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Joseph L. Gerver, Department of Mathematics, Rutgers University, Camden, New Jersey 08102, U.S.A. email: gerver@camden.rutgers.edu

A NICE EXAMPLE OF LEBESGUE INTEGRATION

Abstract

We explore the properties of an interesting new example of a function which is Lebesgue integrable but not Riemann integrable.

1 Introduction

Some years ago, while I was teaching Lebesgue's theory of integration to my real analysis class, one of the students, Michael Machuzak, asked for an honest example of a function that was Lebesgue integrable but not Riemann integrable. He pointed out that all of my examples were the characteristic functions of Cantor sets, which he said was like developing Riemann's theory of integration, and then using it only to find the areas of rectangles.

No such example came immediately to mind, and I told Machuzak that I would get back to him. Nor could I find any examples on the shelf of analysis textbooks in my office. To be sure, the historical archetype of a function which is Lebesgue integrable but not Riemann integrable is the derivative of Volterra's function [1] (pp. 89-94). But I would have had to spend some time constructing that function in class, and I felt that a one-line question ought to have a one-line answer. So the following week, I gave the class the function

$$f(x) = \prod_{n=0}^{\infty} [\sin(2^n x)]^{2/(2n+1)^2}. \tag{1}$$

Note that $f(x)$ is not the characteristic function of a Borel set, nor can its integral be transformed by a change of variables into the integral of the

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characteristic function of a Borel set. (In particular, it is not a Riemann integrable function times the characteristic function of a Borel set.) Thus $f(x)$ respects, in spirit, the principle that the full machinery of the integral calculus should not be employed to find the area of a rectangle.

Over the next few years, I came to realize that this function has a number of interesting properties, and I thought it ought to be more well known, which is my reason for writing this paper.

Figure 1 shows the graph of $f(x)$, as plotted by Maple. However, as we shall see, there is no truly satisfactory way to picture this graph, although fig. 1 may be as good as any.

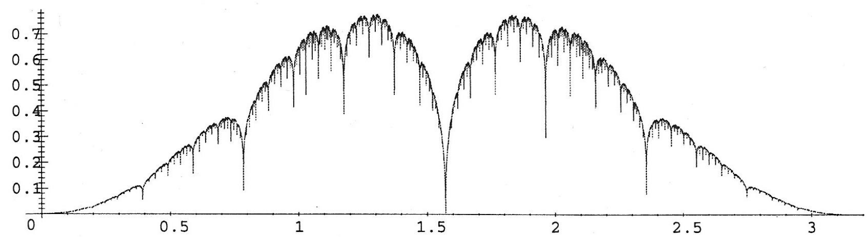

Figure 1: The function

Some properties of $f(x)$ are immediately apparent. For each factor of the infinite product, the exponent is a positive rational number with even numerator and odd denominator, so each factor is ≥ 0 for all x. Because the factors are positive powers of sine functions, they are also ≤ 1. For each x, the partial products are a monotonically decreasing sequence on the interval $[0, 1]$, which must approach a limiting value. In other words, the partial products either converge to a number between 0 and 1, or they diverge to 0. Either way, $f(x)$ is a well-defined function with values in the range $0 \leq f(x) \leq 1$ (in fact, $f(x)$ is strictly less than 1).

2 Set of zeroes

Because $\sin(2^n x) = 0$ when $2^n x = m\pi$, *i.e.* $x = m\pi/2^n$, for any integer m, we have

$$f(m\pi/2^n) = 0 \tag{2}$$

for every integer m and non-negative integer n. Thus the zeroes of f are dense on the real line.

But $f(x)$ is not uniformly zero. For example,

$$f(\pi/3) = (3/4)^{\pi^2/8}. \tag{3}$$

This follows from the fact that 2^n is congruent to 1, 2, or 4 mod 6, so that $\sin(2^n\pi/3) = \pm\frac{1}{2}\sqrt{3}$ and

$$[\sin(2^n\pi/3)]^{2/(2n+1)^2} = (3/4)^{1/(2n+1)^2}. \tag{4}$$

Thus

$$f\left(\frac{\pi}{3}\right) = \left(\frac{3}{4}\right)^{\sum_{n=0}^{\infty} 1/(2n+1)^2} \tag{5}$$

($\pi^2/8 = \pi^2/6 - \pi^2/24$, the sum of the reciprocals of all squares minus the sum for even squares.)

