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ANALOGUES OF TRANSITIVENESS AND DECOMPOSITION OF CONTINUITY

Abstract

We consider two conditions that weaken the closed graph condition and we study their properties. We show that if X is a locally connected Baire space, Y is a separable metrizable space and $f : X \to Y$ is a w^*quasi-continuous, almost continuous and weakly Darboux function, then f is continuous.

1 Introduction

Issues related to the decomposition of continuity have been studied in the works of many mathematicians. Smith showed in [19] that the function $f : [0,1] \to \mathbb{R}$ is continuous if and only if f is almost continuous in the sense of Stallings, almost continuous in the sense of Husain and not of the Cesaro type. Later, J. Smital and E. Stanova generalized in [18] this result to the case of functions $f : X \to \mathbb{R}$, where X is a locally connected Baire topological T_3-space. Finally, R. Gibson replaced in [5] the almost continuity in the sense of Stallings with the Darboux property and obtained the following result.

Theorem A. *Let X be a locally connected Baire topological T_3-space. A function $f : X \to \mathbb{R}$ is continuous if and only if*

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(1) f is a Darboux function,
(2) f is almost continuous in the sense of Husain, and
(3) f is not of Cesaro type.

In [13], R. Mimna generalized a well-known result on the continuity of the function $f : \mathbb{R} \to \mathbb{R}$ with connected and closed graph. He established, using the concepts of O-connectedness and local w^*continuity, the following result: if X is a locally connected space and Y a topological space, then the map $f : X \to Y$ is continuous if and only if f is O-connected and locally w^*continuous. Introducing the concept of transitiveness, V. Maslyuchenko and V. Kretsu proved in [8] that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if f is almost continuous in the sense of Stallings and transitional. The result of Mimna from [13] has been improved by V. Maslyuchenko and V. Nesterenko in [11] as the following theorem.

Theorem B. *Let X be a locally connected space and Y a topological space. A function $f : X \to Y$ is continuous if and only if*
1) f is a weakly Darboux function,
2) f is transitional.

Introducing the concept of w^*quasi-continuity, M. Matejdes obtained in [12] another outcome to this thread.

Theorem C. *Let X be a π-connected space, Y a topological space and $f : X \to Y$ an O-connected function. If f is almost continuous at x and w^*quasi-continuous at x, then f is quasi-continuous at x.*

If in Theorem C the space Y is regular, then using the well-known decomposition theorem of [16], we obtain that the function f is continuous.

In all of these results, when Y is a locally compact Hausdorff space, the properties studied (e.g., transitiveness, local w^*continuity and w^*quasi-continuity of a function) are all strictly weaker than the closed graph property. These properties of functions are called analogs of transitiveness. Relations between them are studied in sections 3 and 4.

Later in this article, we find that if Y is a second countable space and $f : X \to Y$ is w^*quasi-continuous, then the set of points of transitiveness of f is residual in X; see Theorem 16. We also get a local version of Theorem B; see Theorem 18. With these tools, we obtain Theorem 20, a result on the set of points of discontinuity of continuous mappings that have the weak Darboux property, and Theorems 22 and 24, which develop Theorem C.

2 Definitions

A function $f : X \to \mathbb{R}$ is said to be *upper* $\{$*lower*$\}$ *transitional at a point* $x \in X$ [11] if, for any $\varepsilon > 0$, there is a neighborhood U of x and a point $y \in (f(x), f(x)+\varepsilon)$ $\{y \in (f(x)-\varepsilon, f(x))\}$ such that $U \cap f^{-1}(y) = \varnothing$. If a function is upper transitional and lower transitional at a point x, then the function is called *transitional at* x. A function is said to be *transitional, upper transitional* or *lower transitional* if it is so at each point.

For a subset A of a topological space, let $\operatorname{int} A$, $\overline{A}$ and $\operatorname{fr} W = \overline{W} \setminus \operatorname{int} W$ denote the interior, the closure and the boundary of A, respectively. Let X and Y be topological spaces. A function $f : X \to Y$ is called

- *transitional at* $x \in X$ [11] if, for each neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X and an open neighborhood W of $f(x)$ in Y such that $W \subseteq V$ and $U \cap f^{-1}(\operatorname{fr} W) = \varnothing$;

- *weakly transitional at* $x \in X$ [11] if, for each neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X and a point $b \in V$ such that $U \cap f^{-1}(b) = \varnothing$;

- *quasi-transitional at* $x \in X$ [11] if, for each neighborhood V of $f(x)$ in Y and each neighborhood U of x in X, there is an open neighborhood W of $f(x)$ in Y and a nonempty open subset G of X such that $W \subseteq V$, $G \subseteq U$ and $G \cap f^{-1}(\operatorname{fr} W) = \varnothing$;

- *weakly quasi-transitional at* $x \in X$ [11] if, for each neighborhood V of $f(x)$ in Y and each neighborhood U of x in X, there is a nonempty open set G in X and a point $b \in V$ such that $G \subseteq U$ and $G \cap f^{-1}(b) = \varnothing$.

