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SUFFICIENT CONDITIONS FOR CONVERGENCE ALMOST EVERYWHERE OF MULTIPLE TRIGONOMETRIC FOURIER SERIES WITH LACUNARY SEQUENCE OF PARTIAL SUMS

Abstract

Sufficient conditions for the convergence (almost everywhere) of multiple trigonometric Fourier series of functions f in the classes L_p, $p > 1$, are obtained in the case where rectangular partial sums $S_n(x; f)$ of this series have numbers $n = (n_1, \dots, n_N) \in \mathbb{Z}^N$, $N \geq 3$, such that of N components only k ($1 \leq k \leq N - 2$) are elements of some lacunary sequences. Earlier, in the case where $N - 1$ components of the number n are elements of lacunary sequences, convergence almost everywhere for multiple Fourier series was obtained for functions in the classes L_p, $p > 1$, by M. Kojima (1977), and for functions in Orlizc classes by D. K. Sanadze, Sh. V. Kheladze (1977) and N. Yu. Antonov (2014).

Note that presence of two or more "free" components in the number n, as follows from the results by Ch. Fefferman (1971) and M. Kojima (1977), does not guarantee the convergence almost everywhere of $S_n(x; f)$ for $N \geq 3$ even in the class of continuous functions.

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1 Introduction

1. Consider the N-dimensional Euclidean space $\mathbb{R}^N$, whose elements will be denoted as $x = (x_1, \dots, x_N)$, and set $(nx) = n_1x_1 + \dots + n_Nx_N$. We introduce $\mathbb{R}^N_\sigma = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_j \geq \sigma,\ j = 1, \dots, N\}$, $\sigma \in \mathbb{R}^1$, and the set $\mathbb{Z}^N \subset \mathbb{R}^N$ of all vectors with integer coordinates, and denote $\mathbb{Z}^N_\sigma = \mathbb{R}^N_\sigma \cap \mathbb{Z}^N$.

Let a 2π-periodic (in each argument) function $f \in L_1(\mathbb{T}^N)$, where $\mathbb{T}^N = \{x \in \mathbb{R}^N : -\pi \leq x_j < \pi, j = 1, \dots, N\}$, be expanded in a multiple trigonometric Fourier series: $f(x) \sim \sum_{k \in \mathbb{Z}^N} c_k e^{i(kx)}$.

For any vector $n = (n_1, \dots, n_N) \in \mathbb{Z}^N_0$ consider the rectangular partial sum of this series

$$S_n(x; f) = \sum_{|k_1| \leq n_1} \cdots \sum_{|k_N| \leq n_N} c_k e^{i(kx)}. \tag{1}$$

The main purpose of our investigation is to study the behavior on $\mathbb{T}^N$ of the partial sum (1) as $n \to \infty$ (i.e. $\min\limits_{1 \leq j \leq N} n_j \to \infty$) depending on the class of functions f, as well as on the restrictions imposed on the components $n_1, \dots, n_N$ of the vector n—the "number" of $S_n(x; f)$.

2. In the case $N = 1$, A. N. Kolmogorov [8] established that for any function $f \in L_2(\mathbb{T}^1)$ $\lim\limits_{\lambda \to \infty} S_{n^{(\lambda)}}(x; f) = f(x)$ almost everywhere (a.e.) on $\mathbb{T}^1$, where $\{n^{(\lambda)}\}$, $n^{(\lambda)} \in \mathbb{Z}^1_1$, $\lambda = 1, 2, \dots$, is a lacunary sequence. (A sequence $\{n^{(s)}\}$, $n^{(s)} \in \mathbb{Z}^1_1$, is called lacunary, if $n^{(1)} = 1$ and $\frac{n^{(s+1)}}{n^{(s)}} \geq q > 1$, $s = 1, 2, \dots$.) This result was extended by J. Littlewood and R. Paley [11] on the classes $L_p(\mathbb{T}^1), p > 1$. Later R. Gosselin [4] and V. Totik [17] established that in $L_1(\mathbb{T}^1)$ this result is not true. Further, S. V. Konyagin [9] showed, first, that the positive result is true for any function $f \in L(\log^+ L)(\mathbb{T}^1)$, and, second, he strengthened the negative result of V. Totik [17] by proving that for any function $\Phi(u) = o(u \log^+ \log^+ u)$ as $u \to \infty$ and for any sequence $\{n^{(\nu)}\}$, $n^{(\nu)} \in \mathbb{Z}^1_0$, $n^{(\nu)} \to \infty$ as $\nu \to \infty$, there exists a function $f \in \Phi(L)(\mathbb{T}^1)$ for which $\overline{\lim\limits_{\nu \to \infty}} |S_{n^{(\nu)}}(x; f)| = +\infty$ everywhere on $\mathbb{T}^1$. Later in the paper by V. Lie [10] it was proved that for any function $f \in L(\log^+ \log^+ L)(\log^+ \log^+ \log^+ L)(\mathbb{T}^1)$ and for any lacunary sequence $\{n^{(\lambda)}\}$, $n^{(\lambda)} \in \mathbb{Z}^1_1$, $\lambda = 1, 2, \dots$, $\lim\limits_{\lambda \to \infty} S_{n^{(\lambda)}}(x; f) = f(x)$ a.e. on $\mathbb{T}^1$. And finally, in 2014 by F. Di Plinio [12] it was proved that the positive result is true for any function $f \in L(\log^+ \log^+ L)(\log^+ \log^+ \log^+ \log^+ L)(\mathbb{T}^1)$.

