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ON THE DISCRETIZATION TECHNIQUE FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATORS

Abstract

We extend the discretization method of de Guzmán to the setting of general metric measure spaces with mild assumptions on their structures. This method allows one to relate the best constants in the weak type $(1,1)$ inequalities for the relevant centered and uncentered Hardy-Littlewood maximal operators with the analogous constants received by applying the maximal operators to sums of Dirac deltas rather than to L^1-functions.

1 Introduction

A discretization method as a tool for the study of maximal operators was introduced by de Guzmán in [4]. His result, in the case of $\mathbb{R}^d$ with Lebesgue measure and the Euclidean metric, allowed one to replace integrable functions by finite sums of Dirac deltas as tested objects to determine the best constant in the weak type $(1,1)$ inequality for maximal convolution operators. The discretization was used for example by Aldaz (see [1]) and Melas (see [5]). The conversion from functions to Dirac deltas enabled problems investigated in [1] and [5] to receive a clear form of a probabilistic nature. The discretization was also crucial in the reasoning of Carlsson, who presented a new proof of the weak type $(1,1)$ of the classical Hardy-Littlewood maximal operator on $\mathbb{R}^d$ (see [2]). It was a very nice argument that does not use the covering lemmas which are standard tools in such proofs. Finally, the technique of de Guzmán was

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generalized in several ways (see [3] and [6]). However, all the proofs justifying the use of the discretization technique are based strongly on the properties of the convolution. Notice that this method may not be sufficient for the Hardy-Littlewood maximal operators since they can be also defined for spaces which have no group structure. In this note we show that the discretization, in the case of Hardy-Littlewood maximal operators, can be successfully applied in the context of a wide class of metric measure spaces.

Let $(\mathbb{X}, \rho, \mu)$ be a metric measure space with a metric ρ and a positive Borel measure μ. Denote an open ball centered at $x \in \mathbb{X}$ with radius $r > 0$ by $B(x, r)$. We will assume that the measure of each ball is finite. We allow that certain balls may have measure zero. However, we use the convention that such balls are omitted in the definitions of maximal operators. We assume that the support of μ, namely

$$\operatorname{supp}(\mu) = \{x \in \mathbb{X}\colon \forall r > 0 \ \mu(B(x,r)) > 0\},$$

is nonempty (in particular, in the case of separable metric measure spaces any nontrivial measure satisfies this condition).

For a Borel-measurable function f we define the *centered Hardy-Littlewood maximal operator*

$$M^c f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu, \qquad x \in \mathbb{X},$$

and the level sets $E_\lambda(f) = \{x \in \mathbb{X}\colon M^c f(x) > \lambda\}$, $\lambda > 0$. Denote

$$C = \sup_{f,\lambda} \frac{\lambda \mu(E_\lambda(f))}{\|f\|_1},$$

where the supremum is taken over all $\lambda > 0$ and $f \in L^1(\mu)$ such that $\|f\|_1 > 0$. Clearly if $C < \infty$, then C is the best constant in the weak type $(1,1)$ inequality for the associated operator M^c. On the other hand, if $C = \infty$, then M^c is not of weak type $(1,1)$.

The action of M^c can be extended to finite sums of Dirac deltas, namely measures on $\mathbb{X}$ of the form

$$\nu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \tag{1}$$

where $x_i \in \operatorname{supp}(\mu)$ are pairwise different and $\alpha_i \in \mathbb{N}$, $i = 1, 2, \ldots, n$, for some $n \in \mathbb{N}$. Denote $\|\nu\|_1 = \sum_{i=1}^{n} \alpha_i$. We define

$$M^c \nu(x) = \sup_{r} \frac{\nu(B(x,r))}{\mu(B(x,r))}, \qquad x \in \mathbb{X},$$

and the sets $E_\lambda(\nu) = \{x \in \mathbb{X} \colon M^c\nu(x) > \lambda\}$, $\lambda > 0$. Denote

$$C' = \sup_{\nu,\lambda} \frac{\lambda\mu(E_\lambda(\nu))}{\|\nu\|_1},$$

where the supremum is taken over all $\lambda > 0$ and ν of the form (1). Similarly, considering the *noncentered Hardy-Littlewood maximal operator*

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| d\mu, \qquad x \in \mathbb{X},$$

in the same way we define the sets $\widetilde{E}_\lambda(f)$, extend the action of M to measures ν of the form (1), define $\widetilde{E}_\lambda(\nu)$ and denote the constants $\widetilde{C}$ and $\widetilde{C}'$.