A function is said to be *transitional, weakly transitional, quasi-transitional, weakly quasi-transitional* if it is so at each point. The class of transitional maps is very extensive. In particular, every continuous mapping is transitional, every monotone function $f : \mathbb{R} \to \mathbb{R}$ or a function with closed graph is transitional. It is clear that a function $f : X \to \mathbb{R}$ is weakly transitional at x if and only if f is upper or lower transitional at x. Obviously, if a mapping is transitional at some point, then it is quasi-transitional at that point.

Similar to as was introduced the notion of the upper or lower transitiveness of function $f : X \to \mathbb{R}$ can introduce the concept of upper or lower quasi-transitiveness.

A set A is *quasi-open* if $A \subseteq \overline{\operatorname{int} A}$. A function $f : X \to Y$ is said to be

- w^**continuous* [9] if $f^{-1}(\operatorname{fr} V)$ is closed in X for each open set V in Y;

- *locally w^* continuous* [14] if there is a base $\mathcal{B}$ for Y such that $f^{-1}(\mathrm{fr}\, B)$ is closed for any $B \in \mathcal{B}$;
- *locally relative continuous* [15] if there is a base $\mathcal{B}$ for Y such that $f^{-1}(B)$ is an open set of a subspace $f^{-1}(\overline{B})$ for any $B \in \mathcal{B}$;
- *w^* quasi-continuous at $x \in X$* [12] if, for each neighborhood V of $f(x)$ in Y, there is an open neighborhood W of $f(x)$ in Y and a quasi-open subset A of X such that $W \subseteq V$, $x \in A$ and $A \cap f^{-1}(\mathrm{fr}\, W) = \varnothing$; and *$w^*$ quasi-continuous* if f is so at every point.

It is said that $f : X \to Y$ is of *the Cesaro type* if there exist nonempty open subsets U of X and V of Y such that $f^{-1}(y)$ is dense in U for each $y \in V$. In [11], it is shown that a function $f : X \to Y$ between topological spaces X and Y is weakly quasi-transitional if and only if f is not of Cesaro type.

Denote by $\mathrm{Gr}(f) = \{(x, f(x)) : x \in X\}$ the graph of f. A function $f : X \to Y$ is called

- *almost continuous in the sense of Stallings* [20] if, for any open subset O of $X \times Y$ containing the graph of f, O contains the graph of a continuous function $g : X \to Y$;
- *a Darboux function* [2] if $f(A)$ is a connected subset of Y whenever A is a connected subset of X;
- *a weakly Darboux function* [11] (also *O-connected* [13]) if $f(G)$ is a connected subset of Y whenever G is an open connected subset of X;
- *almost continuous in the sense of Husain at $x \in X$* [3, 6, 17] if, for each neighborhood V of $f(x)$ in Y, there is a subset A of X such that $x \in \mathrm{int}\, \overline{A}$ and $f(A) \subseteq V$; and *almost continuous in the sense of Husain* if f is so at any point;
- *quasi-continuous at $x \in X$* [7, 12] if, for each neighborhood V of $f(x)$ in Y and each neighborhood U of x in X, there is a nonempty open subset G of X such that $G \subseteq U$ and $f(G) \subseteq V$; and *quasi-continuous* if f is so at any point.

3 Relations between different analogues of transitiveness which are stronger than transitiveness

First we note that local w^*continuity is a weaker condition than w^*continuity. This is shown by the following example.

Example 1. *The function $f : \mathbb{R} \to \mathbb{R}$ defined as*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is locally w^*continuous, but is not w^*continuous.

In [11, Theorem 17], it was established that every locally w^*continuous function $f : X \to Y$ between topological spaces X and Y is transitional. [11, example 3] shows that the inverse statement is not true.

In [15], it was established that for arbitrary topological spaces X and Y, a locally w^*continuous function $f : X \to Y$ is locally relative continuous. There is also an example of a function $f : \mathbb{R} \to \mathbb{R}$ which shows that the converse statement is not true.

Proposition 2. *Let X and Y be topological spaces and $f : X \to Y$ a locally relatively continuous function. Then f is transitional.*

PROOF. Assume the contrary, that f is not transitional at x_0. Then there is a neighborhood V_0 of $f(x_0)$ in Y such that for each open neighborhood U of x_0 in X and for each open neighborhood W of $f(x_0)$ in Y with $W \subseteq V_0$, we have that $U \cap f^{-1}(\operatorname{fr} W) \neq \varnothing$.