The first result for multiple Fourier series with the "lacunary sequence of partial sums" was obtained in 1971 by P. Sjolin in [14] where it was proved that for any lacunary sequence $\{n_1^{(\lambda_1)}\}$, $n_1^{(\lambda_1)} \in \mathbb{Z}^1_1, \lambda_1 = 1, 2, \dots$, and for

$f \in L_p(\mathbb{T}^2)$, $p > 1$,

$$\lim_{\lambda_1, n_2 \to \infty} S_{n_1^{(\lambda_1)}, n_2}(x; f) = f(x) \quad \text{a.e. on} \quad \mathbb{T}^2.$$

(Note, that in 1970 N. R. Tevzadze [16] obtained the following result: For any two given sequences of numbers $\{n_j^{(l)}\}$, $j = 1, 2$, increasing to ∞, $n_j^{(l)} \in \mathbb{Z}_1^1$, $l = 1, 2, \dots$, $S_{n_1^{(l)}, n_2^{(l)}}(x; f)$ converges to $f(x)$ a.e. on $\mathbb{T}^2$ for $f \in L_2(\mathbb{T}^2)$.)

In 1977 M. Kojima [7] generalized P. Sjolin's result by proving that, if a function $f \in L_p(\mathbb{T}^N)$, $p > 1$, $N \geq 2$, and $\{n_j^{(\lambda_j)}\}$, $n_j^{(\lambda_j)} \in \mathbb{Z}_1^1$, $\lambda_j = 1, 2, \dots$, $j = 1, \dots, N-1$, are lacunary sequences, then

$$\lim_{\lambda_1, \dots, \lambda_{N-1}, n_N \to \infty} S_{n_1^{(\lambda_1)}, \dots, n_{N-1}^{(\lambda_{N-1})}, n_N}(x; f) = f(x) \quad \text{a.e. on} \quad \mathbb{T}^N.$$

(In the classes $L(\log^+ L)^{3N-2}(\mathbb{T}^N)$ the analogous result was obtained by D. K. Sanadze, Sh. V. Kheladze [13] in 1977; the other generalization of M. Kojima's result for the classes $L(\log^+ L)^{N-1}(\log^+ \log^+ L)(\log^+ \log^+ \log^+ \log^+ L)(\mathbb{T}^N)$ was made by N. Yu. Antonov [1] in 2014.)

As M. Kojima [7, Theorem 2] has observed using Ch. Fefferman's function from [2], it can be easily proved that the result formulated above can not be strengthened in the following sense: For any sequence $\widetilde{n} = (n_3, n_4, \dots, n_N) \in \mathbb{Z}_0^{N-2}$ (in particular, each component n_j of the vector $\widetilde{n}$ can be an element of a lacunary sequence), there exists a continuous function $f \in \mathbb{C}(\mathbb{T}^N)$ such that

$$\overline{\lim_{n_1, n_2, \widetilde{n} \to \infty}} |S_{n_1, n_2, \widetilde{n}}(x; f)| = +\infty \;\text{ a.e. on }\; \mathbb{T}^N.$$

The last result shows that even the class of functions $\mathbb{C}(\mathbb{T}^N)$, $N \geq 3$, is not the "class of convergence a.e." of multiple Fourier expansions in the case where two components of the vector $n = (n_1, \dots, n_N) \in \mathbb{Z}^N$—the "number"of $S_n(x; f)$—remain "free" (in particular, these two components are not elements of any lacunary sequences).

3. The question arises: In general, is it possible to speak about convergence a.e. of multiple ($N \geq 3$) trigonometric Fourier series of functions f in the classes L_p, $p > 1$, being in the "framework" of rectangular summation, when the "numbers" n of the partial sums $S_n(x; f)$ of this series have two or more "free" components?

Some answer to this question is given in the following theorems.

Let $N \geq 1$, $M = \{1, \dots, N\}$ and $s \in M$. Denote: $J_s = \{j_1, ..., j_s\}$, $j_q < j_l$ for $q < l$, and (in the case $s < N$) $M \setminus J_s = \{m_1, \dots, m_{N-s}\}$, $m_q < m_l$ for

$q < l$; these are nonempty subsets of the set M. We will also consider that $J_0 = M \setminus J_N = \emptyset$.