On the other hand, f has zeroes other than $x = \pi m/2^n$. For example, $f(x) = 0$ if

$$x = \pi \sum_{k=0}^{\infty} (-1)^k 2^{-2^{2^k}}. \tag{6}$$

Indeed, if $n = 2^{2^j}$, then

$$2^n x = \pi \sum_{k=0}^{\infty} (-1)^k 2^{2^{2^j} - 2^{2^k}} = \pi \sum_{k=0}^{j} (-1)^k 2^{2^{2^j} - 2^{2^k}} + \pi \sum_{k=j+1}^{\infty} (-1)^k 2^{2^{2^j} - 2^{2^k}}. \tag{7}$$

But the sum from $k = 0$ to j is an integer, so

$$0 \le [\sin(2^n x)]^2 = \left[\sin\left(\pi \sum_{k=j+1}^{\infty} (-1)^k 2^{2^{2^j} - 2^{2^k}}\right)\right]^2 < \left(\pi \sum_{k=j+1}^{\infty} (-1)^k 2^{2^{2^j} - 2^{2^k}}\right)^2$$

$$= \pi^2 \left| \sum_{k=j+1}^{\infty} (-1)^k 2^{2^{2^j} - 2^{2^k}} \right|^2 < \pi^2 \cdot 2^{(2^{2^j} - 2^{2^{j+1}})2} \text{ (since the series is alternating)}$$

$$= \pi^2 \cdot 2^{2(n-n^2)}. \tag{8}$$

It follows that

$$0 \le [\sin(2^n x)]^{2/(2n+1)^2} < \pi^{2/(2n+1)^2} \cdot 2^{-2(n^2-n)/(2n+1)^2}, \tag{9}$$

where

$$\lim_{n\to\infty} \pi^{2/(2n+1)^2} \cdot 2^{-2(n^2-n)/(2n+1)^2} = \pi^0 \cdot 2^{-1/2} = \frac{1}{2}\sqrt{2}. \tag{10}$$

Thus, for sufficiently large n, the upper bound in (9) gets arbitrarily close to $\frac{1}{2}\sqrt{2}$, and in particular, beyond a certain point it becomes less than $\frac{9}{10}$, say, and stays less than $\frac{9}{10}$ for all larger n. A bit of experimentation reveals that this point occurs when $n = 3$ (that is, $\pi^{2/49} \cdot 2^{-12/49} < \frac{9}{10}$).

Thus there are an infinite number of values of n (namely $n = 2^{2^j}$, where j is any integer ≥ 1, so that $n > 3$) for which

$$[\sin(2^n x)]^{2/(2n+1)^2} < \frac{9}{10}\,. \tag{11}$$

Since there are no values of n for which

$$[\sin(2^n x)]^{2/(2n+1)^2} > 1, \tag{12}$$

it follows that

$$0 \leq f(x) \leq \prod_{j=1}^{\infty} \frac{9}{10} = \lim_{j\to\infty}\left(\frac{9}{10}\right)^j = 0. \tag{13}$$

Note that x is an irrational multiple of π, because the binary expansion of the sum in (6) consists of 2 zeroes, followed by 2 ones, followed by 12 zeroes, 240 ones, 65280 zeroes, etc.

Nevertheless, for "most" x, $f(x) > 0$.

Theorem 1. *The set of zeroes of $f(x)$ in the interval $0 \leq x \leq \pi$ has measure 0.*

Proof. For each positive integer k, let

$$A_k = \left\{x \in [0,\pi] : \left|x - \frac{m\pi}{2^n}\right| > \frac{1}{2^{n+\sqrt{n}+k}} \text{ for all non-negative integers } m \text{ and } n\right\}. \tag{14}$$

Some of the intervals excluded from A_k overlap, but we can obtain a lower bound on the measure of A_k by subtracting from π the lengths of all the excluded intervals. When m is even, $m/2^n$ is equal to an odd integer over a smaller power of 2, so when we add up the lengths of the excluded intervals, we can ignore even values of m, except for the case $m = n = 0$.

Fix $n \geq 1$. There are 2^{n-1} odd values of m for which $m\pi/2^n$ is in the interval $[0,\pi]$, and there is an excluded interval of length $2/2^{n+\sqrt{n}+k}$ for each such m. The total length of all these intervals is $1/2^{\sqrt{n}+k}$.