Since f is locally relatively continuous, there is a base $\mathcal{B}$ of open subsets of Y such that $f^{-1}(V)$ is an open subset of $f^{-1}(\overline{V})$ for each $V \in \mathcal{B}$. For the point $f(x_0)$ and a neighborhood V_0 of $f(x_0)$, there is $V \in \mathcal{B}$ such that $f(x_0) \in V \subseteq V_0$. We take an arbitrary neighborhood U of x_0 in X. Since f is not transitional at x_0, $U \cap f^{-1}(\operatorname{fr} V) \neq \varnothing$. Then there is a point $x \in U$ such that

$$x \in f^{-1}(\operatorname{fr} V) = f^{-1}(\overline{V} \setminus V) = f^{-1}(\overline{V}) \setminus f^{-1}(V).$$

Then $U \cap f^{-1}(\overline{V}) \nsubseteq f^{-1}(V)$. This means that x_0 is not an interior point of $f^{-1}(V)$ in the subspace $f^{-1}(\overline{V})$. Thus, $f^{-1}(V)$ is not an open subset of $f^{-1}(\overline{V})$. This contradicts the fact that f is locally relatively continuous. So our assumption is not true. □

As already noted, a transitional mapping is not required to be locally w^*continuous [11, Example 3]. The same example shows that a transitional mapping is not required to be locally relatively continuous.

Proposition 3. *There is a transitional function $f : \mathbb{R} \to \mathbb{R}$ which is not locally relatively continuous.*

PROOF. Let $A = (-\infty, -1] = \bigsqcup_{r\in\mathbb{Q}} A_r$, where A_r is dense in A for each $r \in \mathbb{Q}$, and $B = [1, +\infty) = \bigsqcup_{\xi\in\mathbb{R}\setminus\mathbb{Q}} B_\xi$, where B_ξ is dense in B for each $\xi \in \mathbb{R}\setminus\mathbb{Q}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} r, & x \in A_r \\ \xi, & x \in B_\xi \\ 0, & x \in (-1, 1) \end{cases}.$$

The function f is transitional at an arbitrary point $x \in \mathbb{R}$. It is clear for points in the interval $(-1, 1)$, because f is constant on this interval. If $x \geq 1$ and $\varepsilon > 0$, then there are rational numbers $y_1 \in (f(x), f(x) + \varepsilon)$ and $y_2 \in (f(x) - \varepsilon, f(x))$ such that

$$(U \times \{y_i\}) \cap \mathrm{Gr}(f) = \varnothing, \qquad i = 1, 2$$

for the neighborhood $U = (0, +\infty)$ of x. If $x \leq -1$, then for irrational numbers $y_1 \in (f(x), f(x) + \varepsilon)$ and $y_2 \in (f(x) - \varepsilon, f(x))$ and the neighborhood $U = (-\infty, 0)$, we have that $(U \times \{y_i\}) \cap \mathrm{Gr}(f) = \varnothing$, $i = 1, 2$.

Let $\mathcal{B}$ be an arbitrary base of $\mathbb{R}$. Then there is a set $V \in \mathcal{B}$ such that $\varnothing \neq V \subseteq (1, +\infty)$. We show that $f^{-1}(V)$ is not an open subset of $f^{-1}(\overline{V})$.

First, we note that $\overline{V} \setminus V \neq \varnothing$. Indeed $\varnothing \neq V \subseteq \overline{V} \subseteq [1, +\infty) \subset \mathbb{R}$. If $\overline{V} = V$, then V is a open-closed subset of $\mathbb{R}$, $V \neq \varnothing$ and $V \neq \mathbb{R}$. This contradicts the connectedness of $\mathbb{R}$.

So there is $y \in \overline{V} \setminus V$. It is clear that $y \neq 0$ because $\overline{V} \subseteq [1, +\infty)$.

Suppose that $y \in \mathbb{Q}$. Then $f^{-1}(y) = A_y$. By construction, we have that $\overline{A_y} = (-\infty, -1]$. We take a rational number y_0 in V and consider an arbitrary point x_0 in $f^{-1}(y_0) = A_{y_0}$. Then $x_0 \in f^{-1}(V)$. We show then x_0 is not a interior point of the subset $f^{-1}(V)$ of $f^{-1}(\overline{V})$. Let U be an arbitrary neighborhood of x_0 in $\mathbb{R}$. Since $x_0 \in A_{y_0} \subseteq (-\infty, -1]$ and $\overline{A_y} \supseteq (-\infty, -1]$, $x_0 \in \overline{A_y}$. So $U \cap A_y \neq \varnothing$, namely there is $u \in U \cap A_y$. Then $u \in U \cap f^{-1}(\overline{V})$ and $u \notin f^{-1}(V)$ because $f(u) = y \notin V$. Hence, $U \cap f^{-1}(\overline{V}) \not\subseteq f^{-1}(V)$ for each neighborhood U of $x_0 \in f^{-1}(V)$. This shows that $f^{-1}(V)$ is not an open subset of $f^{-1}(\overline{V})$.