Fix an arbitrary k, $1 \le k < N$, $N \ge 2$, and define two vectors: the vector $\lambda = \lambda[J_k] = (\lambda_{j_1}, \dots, \lambda_{j_k}) \in \mathbb{Z}_1^k$, $j_s \in J_k$, $s = 1, \dots, k$, and the vector

$$m = m[J_k] = (m_{j_1}, \dots, m_{j_{N-k}}) \in \mathbb{Z}_1^{N-k}, \quad j_s \in M \setminus J_k, \quad s = 1, \dots, N-k.$$

Further, by the symbol $n^{(\lambda,m)} = n^{(\lambda,m)}[J_k] = (n_1, \dots, n_N) \in \mathbb{Z}_1^N$ we will denote the N-dimensional vector, whose components n_j with indices $j \in J_k$ are elements of some (single) lacunary sequences; i.e. for $j \in J_k$: $n_j = n_j^{(\lambda_j)} \in \mathbb{Z}_1^1$, $\frac{n_j^{(\lambda_j+1)}}{n_j^{(\lambda_j)}} \ge q_j > 1$, $\lambda_j = 1, 2, \dots$, and $n_j^{(\lambda_j)} \to \infty$ as $\lambda_j \to \infty$, we set

$$q = q(J_k) = (q_{j_1}, \dots, q_{j_k}) \in \mathbb{R}^k, \quad j_s \in J_k, \quad s = 1, \dots, k. \tag{2}$$

In its turn, the components n_j with indices $j \in M \setminus J_k$ are of the form $n_j = n_0 \cdot m_j$, where m_j are components of the vector $m[J_k]$ and $n_0 \in \mathbb{Z}_0^1$.

Theorem 1. *Let J_k be an arbitrary "sample" from M, $1 \le k \le N-2$, $N \ge 3$. Then for any function $f \in L_p(\mathbb{T}^N)$, $1 < p < \infty$, and for any vector $m[J_k]$*

$$\lim_{\substack{\lambda_j \to \infty, j \in J_k,\\ n_j = n_0 \cdot m_j, j \in M \setminus J_k, n_0 \to \infty}} S_{n^{(\lambda,m)}[J_k]}(x; f) = f(x) \quad \textit{almost everywhere on} \quad \mathbb{T}^N;$$

moreover,

$$\left\| \sup_{\substack{\lambda_j > 0, j \in J_k,\\ n_j = n_0 \cdot m_j, j \in M \setminus J_k, n_0 > 0}} |S_{n^{(\lambda,m)}[J_k]}(x; f)| \right\|_{L_p(\mathbb{T}^N)} \le C \|f\|_{L_p(\mathbb{T}^N)}, \tag{3}$$

where the constant C does not depend on the function f, $C = C(p, m[J_k], q)$, and the quantity q is defined in (2).

In its turn, by the symbol $n^{(\lambda,m(\nu))} = n^{(\lambda,m(\nu))}[J_k] = (n_1, \dots, n_N) \in \mathbb{Z}_1^N$ we will denote the N-dimensional vector, whose components n_j with indices $j \in J_k$ are, as before, the elements of some (single) lacunary sequences, $n_j^{(\lambda_j)} \in \mathbb{Z}_1^1$, $\lambda_j = 1, 2, \dots$, and components n_j with indices $j \in M \setminus J_k$ are of the form $n_j = m_j = n_j(\nu)$, where $n_j(\nu) \in \mathbb{Z}_0^1$, $\nu = 1, 2, \dots$.

Theorem 2. *Let J_k be an arbitrary "sample" from M, $1 \le k \le N-2$, $N \ge 3$. Then for any function $f \in L_2(\mathbb{T}^N)$ and for any sequences $n_j(\nu) \in \mathbb{Z}_0^1$, $\nu = 1, 2, \dots$, $n_j(\nu) \to \infty$ as $\nu \to \infty$, $j \in M \setminus J_k$,*

$$\lim_{\substack{\lambda_j \to \infty, j \in J_k,\\ n_j(\nu), j \in M \setminus J_k, \nu \to \infty}} S_{n^{(\lambda,m(\nu))}[J_k]}(x; f) = f(x) \quad \textit{almost everywhere on} \quad \mathbb{T}^N;$$

moreover,

$$\left\| \sup_{\substack{\lambda_j>0, j\in J_k, \\ n_j(\nu), j\in M\setminus J_k, \nu>0}} |S_{n^{(\lambda, m(\nu))}[J_k]}(x;f)| \right\|_{L_2(\mathbb{T}^N)} \le C\|f\|_{L_2(\mathbb{T}^N)}, \tag{4}$$

where the constant C *does not depend on the function* f, $C = C(J_k, q)$, *and the quantity* q *is defined in* (2).

2 Proofs

In order to prove Theorem 1 it is necessary to prove the following lemma. In the proof of this lemma, some ideas represented in [14] and [7] are used.

Lemma 1. *Let* $J_1 = \{r\}$, $1 \le r \le N$. *Then, for any function* $f \in L_p(\mathbb{T}^N)$, $1 < p < \infty$, $N \ge 3$, *and for any vector* $m[J_1]$

$$\left\| \sup_{\substack{\lambda_r>0, r\in J_1, \\ n_j=n_0\cdot m_j, j\in M\setminus J_1, n_0>0}} |S_{n^{(\lambda, m)}[J_1]}(x;f)| \right\|_{L_p(\mathbb{T}^N)} \le C\|f\|_{L_p(\mathbb{T}^N)}, \tag{5}$$

where the constant C *does not depend on the function* f, $C = C(p, m[J_1], q)$, *and the quantity* q *is defined in* (2).