Summing over all $n \geq 1$, we get

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}+k}} = \frac{1}{2^k}\sum_{n=1}^{\infty} 2^{-\sqrt{n}}, \tag{15}$$

where

$$\sum_{n=1}^{\infty} 2^{-\sqrt{n}} < \int_0^{\infty} 2^{-\sqrt{x}}\,dx. \tag{16}$$

With the change of variables $u = (\log 2)^2 x$ and $z = -\sqrt{u}$, we have

$$\begin{aligned} &\int_0^{\infty} 2^{-\sqrt{x}}\,dx = \int_0^{\infty} e^{-\sqrt{x}\log 2}\,dx = \frac{1}{(\log 2)^2}\int_0^{\infty} e^{-\sqrt{u}}\,du \\ &= \frac{2}{(\log 2)^2}\int_0^{-\infty} ze^z\,dz = \frac{2}{(\log 2)^2}(z-1)e^z\Big|_0^{-\infty} = \frac{2}{(\log 2)^2} < \frac{25}{6}\,. \end{aligned} \tag{17}$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}+k}} < \frac{25}{6\cdot 2^k}\,. \tag{18}$$

For $n = 0$, there are two excluded intervals (around 0 and π), each with length 2^{-k}. So the total length of all excluded intervals is less than

$$\frac{25}{6\cdot 2^k} + \frac{2}{2^k} = \frac{37}{6\cdot 2^k}\,, \tag{19}$$

and the measure of A_k is greater than $\pi - 37/(6\cdot 2^k)$.

Next, we find a lower bound on $f(x)$ for $x \in A_k$. Suppose $0 < \delta \le \frac{1}{2}$. Then

$$\sin\delta > \delta - \tfrac{1}{6}\,\delta^3 = \delta(1 - \tfrac{1}{6}\,\delta^2) > \delta[1 - \tfrac{1}{6}(\tfrac{1}{2})^2] = \tfrac{23}{24}\delta\,. \tag{20}$$

Now suppose $|z - \pi m| > \delta$ for every integer m. Then

$$|\sin z| > \sin\delta > \tfrac{23}{24}\delta. \tag{21}$$

Let n be any non-negative integer and suppose $|2^n x - \pi m| > \delta$. Then $|\sin(2^n x)| > \frac{23}{24}\delta$, and the same conclusion follows from the condition

$$\left|x - \frac{\pi m}{2^n}\right| > \frac{\delta}{2^n}. \tag{22}$$

If k is any positive integer and n is any non-negative integer, then

$$0 < \frac{1}{2^{\sqrt{n}+k}} \le \frac{1}{2}\,, \tag{23}$$

so we can let $\delta = 1/2^{\sqrt{n}+k}$ and conclude that if

$$\left|x - \frac{\pi m}{2^n}\right| > \frac{1}{2^{n+\sqrt{n}+k}} \tag{24}$$

for every integer m, then

$$|\sin(2^n x)| > \frac{23}{24} \cdot \frac{1}{2^{\sqrt{n}+k}}\,. \tag{25}$$

In other words, if $x \in A_k$, so that (24) holds for every integer m and every non-negative integer n, then (25) holds for every non-negative integer n. It follows that

$$\log|\sin(2^n x)| > \log\tfrac{23}{24} - (\sqrt{n}+k)\log 2 > -\tfrac{1}{23} - (\sqrt{n}+k)\log 2, \tag{26}$$

so

$$\begin{aligned}\frac{2}{(2n+1)^2}\log|\sin(2^n x)| &> \frac{-2}{23(2n+1)^2} - \frac{2(\sqrt{n}+k)(\log 2)}{(2n+1)^2}\\ &= -\frac{\frac{2}{23}+2k\log 2}{(2n+1)^2} - \frac{2\sqrt{n}\log 2}{(2n+1)^2}.\end{aligned} \tag{27}$$