If $y \in \mathbb{R}\setminus\mathbb{Q}$, then $f^{-1}(y) = B_y$. By construction, we have that $\overline{B_y} = [1, +\infty)$. Take a irrational number $y_0 \in V$ and consider a point $x_0 \in f^{-1}(y_0) = B_{y_0}$. It is then easy to verify that $U \cap f^{-1}(\overline{V}) \not\subseteq f^{-1}(V)$ for any neighborhood U of x_0 in $\mathbb{R}$. Hence, $f^{-1}(V)$ is not an open subset of $f^{-1}(\overline{V})$. □

4 Relations between different analogues of transitiveness which are weaker than transitiveness

Proposition 4. *Let X and Y be topological spaces and $f : X \to Y$ a function which is transitional at a point $x_0 \in X$. Then f is w^*quasi-continuous at x_0.*

Proof. Let V be any neighborhood of $y_0 = f(x_0)$ in Y. Since f is transitional at x_0, there is a neighborhood U of x_0 in X and an open neighborhood W of y_0 in Y such that $U \cap f^{-1}(\operatorname{fr} W) = \varnothing$. The open set $G = \operatorname{int} U$ is a neighborhood of x_0 in X and $G \cap f^{-1}(\operatorname{fr} W) = \varnothing$ with $G \subseteq U$. The open set G is quasi-open in X; hence, f is w^*quasi-continuous at x_0. □

In [11], it was shown that every continuous mapping is transitional. A similar result holds for the w^*quasi-continuity.

Proposition 5. *Let X and Y be topological spaces and $f : X \to Y$ a function which is quasi-continuous at a point $x \in X$. Then f is w^*quasi-continuous at x.*

Proof. Take any neighborhood V of $f(x)$ in Y. Put $W = \operatorname{int} V$. It is clear that $W \subseteq V$ and W is an open neighborhood of $f(x)$ in Y. Since f is quasi-continuous at x, for each neighborhood U of x there is a nonempty open subset G_U of X such that $G_U \subseteq U$ and $f(G_U) \subseteq \operatorname{int} W$. Consider the set $H = \bigcup\{G_U : U \text{ is a neighborhood of } x\}$. It is clear that $x \in \overline{H}$. It is easy to verify that $A = H \cup \{x\}$ is quasi-open in X. In addition, $x \in A$, $f(A) \subseteq W$ and $W \cap \operatorname{fr} W = \varnothing$, because W is an open subset of Y. Hence, $A \cap f^{-1}(\operatorname{fr} W) = \varnothing$. This shows that f is w^*quasi-continuous at x. □

Obviously, the converse is not true. In fact, the Dirichlet function is transitional and thus w^*quasi-continuous, but is not quasi-continuous at any point in $\mathbb{R}$.

Also, the converse to Proposition 4 is not true.

Example 6. *The function $f : \mathbb{R} \to \mathbb{R}$ defined as*

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

is w^*quasi-continuous because f is quasi-continuous; see Proposition 5. But f is not transitional.

Proposition 7. *Let X and Y be topological spaces and $f : X \to Y$ a function which is w^*quasi-continuous at a point $x \in X$. Then f is quasi-transitional at x.*

Proof. Take arbitrary neighborhoods V of $f(x)$ and U of x in Y and X, respectively. Since f is w^*quasi-continuous at x, there is an open neighborhood W of $f(x)$ in Y and a quasi-open subset A of X such that $W \subseteq V$, $x \in A$ and $A \cap f^{-1}(\operatorname{fr} W) = \varnothing$. The set A is quasi-open; therefore, $x \in A \subseteq \overline{\operatorname{int} A}$. Since U is a neighborhood of x, $G = U \cap \operatorname{int} A \neq \varnothing$. The set G is open and $G \subseteq U$. Since $A \cap f^{-1}(\operatorname{fr} W) = \varnothing$, $G \cap f^{-1}(\operatorname{fr} W) = \varnothing$. This means that f is quasi-transitional at x. □

In Proposition 12, we will show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is quasi-transitional but not w^*quasi-continuous.

In [11, Theorem 7], conditions are given for spaces X and Y in which $f : X \to Y$ is transitional at x implies f is weakly transitional at x. The same conditions on spaces ensure that a mapping is weakly quasi-transitional at a point, provided that this mapping is quasi-transitional at that point.

Proposition 8. *Let X be a topological space, Y a regular space, $x \in X$ and $f : X \to Y$ a function which is quasi-transitional at x such that there is a connected neighborhood V_0 of $y = f(x)$ in Y with $V_0 \neq \{y\}$. Then f is weakly quasi-transitional at x.*

Proof. Suppose that V is a neighborhood of y in Y and U is a neighborhood of x in X. By assumption, there is a point $y_0 \in V_0$ such that $y_0 \neq y$. It follows from T_1-axiom that there is a neighborhood V_1 of y in Y such that $y_0 \notin V_1$. Since Y is regular space, there is a closed neighborhood V_2 of y in Y such that $V_2 \subseteq V \cap V_0 \cap V_1$. It follows from the quasi-transitiveness of f that there is a nonempty open subset G of X and an open neighborhood W of y in Y such that $W \subseteq V_2$, $G \subseteq U$ and $G \cap f^{-1}(\operatorname{fr} W) = \varnothing$. We show that $\operatorname{fr} W \neq \varnothing$. If $\operatorname{fr} W = \varnothing$, then W is an open-closed subset of Y and $W \subseteq V_0$. Moreover, since $y_0 \in V_0 \setminus W$, we obtain $W \subset V_0$. In addition, $W \neq \varnothing$ because $y \in W$. Then the connected set V_0 is the disjoint union of two nonempy sets W and $V_0 \setminus W$, each open in V_0, which is impossible.