The aim of this note is to show that the discretization technique can be applied in the general context of metric measure spaces with very mild restrictions on their structure.

Theorem. *Let $(\mathbb{X}, \rho, \mu)$ be a metric measure space, such that the measure of each open ball is finite and $\operatorname{supp}(\mu)$ is nonempty. Then the inequalities $C \geq C'$ and $\widetilde{C} \geq \widetilde{C}'$ hold. In addition, if $(\mathbb{X}, \rho)$ is separable, then $C = C'$ and $\widetilde{C} = \widetilde{C}'$.*

2 Proof of the theorem

Proof. We will consider only the case of the centered operator. The proof in the noncentered case is similar, with appropriate modifications in the definitions of auxiliary objects that appear in the reasoning below.

First we prove the inequality $C \geq C'$. Consider the case $C < \infty$. Let ν be a measure of the form (1) and take $\lambda > 0$ such that $\mu(E_\lambda) > 0$. Denote $A_r\nu(x) = \frac{\nu(B(x,r))}{\mu(B(x,r))}$ (for the pairs $(x, r) \in \mathbb{X} \times (0, \infty)$ satisfying $\mu(B(x, r)) > 0$) and $T_\lambda(x) = \{r \colon A_r\nu(x) > \lambda\}$. Fix $0 < \epsilon < 1$ and consider $\Delta \colon E_\lambda \to (0, \infty]$ defined by

$$\Delta(x) = \sup\{h > 0 \colon \exists r \in T_\lambda(x) \text{ such that } \mu(B(x, r + h)) < (1 + \epsilon)\mu(B(x, r))\}.$$

We can see that Δ is lower semi-continuous (which means that the sets $E_{\lambda,t} = \{x \in E_\lambda \colon \Delta(x) > t\}$, $t > 0$, are open) and hence Borel-measurable. Observe that $\lim_{t\to 0} \mu(E_{\lambda,t}) = \mu(E_\lambda)$. Choose $\delta > 0$ and take

$$f = \sum_{i=1}^{n} \frac{\alpha_i \chi_{B(x_i,\delta)}}{\mu(B(x_i, \delta))},$$

where χ_A is the characteristic function of A. Of course, we have $f \in L^1(\mu)$ and $\|f\|_1 = \|\nu\|_1$. For each $x \in E_{\lambda,\delta}$ we can choose $r_x \in T_\lambda(x)$ such that $\mu(B(x, r_x + \delta)) < (1+\epsilon)\mu(B(x, r_x))$. Therefore

$$M^c f(x) \geq \frac{1}{\mu(B(x, r_x + \delta))} \int_{B(x, r_x+\delta)} f d\mu > \frac{\lambda}{1+\epsilon},$$

which implies

$$C \geq \frac{\lambda}{1+\epsilon} \frac{\mu(E_{\lambda,\delta})}{\|f\|_1} = \Big(\frac{1}{1+\epsilon}\Big) \frac{\lambda \mu(E_{\lambda,\delta})}{\|\nu\|_1}.$$

Letting first $\delta \to 0^+$ and next $\epsilon \to 0^+$ for fixed ν and $\lambda > 0$, and then taking the supremum over ν and λ, we conclude that $C \geq C'$.