Proof. To simplify the notation let us consider that $r = 1$. Introduce the following notation. Let $\widetilde{x} = (x_2, x_3, \ldots, x_N) \in \mathbb{T}^{N-1}$,

$$\widetilde{\mathbb{T}}^{N-1} = \left\{ \widetilde{x} \in \mathbb{T}^{N-1} : g(x_1) = f(x_1, \widetilde{x}) \in L_p(\mathbb{T}^1) \right\}; \tag{6}$$

it is obvious that

$$\mu_{N-1}\widetilde{\mathbb{T}}^{N-1} = \mu_{N-1}\mathbb{T}^{N-1} = (2\pi)^{N-1}; \tag{7}$$

here μ_{N-1} is the $(N-1)$-dimensional Lebesgue measure.

Fix an arbitrary point $\widetilde{x} \in \widetilde{\mathbb{T}}^{N-1}$ and expand the function $g(x_1)$ in the (single) trigonometric Fourier series

$$g(x_1) \sim \sum_{k\in\mathbb{Z}^1} c_k e^{ikx_1}. \tag{8}$$

Consider the partial sums of this series $S_m(x_1; g)$ with the numbers $m = n_1^{(\lambda_1)} \in \mathbb{Z}_1^1$, $\lambda_1 = 1, 2, \ldots$, where $\left\{ n_1^{(\lambda_1)} \right\}$ is a lacunary sequence; set $n_1^{(0)} = 0$

and define the following difference

$$\Delta_{\lambda_1}(x_1;g)=\begin{cases} S_0(x_1;g) & \text{for} \quad \lambda_1=0,\\ S_{n_1^{(\lambda_1)}}(x_1;g)-S_{n_1^{(\lambda_1-1)}}(x_1;g) & \text{for} \quad \lambda_1=1,2,\dots. \end{cases}$$

Let us split the series in (8) into two series:

$$\sum_{\lambda_1=0}^{\infty}\Delta_{2\lambda_1+1}(x_1;g), \quad \sum_{\lambda_1=0}^{\infty}\Delta_{2\lambda_1}(x_1;g). \tag{9}$$

For the further proof of Lemma 1 we need the following theorem [19, Ch. 15, Theorem (4.11)].

Theorem A. *Let a function $\varphi(t)\in L_p(\mathbb{T}^1)$, $1<p<\infty$, and let $\{n^{(\lambda)}\}$, $n^{(\lambda)}\in\mathbb{Z}_1^1$, $n^{(\lambda+1)}/n^{(\lambda)}\ge q>1$, $\lambda=1,2,\dots$, be a lacunary sequence, and $\{w_\lambda\}$, $w_\lambda\in\mathbb{Z}_0^1$, be any sequence, consisting only of the numbers 0 or 1; then the series*

$$\sum_{\lambda=0}^{\infty}w_\lambda\Delta_\lambda(t;\varphi)$$

is the Fourier series of some function $\varphi_1(t)\in L_p(\mathbb{T}^1)$ and

$$\|\varphi_1\|_{L_p(\mathbb{T}^1)}\le C\|\varphi\|_{L_p(\mathbb{T}^1)},$$

where the constant $C=C(q)$ does not depend on the function φ.

From Theorem A it follows that trigonometric series (9) are Fourier series of some functions $g_1(x_1)=f_1(x_1,\widetilde{x})$ and $g_2(x_1)=f_2(x_1,\widetilde{x})$, $g_1,g_2\in L_p(\mathbb{T}^1)$ (here we took account of notation (6)), and for these functions the following inequalities are true

$$\|g_1\|_{L_p(\mathbb{T}^1)}\le C\|g\|_{L_p(\mathbb{T}^1)}, \quad \|g_2\|_{L_p(\mathbb{T}^1)}\le C\|g\|_{L_p(\mathbb{T}^1)}. \tag{10}$$

In its turn, taking into account Hunt's result [6] for the one-dimensional trigonometric Fourier series, we have

$$g_1(x_1)=\sum_{\lambda_1=0}^{\infty}\Delta_{2\lambda_1+1}(x_1;g), \quad g_2(x_1)=\sum_{\lambda_1=0}^{\infty}\Delta_{2\lambda_1}(x_1;g) \quad \text{for a.e.} \quad x_1\in\mathbb{T}^1.$$

Hence, in view of definition of the functions g, g_1 and g_2 (also notation (6)) we get

$$f(x_1,\widetilde{x})=g(x_1)=g_1(x_1)+g_2(x_1)=f_1(x_1,\widetilde{x})+f_2(x_1,\widetilde{x}) \quad \text{for a.e.} \quad x_1\in\mathbb{T}^1. \tag{11}$$

In its turn, taking into account that, according to the assumption of the lemma, $f \in L_p(\mathbb{T}^N)$, in view of estimates (7), (10) and arbitrariness of the choice of $\widetilde{x} \in \widetilde{\mathbb{T}}^{N-1}$, we obtain the following estimates:

$$\|f_j\|_{L_p(\mathbb{T}^N)} \le C\|f\|_{L_p(\mathbb{T}^N)}, \quad j = 1, 2. \tag{12}$$