So $\operatorname{fr} W \neq \varnothing$, and hence, there is a point $b \in \operatorname{fr} W$. Since $W \subseteq V_2$ and V_2 is closed set, $\operatorname{fr} W \subseteq V_2 \subseteq V$. In this case, $b \in V$ and $b \notin f(G)$. Hence, $G \cap f^{-1}(b) = \varnothing$. This means that f is weakly quasi-transitional at x. □

In [11, Theorem 1], it was established that if $\mathbb{R} \setminus f(X)$ is an everywhere dense set for the function $f : X \to \mathbb{R}$, then f is transitional.

Proposition 9. *Let X and Y be topological spaces and $f : X \to Y$ a function such that $\overline{Y \setminus f(X)} = Y$. Then f is weakly transitional.*

Proof. Let $x \in X$ and V a neighborhood of $f(x)$ in Y. Since $Y \setminus f(X)$ is an everywhere dense subset of Y, there is a point $b \in V \cap (Y \setminus f(X))$. Therefore, $X \cap f^{-1}(b) = \varnothing$. Hence, f is weakly transitional at x. □

Proposition 10. *Let X be a topological space. Then each weakly quasi-transitional function $f : X \to \mathbb{R}$ is quasi-transitional.*

Proof. Let $x_0 \in X$, $y_0 = f(x_0)$, U an open neighborhood of x_0 in X and $V = (y_0 - \varepsilon, y_0 + \varepsilon)$ a neighborhood of y_0. Suppose that there is a point $x_1 \in U$ such that $f(x_1) \in V_1 = (y_0, y_0 + \varepsilon)$. Since f is weakly transitional at x_1, there is a nonempty open subset G_1 of X and a point $b_1 \in V_1$ such that $G_1 \subseteq U$ and $b_1 \notin f(G_1)$. Otherwise, for any point $y \in (y_0, y_0 + \varepsilon)$, we obtain $y \notin f(U)$. Anyway, there is a nonempty open set $G_1 \subseteq U$ and a point $b_1 \in V_1$ such that $b_1 \notin f(G_1)$. Similarly, we establish that there is a nonempty open set $G_2 \subseteq G_1$ and a point $b_2 \in V_2 = (y_0 - \varepsilon, y_0)$ such that $b_2 \notin f(G_2)$.

The interval $W = (b_2, b_1)$ is a neighborhood of $f(x_0)$ in $\mathbb{R}$, $W \subseteq V$ and $G_2 \cap f^{-1}(\operatorname{fr} W) = G_2 \cap f^{-1}(\{b_2, b_1\}) = \varnothing$. Hence, f is quasi-transitional at x_0. This means that f is quasi-transitional. □

Note that the function in Proposition 10 is real-valued; the following proposition shows that this is an important hypothesis.

Proposition 11. *There is a weakly transitional function $f : \mathbb{R} \to \mathbb{R}^2$ (hence, weakly quasi-transitional) which is not quasi-transitional at any point.*

Proof. Let $(A_\alpha)_{\alpha \in \mathbb{R}}$ be a system of disjoint everywhere dense subsets of $\mathbb{R}$ such that $\bigsqcup_{\alpha \in \mathbb{R}} A_\alpha = \mathbb{R}$. The function $f : \mathbb{R} \to \mathbb{R}^2$ defined as $f(x) = (\alpha, 0)$, $x \in A_\alpha$ is weakly transitional, but is not quasi-transitional at any point.

In fact, since $f(\mathbb{R}) = \mathbb{R} \times \{0\}$, by Proposition 9 it follows that f is weakly transitional. Take any point $x \in \mathbb{R}$ and a bounded neighborhood V of $f(x)$ in $\mathbb{R}^2$. Since $\mathbb{R} \times \{0\}$ is a connected subset of $\mathbb{R}^2$, $\operatorname{fr} W \cap (\mathbb{R} \times \{0\}) \neq \varnothing$ for an arbitrary open neighborhood W of $f(x)$ in $\mathbb{R}^2$ such that $W \subseteq V$. Let $(\alpha, 0) \in \operatorname{fr} W \cap (\mathbb{R} \times \{0\})$. Then

$$f^{-1}(\operatorname{fr} W) \supseteq f^{-1}((\alpha, 0)) = A_\alpha,$$

and therefore, $f^{-1}(\operatorname{fr} W)$ is an everywhere dense subset of $\mathbb{R}$. Hence, f is not quasi-transitional at x. □

Proposition 12. *There is a function $f : \mathbb{R} \to \mathbb{R}$ which is quasi-transitional, but not w^*quasi-continuous.*

Proof. Again, let $(A_\alpha)_{\alpha \in \mathbb{R}}$ be a system of disjoint everywhere dense subsets of $\mathbb{R}$ such that $\bigsqcup_{\alpha \in \mathbb{R}} A_\alpha = \mathbb{R}$. Let f_0 be the function of $\mathbb{R}$ to $\mathbb{R}$ such that $f_0(x) = \alpha$ for $x \in A_\alpha$. Note that $f_0^{-1}(\alpha) = A_\alpha$ is an everywhere dense subset for any point $\alpha \in \mathbb{R}$; hence, f_0 is not weakly quasi-transitional at any point.

Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be the set of nonzero rational numbers. Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, & x \in A_{r_n} \setminus (-\frac{1}{n}, \frac{1}{n}) \\ f_0(x), & \text{otherwise} \end{cases}.$$

Let us prove that f is quasi-transitional, but not w^*quasi-continuous. Take any point $x \in \mathbb{R}$, a neighborhood U of x and a neighborhood V of $f(x)$. There is a positive integer n such that $r_n \in V$ and $G = \operatorname{int}(U \setminus (-\frac{1}{n}, \frac{1}{n})) \neq \varnothing$. Then $G \cap f^{-1}(r_n)) = \varnothing$. So, f is weakly quasi-transitional at x. Therefore, f is weakly quasi-transitional, and by proposition 10, it follows that f is quasi-transitional.

We show that f is not w^*quasi-continuous at 0. Consider the neighborhood $V = (y_0 - 1, y_0 + 1)$ of $y_0 = f(0)$ and an arbitrary open neighborhood W of y_0 such that $W \subseteq V$. Take any quasi-open set A such that $0 \in A$. We show that $A \cap f^{-1}(\operatorname{fr} W) \neq \varnothing$. Note that if $\operatorname{fr} E = \varnothing$ for $E \subseteq \mathbb{R}$, then $\overline{E} = \operatorname{int} E$. So E is open-closed in $\mathbb{R}$. Therefore, $E = \varnothing$ or $E = \mathbb{R}$, because $\mathbb{R}$ is a connected space. But $\varnothing \neq W$ and $W \neq \mathbb{R}$. Hence, $\operatorname{fr} W \neq \varnothing$, and so there is a point $b \in \operatorname{fr} W$.

Suppose that $b = r_n$ for some n. Since $A \subseteq \overline{\operatorname{int} A}$, $0 \in A$ and $(-\frac{1}{n}, \frac{1}{n})$ is a neighborhood 0 in $\mathbb{R}$, $U_n = (-\frac{1}{n}, \frac{1}{n}) \cap \operatorname{int} A \neq \varnothing$. The set A_{r_n} is dense in $\mathbb{R}$; therefore, $U_n \cap A_{r_n} \neq \varnothing$. But $(-\frac{1}{n}, \frac{1}{n}) \cap A_{r_n} = f^{-1}(r_n)$; therefore, $f^{-1}(r_n) \cap \operatorname{int} A \neq \varnothing$. So $f^{-1}(r_n) \cap A \neq \varnothing$. Then there is a point $a \in A$ such that $f(a) = r_n = b \in \operatorname{fr} W$. Hence, $A \cap f^{-1}(\operatorname{fr} W) \neq \varnothing$.

Let $b \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}$ be fixed. Since A_b is dense in $\mathbb{R}$ and $\operatorname{int} A \neq \varnothing$, $\operatorname{int} A \cap A_b \neq \varnothing$. So, there is a point $a \in A$ such that $a \in A_b$. But $b \neq r_n$ for each n. Hence, $A_b \cap A_{r_n} = \varnothing$ for each n. Therefore, $f(a) = f_0(a) = b$. In this case, we have that $A \cap f^{-1}(\operatorname{fr} W) \neq \varnothing$.

Hence, f is not w^*quasi-continuous at 0. □

So, for arbitrary topological spaces X and Y, we have the following implications:

$$w^*\text{continuity} \Rightarrow \text{local } w^*\text{continuity} \Rightarrow \text{transitiveness} \Rightarrow$$
$$\Rightarrow w^*\text{quasi-continuity} \Rightarrow \text{quasi-transitiveness}.$$

And none of these implications can not be reversed.

Proposition 13. *If X is a topological space and Y has a basis consisting of clopen sets, then each mapping $f : X \to Y$ is transitional.*

Proof. Indeed, for any point $x \in X$ and any neighborhood V of $y = f(x)$ in Y, there is a clopen neighborhood W of y in Y such that $W \subseteq V$. Then for the neighborhood $U = X$ of x in X, we have that $U \cap f^{-1}(\operatorname{fr} W) = \varnothing$, because $\operatorname{fr} W = \varnothing$. Hence, f is transitional at x. □

In particular, all functions $f : X \to Y$, where Y is the set of rational numbers $\mathbb{Q}$ or the Sorgenfrey plane, are transitional.