Now we will show the inequality $C' \geq C$ assuming that $(\mathbb{X}, \rho)$ is separable, which means that $\mathbb{X}$ has a countable dense subset $\{p_i\}_{i=1}^\infty$. Consider the case $C' < \infty$. Let $f \in L^1(\mu)$ with $\|f\|_1 \neq 0$ and take $\lambda > 0$ such that $\mu(E_\lambda) > 0$. Without any loss of generality we can take $f \geq 0$. Denote $A_r f(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu$ (for the pairs $(x, r) \in \mathbb{X} \times (0, \infty)$ satisfying $\mu(B(x, r)) > 0$) and $T_\lambda(x) = \{r > 0\colon A_r f(x) > \lambda\}$. Fix $0 < \epsilon < 1$ and consider $\Delta\colon E_\lambda \to (0, \infty]$ and $E_{\lambda,t}$, $t > 0$, both defined analogously as it was done earlier (but now in the context of f instead of ν). Observe that Δ is lower semi-continuous on E_λ and $\lim_{t\to 0} \mu(E_{\lambda,t}) = \mu(E_\lambda)$. Choose $\delta > 0$ and denote $I = I(x, \delta, r, n) = \{i \leq n\colon B(p_i, \delta/2) \cap B(x, r) \neq \emptyset\}$, $P_{x,\delta,r,n} = \bigcup_{i\in I} B(p_i, \delta/2)$ and $T_{\lambda,\delta}(x) = \{r \in T_\lambda(x)\colon \mu(B(x, r + \delta)) < (1 + \epsilon)\mu(B(x, r))\}$. Consider $N\colon E_{\lambda,\delta} \to \mathbb{N}$ defined by

$$N(x) = \min\Big\{n \in \mathbb{N}\colon \exists r \in T_{\lambda,\delta}(x) \text{ such that } \int_{P_{x,\delta,r,n}} f d\mu > (1-\epsilon) \int_{B(x,r)} f d\mu\Big\}.$$

Note that N is well-defined by the fact that $\bigcup_{i\in\mathbb{N}} B(p_i, \delta/2) = \mathbb{X}$ and the monotone convergence theorem. We can see that N is upper semi-continuous (in particular, the sets $E_{\lambda,\delta,n} = \{x \in E_{\lambda,\delta}\colon N(x) < n + 1\}$, $n \in \mathbb{N}$, are open) and hence Borel-measurable. Observe that $\lim_{n\to\infty} \mu(E_{\lambda,\delta,n}) = \mu(E_{\lambda,\delta})$. Choose $k \in \mathbb{N}$ and define $G_i = B(p_i, \delta/2) \setminus \bigcup_{j<i} B(p_j, \delta/2)$, $i = 1, 2, \ldots, k$. For each i satisfying $\mu(G_i) > 0$ denote $m_i = \int_{G_i} f d\mu$ and choose $\xi_i \in \operatorname{supp}(\mu) \cap G_i$. Let $L > 0$ be such that $L m_i > \frac{1}{\epsilon}$ for each i satisfying $\mu(G_i) > 0$. Take

$$\nu = \sum_{i\colon \mu(G_i)>0} \lfloor L m_i \rfloor \delta_{\xi_i}.$$

We can see that ν is of the form (1) and $\|\nu\|_1 \leq L\|f\|_1$ holds. In addition, L is so large that $\lfloor L m_i \rfloor > (1-\epsilon) L m_i$ for every i satisfying $\mu(G_i) > 0$. For each

$x \in E_{\lambda,\delta,k}$ choose $r_x \in T_{\lambda,\delta}(x)$ such that $\int_{P_{x,\delta,r_x,k}} f d\mu > (1-\epsilon)\int_{B(x,r_x)} f d\mu$. Then

$$M^c\nu(x) \geq \frac{\nu(B(x,r_x+\delta))}{\mu(B(x,r_x+\delta))} > \frac{(1-\epsilon)L\int_{P_{x,\delta,r_x,k}} f d\mu}{(1+\epsilon)\mu(B(x,r_x))} > \frac{(1-\epsilon)^2}{1+\epsilon}L\lambda,$$

which implies

$$C' \geq \frac{(1-\epsilon)^2}{1+\epsilon}L\lambda\frac{\mu(E_{\lambda,\delta,k})}{\|\nu\|_1} \geq \frac{(1-\epsilon)^2}{1+\epsilon}\frac{\lambda\mu(E_{\lambda,\delta,k})}{\|f\|_1}.$$

Letting first $k \to \infty$, next $\delta \to 0^+$, and then $\epsilon \to 0^+$ for fixed f and $\lambda > 0$, and finally taking the supremum over f and λ, we conclude that $C' \geq C$. □

At the end note that, for example, the space $\mathbb{R}^d$, $d \geq 1$, with the Euclidean or supremum metric and any measure defined by the weight w, which is some non-negative Borel locally finite function, satisfies the conditions of the theorem. In this context the discretization technique (sometimes implicitly) appeared in many articles devoted to the study of the weak type $(1,1)$ of associated maximal operators.

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