Further, by the symbol n_0m we denote the vector $n_0m = n_0m[J_1] = (n_0m_2, \ldots, n_0m_N) \in \mathbb{Z}_0^{N-1}$. We denote the functions $G_{n_0m}(x_1, \widetilde{x})$, $G^{(1)}_{n_0m}(x_1, \widetilde{x})$ and $G^{(2)}_{n_0m}(x_1, \widetilde{x})$ as follows:

$$G_{n_0m}(x_1, \widetilde{x}) = S_{n_0m}(\widetilde{x}; f(x_1, \cdot)), \quad G^{(1)}_{n_0m}(x_1, \widetilde{x}) = S_{n_0m}(\widetilde{x}; f_1(x_1, \cdot))$$

and

$$G^{(2)}_{n_0m}(x_1, \widetilde{x}) = S_{n_0m}(\widetilde{x}; f_2(x_1, \cdot)). \tag{13}$$

From equality (11) we get

$$\begin{aligned} S_{n^{(\lambda,m)}[J_1]}(x; f) &= S_{n_1^{(\lambda_1)}}(x_1; G_{n_0m}(\cdot, \widetilde{x})) \\ &= S_{n_1^{(\lambda_1)}}(x_1; G^{(1)}_{n_0m}(\cdot, \widetilde{x})) + S_{n_1^{(\lambda_1)}}(x_1; G^{(2)}_{n_0m}(\cdot, \widetilde{x})). \end{aligned} \tag{14}$$

The following theorem [19, Ch. 13, Lemma (1.19)] holds.

Theorem B. *Let trigonometric Fourier series of a function $\varphi(t)$, $\varphi \in L_1(\mathbb{T}^1)$,*

$$\varphi(t) \sim \sum_{k \in \mathbb{Z}^1} c_k e^{ikt},$$

satisfy the following condition: there exist two sequences $\{n^{(\nu)}\}$ and $\{m(\nu)\}$, $n^{(\nu)},\ m(\nu) \in \mathbb{Z}_1^1$, $\nu = 1, 2, \ldots$, $n^{(\nu)} \to \infty$ as $\nu \to \infty$, such that, first,

$$n^{(\nu)} + m(\nu) \le n^{(\nu+1)},$$

second,

$$c_k = 0 \quad \text{for} \quad n^{(\nu)} < |k| \le n^{(\nu)} + m(\nu),$$

and third,

$$\frac{n^{(\nu)} + m(\nu)}{n^{(\nu)}} \ge q > 1, \quad \nu = 1, 2, \ldots.$$

Then the partial sums $S_{n^{(\nu)}}(t; \varphi)$ and $S_{n^{(\nu)}+m(\nu)}(t; \varphi)$ converge a.e. to $\varphi(t)$, and we have the inequality

$$\sup_{\nu > 0} \left\{ |S_{n^{(\nu)}}(t; \varphi)| + |S_{n^{(\nu)}+m(\nu)}(t; \varphi)| \right\} \le C \sup_{n > 0} |\sigma_n(t; \varphi)|,$$

where the constant $C = C(q)$ does not depend on the function φ, and $\sigma_n(t;\varphi)$ are the Cezaro means,

$$\sigma_n(t;\varphi) = \frac{1}{n+1}\sum_{r=0}^{n} S_r(t;\varphi).$$

Remark. The series, which satisfy the first two conditions are called (see, e.g., [5, Ch. VI, p. 73] or [18, Ch. III, p. 79]) "the series with infinitely many gaps".

Note that in view of the definition of the functions $f_j(x_1,\widetilde{x})$, $j=1,2$ in (11), for any fixed $\widetilde{x} \in \widetilde{\mathbb{T}}^{N-1}$ the Fourier coefficients of the function $f_1(x_1,\widetilde{x})$ (over the variable x_1): $c_k(f_1) = 0$ for $n_1^{(2\lambda_1+1)} < |k| \le n_1^{(2\lambda_1+2)}$; and the Fourier coefficients of the function $f_2(x_1,\widetilde{x})$ (over the variable x_1): $c_k(f_2)=0$ for $n_1^{(2\lambda_1)} < |k| \le n_1^{(2\lambda_1+1)}$. In its turn, taking account of the definition of the functions $G^{(j)}_{n_0 m}(x_1,\widetilde{x})$, $j=1,2$ (see (13)), the Fourier coefficients of the function $G^{(1)}_{n_0 m}(x_1,\widetilde{x})$ (over the variable x_1): $c_k(G^{(1)}_{n_0 m}) = 0$ for $n_1^{(2\lambda_1+1)} < |k| \le n_1^{(2\lambda_1+2)}$; and the Fourier coefficients of the function $G^{(2)}_{n_0 m}(x_1,\widetilde{x})$ (over the variable x_1): $c_k(G^{(2)}_{n_0 m}) = 0$ for $n_1^{(2\lambda_1)} < |k| \le n_1^{(2\lambda_1+1)}$. Hence, both functions $G^{(j)}_{n_0 m}(x_1,\widetilde{x})$, $j=1,2$ (over the variable x_1) satisfy conditions of Theorem B.

Hence, the following estimates hold true:

$$\sup_{\lambda_1>0} |S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{n_0 m}(\cdot,\widetilde{x}))| \le C \sup_{n_1>0} |\sigma_{n_1}(x_1; G^{(j)}_{n_0 m}(\cdot,\widetilde{x}))|, \quad j=1,2. \tag{15}$$

The result of the following theorem [18, Ch. 4, Theorem (7.8)] permits to estimate the right part in inequality (15).