Therefore

$$\begin{aligned}\log f(x) &= \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2}\log|\sin(2^n x)|\\ &> -\sum_{n=0}^{\infty} \frac{\frac{2}{23}+2k\log 2}{(2n+1)^2} - (\log 2)\sum_{n=0}^{\infty}\frac{2\sqrt{n}}{(2n+1)^2}\,,\end{aligned} \tag{28}$$

where

$$\sum_{n=0}^{\infty} \frac{\frac{2}{23}+2k\log 2}{(2n+1)^2} = \frac{\pi^2}{8}\Big(\frac{2}{23}+2k\log 2\Big) \tag{29}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2\sqrt{n}}{(2n+1)^2} &= \sum_{n=1}^{\infty}\frac{2\sqrt{n}}{(2n+1)^2} < \sum_{n=1}^{\infty}\frac{2\sqrt{n}}{(2n)^2} = \sum_{n=1}^{\infty}\frac{1}{2n^{3/2}}\\ &= \frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{2n^{3/2}} < \frac{1}{2} + \int_1^{\infty}\frac{1}{2x^{3/2}}\,dx = \frac{1}{2}+1 = \frac{3}{2}\,.\end{aligned} \tag{30}$$

Thus

$$\log f(x) > \frac{-\pi^2}{8}\Big(\frac{2}{23}+2k\log 2\Big) - \frac{3}{2}\log 2 > -1.147 - 1.7103k \tag{31}$$

and

$$f(x) > e^{-1.147-1.7103k} > \frac{1}{(3.15)(5.531)^k}\,. \tag{32}$$

It follows that if $f(x) \leq 1/(3.15)(5.531)^k$, then $x \notin A_k$. Therefore, for every positive integer k, the measure of the set of all x in $[0, \pi]$ for which $f(x) \leq 1/(3.15)(5.531)^k$ is less than $37/(6 \cdot 2^k)$, the measure of the complement of A_k. The set of x for which $f(x) = 0$ is a subset of the set of x for which $f(x) \leq 1/(3.15)(5.531)^k$ for every positive integer k. Therefore the measure of the set of x for which $f(x) = 0$ is less than $37/(6 \cdot 2^k)$ for every k, and is thus 0. □

3 Points of continuity

A function $f(x)$ is said to be upper semicontinuous [2] (p. 22) at $x = a$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) < f(a) + \varepsilon$ whenever $|x - a| < \delta$. Note the asymmetry of this definition: $f(x)$ must be less than $f(a) + \varepsilon$ but need not be greater than $f(a) - \varepsilon$. Note also that continuity implies upper semicontinuity.

We shall prove that our function $f(x)$ is upper semicontinuous at all x. Two corollaries are that $f(x)$ is continuous at x if and only if $f(x) = 0$, and that $f(x)$ is Lebesgue integrable.

In proving upper semicontinuity, and elsewhere, we will make use of the partial products

$$f_k(x) = \prod_{n=0}^{k} [\sin(2^n x)]^{2/(2n+1)^2}. \tag{33}$$

Theorem 2. *$f(x)$ is upper semicontinuous at all x.*

Proof. Because

$$0 \leq [\sin(2^n x)]^{2/(2n+1)^2} \leq 1 \tag{34}$$

for all n and all x, it follows that

$$f_{k+1}(x) \leq f_k(x) \tag{35}$$

for all k, and because $f_k(x)$ converges to $f(x)$ as $k \to \infty$, we also have

$$f(x) \leq f_k(x) \tag{36}$$

for all k.

Now each $f_k(x)$, being the finite product of continuous functions, is continuous, and therefore upper semicontinuous. Therefore, for every $\varepsilon > 0$, there exists $\delta = \delta(k, x, \varepsilon) > 0$ such that if $|t - x| < \delta$ then $f_k(t) < f_k(x) + \varepsilon$. Because $f_k(x)$ converges to $f(x)$, we know that for every $\varepsilon > 0$, there exists $K = K(x, \varepsilon)$ such that if $k \geq K$, then $f_k(x) < f(x) + \varepsilon$.

Fix x. Given $\varepsilon > 0$, let $k = K(x, \frac{1}{2}\varepsilon)$ and let $\delta = \delta(k, x, \frac{1}{2}\varepsilon)$. Suppose $|t - x| < \delta$. Then

$$f(t) \le f_k(t) < f_k(x) + \frac{1}{2}\varepsilon < f(x) + \varepsilon. \tag{37}$$

In other words, $f(x)$ is upper semicontinuous. (See [4], Ch. XV, §4, Theorem 9.) □

Corollary. *$f(x)$ is continuous at x if and only if $f(x) = 0$.*

Proof. Because the set of zeroes of $f(x)$ is everywhere dense, $f(x)$ cannot be continuous if $f(x) \ne 0$. On the other hand, $f(x)$ is upper semicontinuous everywhere, so given x, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - x| < \delta$, then $f(t) < f(x) + \varepsilon$. But $f(t)$ is never negative, so if $f(x) = 0$, then $f(t) > f(x) - \varepsilon$. Therefore $f(x)$ is continuous at x if $f(x) = 0$. □

