-Moody algebra $\mathfrak{g}(A_3^{(1)})$.

and the automorphism group of the matrix $A_3^{(1)}$ is

$$\text{Aut}\,(A_3^{(1)}) = \{\sigma_1, \sigma_1^{-1}, \sigma_1 \circ \sigma_1, id\} = \langle \sigma_1 \rangle$$

where $\sigma_1(\alpha_0) = \alpha_1$, $\sigma_1(\alpha_1) = \alpha_2$, $\sigma_1(\alpha_2) = \alpha_3$ and $\sigma_1(\alpha_3) = \alpha_0$.

The action of Aut $(A_3^{(1)})$ on G_0 is given by:

$$\sigma_1(\alpha_4) = \alpha_6, \quad \sigma_1(\alpha_5) = \alpha_4, \quad \sigma_1(\alpha_6) = \alpha_7, \quad \sigma_1(\alpha_7) = \alpha_5,$$
$$\sigma_1(\alpha_8) = \alpha_{11}, \quad \sigma_1(\alpha_9) = \alpha_8, \quad \sigma_1(\alpha_{10}) = \alpha_9, \quad \sigma_1(\alpha_{11}) = \alpha_{10},$$
$$\sigma_1(\delta) = \delta.$$

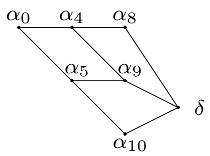
The algorithm calculates all the subgraphs I of G_0, up to the action of $\mathrm{Aut}\,(A_3^{(1)})$, which verify the following properties:

1. $\delta \in V(I)$ and $0 \notin V(I)$
2. if $\alpha \in V(I)$ and $v + \alpha \in V(G_0)$, then $v + \alpha \in V(I)$ and the edge from α to $v + \alpha$ belongs to I.

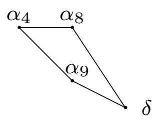
These subgraphs are obtained by recurrence on the number k of vertices which generate the subgraph.

For $A_3^{(1)}$ we have obtained 25 subgraphs:

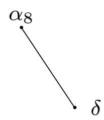
$\boxed{k = 1}$ The subgraphs which are generated by 1 vertex:



$I_0 = \langle \alpha_0 \rangle$



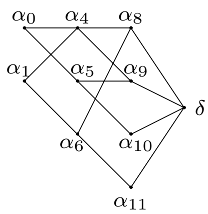
$I_1 = \langle \alpha_4 \rangle$



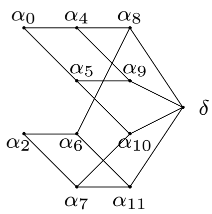
$I_3 = \langle \alpha_8 \rangle$

$I_4 = \langle \delta \rangle$

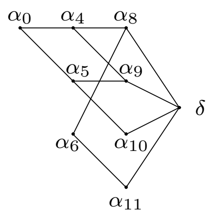
$\boxed{k = 2}$ The subgraphs which are generated by 2 vertices:



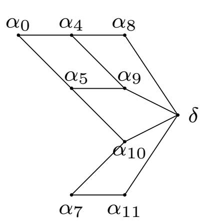
$I_5 = \langle \alpha_0, \alpha_1 \rangle$



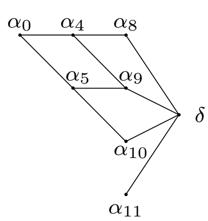
$I_6 = \langle \alpha_0, \alpha_2 \rangle$



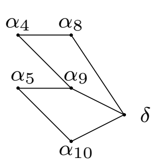
$I_7 = \langle \alpha_0, \alpha_6 \rangle$



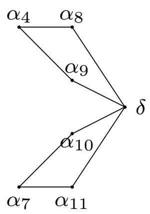
$I_8 = \langle \alpha_0, \alpha_7 \rangle$



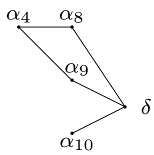
$I_9 = \langle \alpha_0, \alpha_{11} \rangle$