Theorem C. *Let a function $\varphi(t) \in L_p(\mathbb{T}^1)$, $1<p<\infty$. Then*

$$\left\| \sup_{n>0} |\sigma_n(t;\varphi)| \right\|_{L_p(\mathbb{T}^1)} \le C\|\varphi\|_{L_p(\mathbb{T}^1)},$$

where the constant C does not depend on the function φ.

Applying Theorems B and C, we can estimate $S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{n_0 m}(\cdot,\widetilde{x}))$, $j=$

$1, 2$. We have

$$\begin{aligned}\Big\| \sup_{\lambda_1, n_0>0} |S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{n_0 m}(\cdot, \widetilde{x}))| \Big\|_{L_p(\mathbb{T}^N)} &\le C \Big\| \sup_{n_1>0} |\sigma_{n_1}(x_1; \sup_{n_0>0} |S_{n_0 m}(\widetilde{x}; f_j(x_1, \cdot))|)| \Big\|_{L_p(\mathbb{T}^N)} \\ &\le C \Big\| \sup_{n_0>0} |S_{n_0 m}(\widetilde{x}; f_j(x_1, \cdot))| \Big\|_{L_p(\mathbb{T}^N)}, \quad j = 1, 2. \end{aligned} \tag{16}$$

If a function $\varphi(t) \in L_p(\mathbb{T}^\kappa)$, $1 < p < \infty$, $\kappa \ge 2$, then the following estimate [3] holds:

$$\Big\| \sup_{n>0} |S_{\delta_1 n, \dots, \delta_\kappa n}(t; \varphi)| \Big\|_{L_p(\mathbb{T}^\kappa)} \le C(p; \delta_1, \dots, \delta_\kappa) \|\varphi\|_{L_p(\mathbb{T}^\kappa)}, \tag{17}$$

where $\delta_1, \dots, \delta_\kappa \in \mathbb{Z}_1^1$ are the fixed numbers, $n \in \mathbb{Z}_0^1$.

From (16) and (17) we obtain

$$\Big\| \sup_{\lambda_1, n_0>0} |S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{n_0 m}(\cdot, \widetilde{x}))| \Big\|_{L_p(\mathbb{T}^N)} \le C(p, m[J_1]) \|f_j\|_{L_p(\mathbb{T}^N)}, \quad j = 1, 2. \tag{18}$$

Further, from equality (14) and estimates (12) and (18) it follows that

$$\Big\| \sup_{\lambda_1, n_0>0} |S_{n^{(\lambda,m)}[J_1]}(x; f)| \Big\|_{L_p(\mathbb{T}^N)} \le C(p, m[J_1], q) \|f\|_{L_p(\mathbb{T}^N)}.$$

Thus, taking account of our assumptions, we prove estimate (5). □

Proof of Theorem 1. Note that the convergence a.e. of the partial sums $S_{n^{(\lambda,m)}[J_k]}(x; f)$ can be deduced from the majorant estimate (3) by means of the standard argumentation; e.g., [15, p. 58-59]. So, in order to prove the theorem, it is sufficient to prove the validity of this estimate. In its turn, the proof of estimate (3) will be conducted by induction on N, $N \ge 3$.

The first step of induction is $N = 3$. In this case we must prove that for any $J_1 = \{r\}$, $1 \le r \le 3$, for any function $f \in L_p(\mathbb{T}^3)$, $1 < p < \infty$, and for any vector $m[J_1]$

$$\Big\| \sup_{\substack{\lambda_r>0,\\ n_j = n_0 \cdot m_j, j \in M \setminus J_1, n_0>0}} |S_{n^{(\lambda,m)}[J_1]}(x; f)| \Big\|_{L_p(\mathbb{T}^3)} \le C(p, m[J_1], q) \|f\|_{L_p(\mathbb{T}^3)}. \tag{19}$$

As we see, the validity of estimate (19) follows from the validity of Lemma 1; i.e., of estimate (5) for $N = 3$.

Further, suppose that estimate (3) is true for some $N = l$, $l \geq 3$; i.e., for any J_k from $M = \{1, \ldots, l\}$, $1 \leq k \leq l-2$, for any function $f \in L_p(\mathbb{T}^l)$, $1 < p < \infty$, and for any vector $m[J_k]$

$$\left\| \sup_{\substack{\lambda_j > 0, j \in J_k, \\ n_j = n_0 \cdot m_j, j \in M \setminus J_k, n_0 > 0}} |S_{n^{(\lambda,m)}[J_k]}(x; f)| \right\|_{L_p(\mathbb{T}^l)} \leq C(p, m[J_k], q) \|f\|_{L_p(\mathbb{T}^l)}. \quad (20)$$

Let us prove that estimate (3) is true for $N = l+1$, i.e., for any J_d in $M = \{1, \ldots, l+1\}$, $1 \leq d \leq l-1$, for any function $f \in L_p(\mathbb{T}^{l+1})$, $1 < p < \infty$, and for any vector $m[J_d]$

$$\left\| \sup_{\substack{\lambda_j > 0, j \in J_d, \\ n_j = n_0 \cdot m_j, j \in M \setminus J_d, n_0 > 0}} |S_{n^{(\lambda,m)}[J_d]}(x; f)| \right\|_{L_p(\mathbb{T}^{l+1})} \leq C(p, m[J_d], q) \|f\|_{L_p(\mathbb{T}^{l+1})}. \quad (21)$$

If $d = 1$, then estimate (21) follows from the result of Lemma 1.