Another corollary of Theorem 2 is that $f(x)$ is Lebesgue integrable, because a function that is bounded from below and upper semicontinuous on a closed interval is Lebesgue integrable over that interval [3] (p. 151). Indeed, suppose that $f(x)$ is upper semicontinuous on $[a, b]$, and let r be a lower bound. Let s be an upper bound of $f(x)$, which must exist, because if $\{x_i\}$ is a sequence of real numbers on which f is unbounded, then f cannot be upper semicontinuous on an accumulation point of $\{x_i\}$. Now suppose $f(x) < y$ for some $x \in [a, b]$ and $y \in [r, s]$. Let $\varepsilon = y - f(x)$. There exists $\delta > 0$ such that $f(t) < f(x) + \varepsilon = y$ whenever $|t-x| < \delta$. In other words, if $f(x) < y$, then there is a neighborhood U of x such that $f(t) < y$ for all t in U. It follows that for every y in $[r, s]$, the set

$$S_y = \{x \in [a, b] : f(x) < y\} \tag{38}$$

is an open set of $[a, b]$. Let $g(y)$ be the measure of S_y. Then $g(y)$ is monotone on the interval $[r, s]$, so $g(y)$ is Riemann integrable. But every Lebesgue sum of $f(x)$, whether upper or lower, is a Riemann sum of $g(y)$. Thus the Lebesgue integral $\int_a^b f(x)\,dx$ is equal to the Riemann integral $\int_r^s g(y)\,dy$.

4 A lower bound on the Lebesgue integral

An immediate consequence of Theorem 1 is that the Lebesgue integral of $f(x)$ is strictly positive. Indeed, if we let $g(y)$ be the measure of the set of $x \in [0, \pi]$ for which $f(x) < y$, then g is monotone increasing with $g(1) = \pi$ and $\lim_{y\to 0} g(y) = 0$. It follows that for some $\varepsilon > 0$, we have $g(\varepsilon) < \pi - 1$, say. Therefore the set of x for which $f(x) > \varepsilon$ has measure > 1, and $\int_0^\pi f(x)\,dx > \varepsilon$.

With a bit more care, we can find an effective lower bound on $\int_0^\pi f(x)\,dx$.

Theorem 3. *The Lebesgue integral* $\int_0^\pi f(x)\,dx$ *is greater than* $\frac{1}{89}$.

PROOF. First, we prove that for all k, the improper integral

$$\int_0^\pi \log f_k(x)\,dx \tag{39}$$

converges to a value $> -\pi^3/4$. We have

$$\log f_k(x) = \sum_{n=0}^{k} \log[\sin(2^n x)]^{2/(2n+1)^2} = \sum_{n=0}^{k} \frac{2}{(2n+1)^2} \log|\sin(2^n x)| \tag{40}$$

(where we take both sides to be $-\infty$ when x is a multiple of $2^{-k}\pi$), so

$$\int_0^\pi \log f_k(x)\,dx = \sum_{n=0}^{k} \frac{2}{(2n+1)^2} \int_0^\pi \log|\sin(2^n x)|\,dx, \tag{41}$$

where the integrals on both sides are improper. By the change of variables $u = 2^n x$, we have

$$\int_0^\pi \log|\sin(2^n x)|\,dx = 2^{-n} \int_0^{2^n\pi} \log|\sin u|\,du, \tag{42}$$

which, by the symmetry and periodicity of the sine function, is equal to

$$2\int_0^{\pi/2} \log|\sin u|\,du. \tag{43}$$

For $0 \le u \le \pi/2$, we have $|\sin u| = \sin u \ge 2u/\pi$, so

$$\log|\sin u| \ge \log\left(\frac{2u}{\pi}\right), \tag{44}$$

and, by the change of variables $z = 2u/\pi$, we have

$$2\int_0^{\pi/2} \log|\sin u|\,du \ge 2\int_0^{\pi/2} \log\left(\frac{2u}{\pi}\right)du = \pi\int_0^1 \log z\,dz = -\pi. \tag{45}$$