Proposition 14. *Each function $f : \mathbb{Q} \to \mathbb{R}$ is transitional.*

Proof. Take any point $x \in \mathbb{Q}$, a neighborhood $U = (x - \delta, x + \delta) \cap \mathbb{Q}$ of x in $\mathbb{Q}$ and a neighborhood $V = (f(x) - \varepsilon, f(x) + \varepsilon)$ of $f(x)$ in $\mathbb{R}$. We put $V_1 = (f(x), f(x) + \varepsilon)$. By the symbol $|E|$, we denote the power of a set E. Since $\aleph_0 = |U| < |V_1| = \mathfrak{c}$, there is a point $y_1 \in V_1$ such that $y_1 \notin f(U)$. So, f is upper transitional at x. Similarly, we prove that f is lower transitional at x. Hence, f is transitional at x. □

5 On the set of points of transitiveness

In [11], the following results were established.

Theorem 15. *If X is a topological space, a space Y has a countable pseudo-base and $f : X \to Y$ is a weakly quasi-transitional mapping, then $\{x \in X :$ f is weakly transitional at $x\}$ is a residual subset of X.*

Theorem 16. *Let X be a topological space and $f : X \to \mathbb{R}$ a weakly quasi-transitional function. Then the set of points of transitiveness of f is residual in X.*

A similar result can be obtained for w^*quasi-continuous mappings.

Theorem 17. *Let X be a topological space, Y a second countable space and $f : X \to Y$ a w^*quasi-continuous mapping. Then the set A of points of transitiveness of f is residual in X.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be a base of Y. Assume the contrary. Let $E = X \setminus A$ be a non-meagre subset of X. Then for any point $x \in E$, there is a neighborhood $V(x)$ of $f(x)$ such that $f(U) \cap \operatorname{fr} W \neq \varnothing$ for each neighborhood U of x in X and for each open neighborhood W of $f(x)$ with $W \subseteq V(x)$.

For each $n \in \mathbb{N}$, consider the set

$$E_n = \{x \in E : f(x) \in V_n \subseteq V(x)\}.$$

It is clear that $\bigcup_{n=1}^{\infty} E_n = E$. Since E is a non-meagre set, there is a positive integer n_0 such that E_{n_0} is a non-meagre set. By w^*quasi-continuity of f, it follows that for each point $x \in E_{n_0}$, there is an open neighborhood $W(x)$ of $f(x)$ in Y and a quasi-open subset $A(x)$ of X such that $W(x) \subseteq V_{n_0}$, $x \in A(x)$ and $A(x) \cap f^{-1}(\operatorname{fr} W(x)) = \varnothing$.

For each $m \in \mathbb{N}$, we consider sets

$$E_{n_0,m} = \{x \in E_{n_0} : f(x) \in V_m \subseteq W(x)\}.$$

Then $\bigcup_{m=1}^{\infty} E_{n_0,m} = E_{n_0}$. Since E_{n_0} is non-meagre, there is a positive integer m_0 such that E_{n_0,m_0} is dense in a nonempty open subset U_0; i.e., $U_0 \subseteq \overline{E_{n_0,m_0}}$. Take a point $x_0 \in U_0 \cap E_{n_0,m_0}$. Since $A(x_0)$ is quasi-open, $U = U_0 \cap \operatorname{int} A(x_0) \neq \varnothing$. Take a point $a \in U \cap E_{n_0,m_0}$. Then U is a neighborhood of a. Since $U \subseteq A(x_0)$, $f(a) \in V_{m_0} \subseteq W(x_0) \subseteq V_{n_0} \subseteq V(a)$ and $A(x_0) \cap f^{-1}(\operatorname{fr} W(x_0)) = \varnothing$, $U \cap f^{-1}(\operatorname{fr} W(x_0)) = \varnothing$. In addition, $W(x_0)$ is an open neighborhood of $f(a)$. Hence, for $a \in E$, we found the neighbohood U of a and the open neighborhood $W = W(x_0)$ of a such that $W \subseteq V(a)$ and $U \cap f^{-1}(\operatorname{fr} W) = \varnothing$. We obtained a contradiction. □

6 Main results

The following result is a local version of Theorem B.

Theorem 18. *Let X be a locally connected space, Y a topological space and $f : X \to Y$ a weakly Darboux function. Then f is continuous at $x_0 \in X$ if and only if f is transitional at x_0.*

Proof. Since $f : X \to Y$ is continuous at x_0, f is transitional at x_0 for arbitrary topological spaces X and Y [11, Theorem 5]. In fact, let V be any neighborhood of $y_0 = f(x_0)$ in Y. Put $W = \operatorname{int} V$ and $U = f^{-1}(W)$. Then $\operatorname{fr} W = \overline{W} \setminus W$, $y_0 \in W \subseteq V$ and U is a neighborhood of x_0 in X. Thus,

$$U \cap f^{-1}(\operatorname{fr} W) = f^{-1}(W) \cap f^{-1}(\overline{W} \setminus W) = f^{-1}(W \cap (\overline{W} \setminus W)) = f^{-1}(\varnothing) = \varnothing.$$