Consider now $d \geq 2$, and, to simplify the notation, let us assume that the sample J_d is of the form $J_d = \{1, 2, \ldots, d\}$. In this case, the vector $n^{(\lambda,m)}[J_d]$ is of the form: $n^{(\lambda,m)}[J_d] = (n_1^{(\lambda_1)}, n_2^{(\lambda_2)}, \ldots, n_d^{(\lambda_d)}, n_0 m_{d+1}, \ldots, n_0 m_{l+1}) \in \mathbb{Z}_0^{l+1}$. Denote $\widetilde{n}^{(\lambda,m)} = \widetilde{n}^{(\lambda,m)}[J_d] = (n_2^{(\lambda_2)}, \ldots, n_d^{(\lambda_d)}, n_0 m_{d+1}, \ldots, n_0 m_{l+1}) \in \mathbb{Z}_0^l$.

Let the set $\widetilde{\mathbb{T}}^l$ be defined analogously to (6), precisely,

$$\widetilde{\mathbb{T}}^l = \left\{ \widetilde{x} = (x_2, x_3, \ldots, x_{l+1}) \in \mathbb{T}^l : g(x_1) = f(x_1, \widetilde{x}) \in L_p(\mathbb{T}^l) \right\}. \quad (22)$$

It is obvious that

$$\mu_l \widetilde{\mathbb{T}}^l = \mu_l \mathbb{T}^l = (2\pi)^l;$$

here μ_l is the l-dimensional Lebesgue measure. Fixing an arbitrary point $\widetilde{x} \in \widetilde{\mathbb{T}}^l$, by the same argumentation as in Lemma 1 (see (9) − (11)), we define two functions $g_1(x_1) = f_1(x_1, \widetilde{x})$ and $g_2(x_1) = f_2(x_1, \widetilde{x})$, $g_1, g_2 \in L_p(\mathbb{T}^1)$, for which

$$f(x_1, \widetilde{x}) = g(x_1) = g_1(x_1) + g_2(x_1) = f_1(x_1, \widetilde{x}) + f_2(x_1, \widetilde{x}) \quad \text{for a.e.} \quad x_1 \in \mathbb{T}^1. \quad (23)$$

In account of Theorem A, the estimates hold true:

$$\|f_j\|_{L_p(\mathbb{T}^{l+1})} \leq C \|f\|_{L_p(\mathbb{T}^{l+1})}, \qquad j = 1, 2. \quad (24)$$

Further, analogously to (13) we define the following functions:

$$G_{\widetilde{n}^{(\lambda,m)}}(x_1,\widetilde{x}) = S_{\widetilde{n}^{(\lambda,m)}}(\widetilde{x}; f(x_1,\cdot)), \quad G^{(1)}_{\widetilde{n}^{(\lambda,m)}}(x_1,\widetilde{x}) = S_{\widetilde{n}^{(\lambda,m)}}(\widetilde{x}; f_1(x_1,\cdot))$$

and

$$G^{(2)}_{\widetilde{n}^{(\lambda,m)}}(x_1,\widetilde{x}) = S_{\widetilde{n}^{(\lambda,m)}}(\widetilde{x}; f_2(x_1,\cdot)).$$

From equality (23) we have

$$\begin{aligned} S_{n^{(\lambda,m)}[J_d]}(x;f) &= S_{n_1^{(\lambda_1)}}(x_1; G_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x})) \\ &= S_{n_1^{(\lambda_1)}}(x_1; G^{(1)}_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x})) + S_{n_1^{(\lambda_1)}}(x_1; G^{(2)}_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x})). \end{aligned} \tag{25}$$

In view of the definition of the functions $f_j(x_1,\widetilde{x})$ (see (23), (24)), for any fixed $\widetilde{x} \in \widetilde{\mathbb{T}}^l$ (see (22)) the Fourier series of the functions $G^{(j)}_{\widetilde{n}^{(\lambda,m)}}(x_1,\widetilde{x})$, $j = 1,2$, over the variable x_1 satisfy the conditions of Theorem B. Thus, we have

$$\sup_{\lambda_1>0} |S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x}))| \le C \sup_{n_1>0} |\sigma_{n_1}(x_1; G^{(j)}_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x}))|, \quad j=1,2.$$

The same as in the proof of Lemma 1, we use Theorems B and C to estimate $S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x}))$, $j = 1,2$. We have

$$\begin{aligned} &\Big\| \sup_{\lambda_1,\dots,\lambda_d,n_0>0} |S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x}))| \Big\|_{L_p(\mathbb{T}^{l+1})} \\ &\quad \le C \Big\| \sup_{n_1>0} |\sigma_{n_1}(x_1; \sup_{\lambda_2,\dots,\lambda_d,n_0>0} |S_{\widetilde{n}^{(\lambda,m)}}(\widetilde{x}; f_j(x_1,\cdot))|)| \Big\|_{L_p(\mathbb{T}^{l+1})} \\ &\quad \le C \Big\| \sup_{\lambda_2,\dots,\lambda_d,n_0>0} |S_{\widetilde{n}^{(\lambda,m)}}(\widetilde{x}; f_j(x_1,\cdot))| \Big\|_{L_p(\mathbb{T}^{l+1})} \\ &\quad = C \Big\| \sup_{\lambda_2,\dots,\lambda_d,n_0>0} |S_{n_2^{(\lambda_2)},\dots,n_d^{(\lambda_d)},n_0 m_{d+1},\dots,n_0 m_{l+1}}(\widetilde{x}; f_j(x_1,\cdot))| \Big\|_{L_p(\mathbb{T}^{l+1})}, \\ &\qquad\qquad j=1,2. \end{aligned} \tag{26}$$