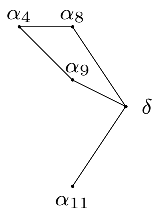
$I_{10} = \langle \alpha_4, \alpha_5 \rangle$



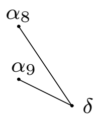
$I_{11} = \langle \alpha_4, \alpha_7 \rangle$



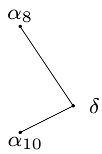
$I_{12} = \langle \alpha_4, \alpha_{10} \rangle$



$I_{13} = \langle \alpha_4, \alpha_{11} \rangle$

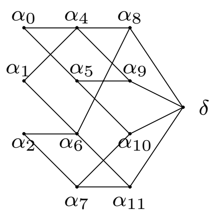


$I_{14} = \langle \alpha_8, \alpha_9 \rangle$

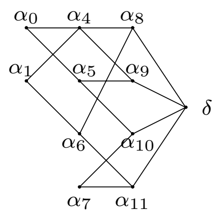


$I_{15} = \langle \alpha_8, \alpha_{10} \rangle$

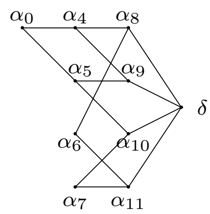
$\boxed{k = 3}$ The subgraphs which are generated by 3 vertices:



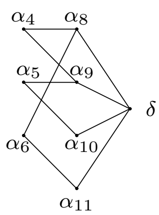
$I_{16} = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$



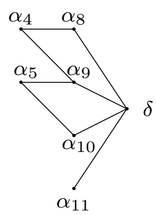
$I_{17} = \langle \alpha_0, \alpha_1, \alpha_7 \rangle$



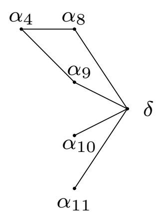
$I_{18} = \langle \alpha_0, \alpha_6, \alpha_7 \rangle$



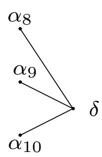
$I_{19} = \langle \alpha_4, \alpha_5, \alpha_6 \rangle$



$I_{20} = \langle \alpha_4, \alpha_5, \alpha_{11} \rangle$

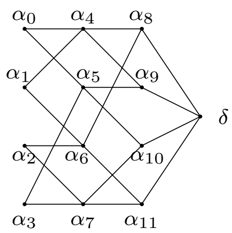


$I_{21} = \langle \alpha_4, \alpha_{10}, \alpha_{11} \rangle$

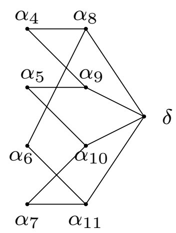


$I_{22} = \langle \alpha_8, \alpha_9, \alpha_{10} \rangle$

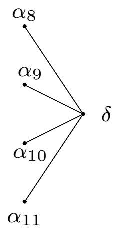
$\boxed{k=4}$ The subgraphs which are generated by 4 vertices:



$I_{23} = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$



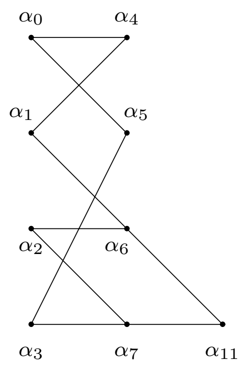
$I_{24} = \langle \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$



$I_{25} = \langle \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11} \rangle$

The Lie algebra $\mathfrak{M}$ of the example 1 is a nilpotent Lie algebra of maximal rank and of affine type $A_3^{(1)}$. Then the digraph $G_{\mathfrak{M}}$ is isomorphic to a subgraph $G'_{\mathfrak{M}}$ of $G_{A_3^{(1)}}$. This subgraph is:

$G'_{\mathfrak{M}} \equiv$



$G'_{\mathfrak{M}}$ is the subgraph of $G_{A_3^{(1)}}$ corresponding to $j = 0$ and the subgraph I_{22} of G_0 (see the list of the subgraphs of G_0 verify the properties 1 and 2 which we have obtained by the algorithm).

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