We establish the sufficiency. Since f is transitional at x_0, there is a neighborhood U of x_0 in X and an open neighborhood W of $f(x_0)$ in Y such that $W \subseteq V$ and $f(U) \subseteq W \sqcup (Y \setminus \overline{W})$. There is an open connected neighborhood U_0 of x_0 such that $U_0 \subseteq U$. This neighborhood will be a connected component of any open neighborhood G of x_0 containing that point and contained in U. It is clear that $f(U_0) \subseteq W \sqcup (Y \setminus \overline{W})$. Since f is a weakly Darboux function, we obtain that $f(U_0)$ is a connected set. Then from the condition $f(x_0) \in W$, it follows that $f(U_0) \subseteq W$. Hence, we obtain that $f(U_0) \subseteq V$. This means that f is continuous at x_0. □

It is clear that Theorem 18 immediately implies Theorem B.

For a mapping $f : X \to Y$, we denote by $C(f)$ and $D(f)$ the set of continuity points of f and the set of discontinuity points of f, respectively.

Theorem 19. *Let X be a locally connected space, Y a second countable space and $f : X \to Y$ a w^*quasi-continuous and weakly Darboux function. Then $D(f)$ is a meagre subset of X.*

Proof. By Theorem 17, it follows that the set E of points of transition of f is a residual, and by Theorem 18, it follows that $C(f) = E$. Hence, $D(f)$ is a meagre subset. □

Theorem 20. *Let X be a locally connected space and $f : X \to \mathbb{R}$ a weakly quasi-transitional and weakly Darboux function. Then $D(f)$ is meagre subset of X.*

Proof. By Theorem 16, it follows that the set E of points of transition f is a residual, and by Theorem 18, it follows that $C(f) = E$. Hence, $D(f)$ is a meagre subset. □

A function f is called *almost quasi-continuous at* $x \in X$ [1], if for each neighborhood V of $y = f(x)$ in Y and for each neighborhood U of x in X, there is a subset A of X such that $A \subseteq U$, $\operatorname{int}\overline{A} \neq \varnothing$ and $f(A) \subseteq V$; and *almost quasi-continuous* if f is so at every point. Let Y be a metric space. A function $f : X \to Y$, where Y is equipped with metric d, is said to be *cliquish at a point* $x \in X$ [21] if, for any $\varepsilon > 0$ and any neighborhood U of $x \in X$, there exists a nonempty open subset G of X such that $G \subseteq U$ and $d(f(u), f(v)) < \varepsilon$ for each $u, v \in G$; and *cliquish* if f is so at every point. We need the following two results that are given for example in [4] and [1], respectively:

(1) *for a metric space Y, a function $f : X \to Y$ is continuous if and only if f is almost continuous in the sense of Husain and cliquish;*

(2) *for a metric space Y, a function $f : X \to Y$ is quasi-continuous if and only if f is almost quasi-continuous and cliquish.*

Recall that a function $f : X \to Y$ is called *pointwise discontinuous* if $\overline{C(f)} = X$. It is easy to verify that for a topological space X and a metric space Y, each pointwise discontinuous function is cliquish.

Theorem 21. *Let X be a locally connected Baire space, Y a separable metrizable space and $f : X \to Y$ a function satisfying*

1) f is w^*quasi-continuous;

2) f is a weakly Darboux function;

3) f is almost continuous in the sense of Husain.

Then f is continuous.

Proof. Fix a metric on the space Y which generates its topology. By Theorem 19, it follows that $C(f)$ is a residual set in X, and therefore, $C(f)$ is an everywhere dense subset. Then f is pointwise discontinuous, and hence, f is cliquish. By (1), it follows that f is continuous. □

Theorem 22. *Let X be a locally connected Baire space and $f : X \to \mathbb{R}$ a function satisfying*

1) f is weakly quasi-transitional;

2) f is a weakly Darboux function;

3) f is almost continuous in the sense of Husain.

Then f is continuous.

Proof. By Theorem 20, it follows that $C(f)$ is a residual set in X. Since X is a Baire space, $\overline{C(f)} = X$. Then f is pointwise discontinuous, and hence, f is cliquish. By (1), it follows that f is continuous. □

Theorem 23. *Let X be a locally connected Baire space, Y a separable metrizable space and $f : X \to Y$ a function satisfying*

1) f is w^*quasi-continuous;

2) f is a weakly Darboux function;

3) f is almost quasi-continuous.

Then f is quasi-continuous.

Proof. Similar to in the proof of Theorem 21, one can establish that f is cliquish. By (2), it follows that f is quasi-continuous. □

Theorem 24. *Let X be a locally connected Baire space and $f : X \to \mathbb{R}$ a function satisfying*

1) f is weakly quasi-transitional;

2) f is a weakly Darboux function;

3) f is almost quasi-continuous.

Then f is quasi-continuous.

Proof. Similar to in the proof of Theorem 22, one can establish that f is cliquish. By (2), it follows that f is quasi-continuous. □

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