Because $\{n_j^{(\lambda_j)}\}$, $n_j^{(\lambda_j)} \in \mathbb{Z}_1^1$, $\lambda_j = 1,2,\dots,$ $j = 2,\dots,d$, are lacunary sequences, and also $1 \le d-1 \le l-2$, and the functions $f_j(x_1,\widetilde{x}) \in L_p(\mathbb{T}^{l+1})$,

$j = 1, 2$, $1 < p < \infty$, in order to estimate the right part of (26) we can use the inductive proposition; i.e., the majorant estimate (20), namely

$$\left\| \sup_{\lambda_2,\dots,\lambda_d,n_0>0} |S_{n_2^{(\lambda_2)},\dots,n_d^{(\lambda_d)},n_0m_{d+1},\dots,n_0m_{l+1}}(\widetilde{x}; f_j(x_1,\cdot))| \right\|^p_{L_p(\mathbb{T}^{l+1})}$$
$$= \int\limits_{\mathbb{T}^1} \left\{ \int\limits_{\mathbb{T}^l} \left\{ \sup_{\lambda_2,\dots,\lambda_d,n_0>0} |S_{n_2^{(\lambda_2)},\dots,n_d^{(\lambda_d)},n_0m_{d+1},\dots,n_0m_{l+1}}(\widetilde{x}; f_j(x_1,\cdot))| \right\}^p d\widetilde{x} \right\} dx_1$$
$$\le C \int\limits_{\mathbb{T}^1} \left\{ \int\limits_{\mathbb{T}^l} |f_j(x_1,\widetilde{x})|^p d\widetilde{x} \right\} dx_1 = C\|f_j\|^p_{L_p(\mathbb{T}^{l+1})}, \quad j = 1, 2.$$

From this and from (26) we have

$$\left\| \sup_{\lambda_1,\dots,\lambda_d,n_0>0} |S_{n_1^{(\lambda_1)}}(x_1; G^{(j)}_{\widetilde{n}^{(\lambda,m)}}(\cdot,\widetilde{x}))| \right\|_{L_p(\mathbb{T}^{l+1})} \le C\|f_j\|_{L_p(\mathbb{T}^{l+1})}, \quad j = 1, 2. \tag{27}$$

Further, from equality (25) and estimates (24) and (27) it follows the validity of estimate (21):

$$\left\| \sup_{\substack{\lambda_j>0, j\in J_d,\\ n_0>0}} |S_{n^{(\lambda,m)}[J_d]}(x; f)| \right\|_{L_p(\mathbb{T}^{l+1})}$$
$$\le C\|f_1\|_{L_p(\mathbb{T}^{l+1})} + C\|f_2\|_{L_p(\mathbb{T}^{l+1})} \le C\|f\|_{L_p(\mathbb{T}^{l+1})}.$$

In view of the induction method, we get that estimate (3) is true for any $N \ge 3$ and any k (the number of lacunary components in the vector $n^{(\lambda,m)}[J_k] \in \mathbb{Z}^N_0$), $1 \le k \le N-2$. $\square$

The proof of Theorem 2 can be conducted by the same scheme, as the proof of Theorem 1. Instead of Lemma 1, the validity of the following statement can be proved.

Lemma 2. *Let $J_1 = \{r\}$, $1 \le r \le N$. Then for any function $f \in L_2(\mathbb{T}^N)$*

$$\left\| \sup_{\substack{\lambda_r>0, r\in J_1,\\ n_j=n_j(\nu), j\in M\setminus J_1, \nu>0}} |S_{n^{(\lambda,m(\nu))}[J_1]}(x; f)| \right\|_{L_2(\mathbb{T}^N)} \le C(J_1,q)\|f\|_{L_2(\mathbb{T}^N)},$$

where the constant $C(J_1,q)$ does not depend on the function f.

Lemma 2 is proved analogously to Lemma 1 with the difference that, instead of inequality (17), the following majorant estimate [7, Theorem 1] is used: For any function $\varphi(t) \in L_2(\mathbb{T}^\kappa)$, $\kappa \geq 2$, and for any sequences $n_j(\nu) \in \mathbb{Z}_0^1$, $\nu = 1, 2, \ldots$, $n_j(\nu) \to \infty$ as $\nu \to \infty$, $j = 1, \ldots, \kappa$,

$$\left\| \sup_{\nu > 0} |S_{n_1(\nu), n_2(\nu), \ldots, n_\kappa(\nu)}(t; \varphi)| \right\|_{L_2(\mathbb{T}^\kappa)} \leq C \|\varphi\|_{L_2(\mathbb{T}^\kappa)},$$

where the constant C does not depend on the function φ.

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