Therefore

$$\int_0^\pi \log f_k(x)\,dx \ge \sum_{n=0}^{k} \frac{-2\pi}{(2n+1)^2} > \sum_{n=0}^{\infty} \frac{-2\pi}{(2n+1)^2} = -2\pi\cdot\frac{\pi^2}{8} = -\frac{\pi^3}{4}. \tag{46}$$

Now, for each positive integer k, let

$$B_k = \{x \in [0,\pi] : f_k(x) > e^{-\pi^2/2}\}. \tag{47}$$

Because $f_k(x)$ is continuous, B_k is open, and hence measurable. We want to show that the measure of B_k is $> \pi/2$ for all k. The complement of B_k in $[0,\pi]$ is

$$\bar{B}_k = \{x \in [0,\pi] : f_k(x) \le e^{-\pi^2/2}\} = \{x \in [0,\pi] : \log f_k(x) \le -\pi^2/2\}, \tag{48}$$

a closed set. Suppose the measure of $\bar{B}_k$ is $\ge \pi/2$ for some k. Then, because $\log f_k(x) \le 0$ (incl. $-\infty$) for all x, and $\bar{B}_k$ is a subset of $[0,\pi]$, we have

$$\int_0^\pi \log f_k(x)\,dx \le \int_{\bar{B}_k} \log f_k(x)\,dx \le -\frac{\pi^2}{2}\cdot\frac{\pi}{2} = -\frac{\pi^3}{4}\,, \tag{49}$$

but by (46), this integral is $> -\pi^3/4$, and this contradiction establishes that the measure of $\bar{B}_k$ is $< \pi/2$, and the measure of B_k is $> \pi/2$.

Since $f_k(x) \ge 0$ for all x, we have

$$\int_0^\pi f_k(x)\,dx \ge \int_{B_k} f_k(x)\,dx > \frac{\pi}{2}\,e^{-\pi^2/2} \tag{50}$$

for all k.

Now the sequence $\{f_k(x)\}_{k=0}^\infty$ converges pointwise to $f(x)$ on the interval $0 \le x \le \pi$, and $0 \le f_k(x) \le 1$ for all x and all k. It follows from the Lebesgue dominated convergence theorem [1] (p. 183, Theorem 6.19) that

$$\int_0^\pi f(x)\,dx = \lim_{k\to\infty} \int_0^\pi f_k(x)\,dx. \tag{51}$$

But $\int_0^\pi f_k(x)\,dx$ exists and is greater than $\frac{\pi}{2}e^{-\pi^2/2}$ for each k. It follows that $\int_0^\pi f(x)\,dx$ exists and is $\ge \frac{\pi}{2}e^{-\pi^2/2} > \frac{1}{89}$. □

An immediate consequence of Theorem 3 is that $f(x)$ is not Riemann integrable. If it were, then the Riemann integral would be equal to the Lebesgue integral, but because the zeroes of $f(x)$ are dense on the interval $[0,\pi]$, every lower Riemann sum is zero.

However, we do have

Theorem 4. *The lim inf of the upper Riemann sums of $f(x)$ is equal to the Lebesgue integral.*

Proof. Let R be the set of partitions of the interval $[0,\pi]$ into a finite number of intervals, and let L be the set of partitions of $[0,\pi]$ into a finite number of Borel sets. Let

$$U_R = \liminf_{\mathcal{P}\in R} \sum_{S\in\mathcal{P}} \mu(S) \limsup_{x\in S} f(x) \tag{52}$$

and let

$$U_L = \liminf_{\mathcal{P}\in L} \sum_{S\in\mathcal{P}} \mu(S) \limsup_{x\in S} f(x) \tag{53}$$

where μ is Borel measure. Because every interval is a Borel set, R is a subset of L, and $U_R \geq U_L$.

For each k, let

$$U_{R,k} = \liminf_{\mathcal{P}\in R} \sum_{S\in\mathcal{P}} \mu(S) \limsup_{x\in S} f_k(x). \tag{54}$$

For each k, $f(x) \leq f_k(x)$ for all x, so $U_R \leq U_{R,k}$. Also, for each k, $f_k(x)$ is continuous, and hence Riemann integrable. By the Lebesgue dominated convergence theorem [1] (p. 183), the Lebesgue integral of $f(x)$, and hence U_L, is equal to the limit as $k \to \infty$ of the Lebesgue integral of $f_k(x)$ (and hence the Riemann integral of $f_k(x)$, and $U_{R,k}$). Therefore $U_R \leq U_L$. Since U_R is both $\geq$ and $\leq U_L$, $U_R = U_L$, and since $f(x)$ is Lebesgue integrable, U_R is equal to the Lebesgue integral. □

5 A numerical estimate

How can we find a decimal value for $\int_0^\pi f(x)\,dx$? The usual numerical integration methods, such as Simpson's rule, are unstable for this function. However, $\int_0^\pi f_k(x)\,dx$ converges to $\int_0^\pi f(x)\,dx$ as $k \to \infty$, and $f_k(x)$ is continuous, so we can estimate $\int_0^\pi f(x)\,dx$ by estimating $\int_0^\pi f_k(x)\,dx$.

Let M_k be the midpoint estimate of $2\int_0^{\pi/2} f_{k+1}(x)\,dx$ (which is equal to $\int_0^\pi f_{k+1}(x)\,dx$) with 2^k intervals. Then

$$M_k = \frac{\pi}{2^k} \sum_{j=1}^{2^k} \prod_{n=0}^{k} \left[\sin\left((2j-1)2^{n-k-2}\pi\right)\right]^{2/(2n+1)^2}. \tag{55}$$

(In the above equation, we need only compute the product up to $n = k$, instead of $n = k+1$, because the sine of any odd multiple of $\frac{\pi}{2}$ is 1.) Let

$$M_\infty = \lim_{k\to\infty} M_k. \tag{56}$$

k	M_k	$(M_{k-1}-M_k)^{-1/2}$	$M_k - .4044/(k-.27)$
6	1.2419727451		1.1713968
7	1.2311527243	9.613598	1.1710636
8	1.2230892609	11.136255	1.1707736
9	1.2168748353	12.685264	1.1705518
10	1.2119511226	14.251272	1.1703889
11	1.2079596568	15.828283	1.1702709
12	1.2046613111	17.412130	1.1701856
13	1.2018911808	18.999838	1.1701237
14	1.1995322446	20.589315	1.1700785
15	1.1974993737	22.179160	1.1700452
16	1.1957292786	23.768496	1.1700204
17	1.1941739924	25.356823	1.1700019
18	1.1927965318	26.943897	1.1699877
19	1.1915679404	28.529638	1.1699769
20	1.1904652307	30.114067	1.1699685
21	1.1894699246	31.697256	1.1699620
22	1.1885669999	33.279304	1.1699568
23	1.1877441184	34.860317	1.1699527
24	1.1869910513	36.440403	1.1699493
25	1.1862992466	38.019661	1.1699466
26	1.1856614980	39.598181	1.1699444
27	1.1850716898	41.176044	1.1699426
28	1.1845245979	42.753322	1.1699411
29	1.1840157324	44.330078	1.1699399

Table 1: M_k Estimates.

We conjecture that M_∞ exists and is equal to $\int_0^\pi f(x)\,dx$. This does not, of course, follow from the fact that for fixed k, the midpoint estimate of $\int_0^\pi f_{k+1}(x)\,dx$ with 2^m intervals converges to this integral as $m \to \infty$, and $\int_0^\pi f_{k+1}(x)\,dx$ converges to $\int_0^\pi f(x)\,dx$ as $k \to \infty$.

Table 1 shows the values of M_k for $6 \le k \le 29$, in column 2. Column 3 shows the reciprocal square roots of the differences $M_{k-1} - M_k$. The fact that these grow linearly with k means that the differences decrease as $1/k^2$, which is what we would expect, given that $\int_0^\pi [\sin(2^n x)]^{2/(2n+2)^2}\,dx = \pi - O(1/n^2)$. This in turn suggests that the errors $M_k - M_\infty$ decrease as $1/k$. We might expect that for a suitable choice of constants a and b, $M_k - a(k-b)^{-1}$ should converge to M_∞ much more rapidly than M_k itself. A bit of trial and error reveals that the values $a = .4044$ and $b = .27$ work nicely. Column 4 shows the values of $M_k - .4044(k - .27)^{-1}$. These numbers appear to be converging slightly more slowly than the sum of a geometric progression, with the ratio of the differences increasing from around .725 near $k = 10$ to a bit more than .8 near the bottom of the column. Extrapolating from the last few numbers in column 4, we can guess that to 5 decimal places, $\int_0^\pi f(x)\,dx = 1.16993\ldots$

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