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ON THE DIFFERENCES OF LOWER SEMICONTINUOUS FUNCTIONS

Abstract

Answering one of the real function problems suggested by A. Maliszewski, the existence of a bounded Darboux function of the Sierpiński first class which cannot be expressed as a difference of two bounded lower semicontinuous functions is proved. As the reply to the other Maliszewski question, we show there exists an almost everywhere continuous Darboux function of the Sierpiński first class which is not a difference of two almost everywhere continuous lower semicontinuous functions.

1 Maliszewski's problems

In [3], A. Maliszewski was concerned with the class of real functions which can be written as the difference of two upper semicontinuous functions; i.e., the Sierpiński first class. Clearly, this class corresponds to the class of all differences of two lower semicontinuous functions and to the class of all sums of a lower semicontinuous function and an upper semicontinuous function, too. Concluding his paper (similarly in [2]), A. Maliszewski proposed to solve the following two problems which can be reformulated in the language of lower semicontinuous functions as follows:

Problem 1. *Is there a bounded Darboux function in the class of all differences of lower semicontinuous functions which cannot be written as the difference of two bounded lower semicontinuous functions?*

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Problem 2. *Is each almost everywhere continuous Darboux function in the class of all differences of lower semicontinuous functions the difference of two almost everywhere continuous lower semicontinuous functions?*

Answering the first problem, the existence of a bounded Darboux function of the Sierpiński first class which cannot be expressed as a difference of two bounded lower semicontinuous functions is proved. As the reply to the other Maliszewski question, we show there exists an almost everywhere continuous Darboux function of the Sierpiński first class which is not a difference of two almost everywhere continuous lower semicontinuous functions.

2 Difference of two lower semicontinuous functions

We will deal with the classes of real functions defined on the unit interval $I = [0, 1]$. Let C, D, B_1, lsc, usc and S_1, stand for the class of *continuous, Darboux, Baire one, lower semicontinuous, upper semicontinuous functions* and for the *Sierpiński first class of functions*, respectively. The intersection $D \cap lsc$ will be denoted by $Dlsc$, and applying the same principle we will use the notation DS_1, too.

Let C_f be the set of all points of continuity of the function f and D_f be the set of all points of discontinuity of the function f. A point x is said to be a *bilateral* $\mathfrak{c}$*-point* of a set A iff for every $\delta > 0$, both $(x; x + \delta) \cap A$ and $(x - \delta; x) \cap A$ have the cardinality of continuum; i.e.,

$$\operatorname{card}\big((x; x+\delta) \cap A\big) = \operatorname{card}\big((x-\delta; x) \cap A\big) = \mathfrak{c}.$$

A set A is said to be *bilaterally* $\mathfrak{c}$*-dense* in a set B, $B \subset_{\mathfrak{c}} A$, iff each point $x \in B$ is a bilateral $\mathfrak{c}$-point of the set A.

If a function $f\colon\ I \to \mathbb{R}$ maps connected sets onto connected sets, then f is said to be *Darboux*.

With respect to [7], let us recall that the Sierpiński first class S_1 of functions is defined as follows:

$$S_1 = \Big\{ \sum_{n=1}^{\infty} f_n\colon\ \sum_{n=1}^{\infty} |f_n(x)| < \infty \ \text{ for every } x \in [0,1] \text{ and each } f_n \in C \Big\}.$$

Due to [8], the Sierpiński first class coincides with the class of sums of lower semicontinuous and upper semicontinuous functions. For this reason, the Sierpiński first class will be denoted either by S_1 or, in accordance with [6], by $lsc - lsc$. It is obvious that $S_1 \subset B_1$, but $S_1 \subsetneqq B_1$; see [6], [8].

To prove the two main results we will use Theorem 1 in [5]. Suiting our purpose, we will use a revised version of this theorem:

Theorem 3. *Let f be a function such that $f \in lsc$ and let E be an arbitrary F_σ set which is bilaterally $\mathfrak{c}$-dense in itself. If the set E is bilaterally $\mathfrak{c}$-dense in the set of points of discontinuity of the function f, then there exists a function $g \in Dlsc$ such that*

$$g(x) < f(x), \textit{ for } x \in E$$
$$g(x) = f(x), \textit{ for } x \in I \setminus E.$$

Proof. Let D_f be the set of all points of discontinuity of the function f such that

$$D_f = \bigcup_{n=1}^{\infty} D_n,$$

where $\{D_n\}_{n\in\mathbb{N}}$, is an increasing sequence of closed nowhere dense sets. The set E is the set of type F_σ, and thus let

$$E = \bigcup_{n=1}^{\infty} F_n,$$

where $\{F_n\}_{n\in\mathbb{N}}$, is again an increasing sequence of closed sets. Due to $f \in lsc$, there exists a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$,

$$f_1 \le f_2 \le f_3 \le ...,$$

which converges to the function f pointwise. Let us define the increasing sequence of functions $\{g_n\}_{n\in\mathbb{N}}$, by the formula

$$g_n = f_n - \frac{1}{n}.$$

Obviously, the inequality $g_n < f$ holds true for every $n \in \mathbb{N}$, and the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges to the function f. Moreover, let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. The functions g_n are uniformly continuous on $[0,1]$. Thus the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ determines a sequence of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ such that for every $x_1, x_2 \in [0,1]$

$$|x_1 - x_2| < \delta_n \Rightarrow |g_n(x_1) - g_n(x_2)| < \varepsilon_n \tag{$*$}$$

holds. Applying Lemma 2 in [5], let P_1 be a perfect set,

$$F_1 \subset_c P_1 \subset E,$$

and let P_1 be associated to the function g_1. Similarly, let P_2 be a perfect set,

$$(F_2 \cup P_1) \subset_c P_2 \subset E$$

and let P_2 be associated to the function g_2. Inductively for $i = 3, 4, \dots$, let P_i be a perfect set associated to the function g_i,

$$(F_i \cup P_{i-1}) \subset_c P_i \subset E.$$

Supposing P_i is already defined as a perfect set associated with the function g_i, the P_i satisfy the conditions

$$P_1 \subset_c P_2 \subset_c P_3 \subset_c \dots, \quad E = \bigcup_{i=1}^{\infty} P_i,$$

and the sequence of associated functions g_i satisfies the condition

$$g_1 < g_2 < g_3 < \dots < f, \quad g_i \to f.$$

Since the set E is $\mathfrak{c}$-dense in D_f, it is possible to require that

$$\forall\, x \in D_i \text{ there exist } a, b \in P_i \text{ such that } a < x < b \wedge b - a < \delta_i. \tag{$**$}$$

Applying the method of the proof of Theorem 1 in [5], let us construct a system of closed sets P_α, $\alpha \geq 1$, such that

$$\alpha_1 < \alpha_2 \Rightarrow P_{\alpha_1} \subset_c P_{\alpha_2},$$

and for each α, $i \leq \alpha < i + 1$, let g_α be a function defined by the formula

$$g_\alpha = g_i + (\alpha - i)(g_{i+1} - g_i),$$

which is associated with the set P_α. Clearly,

$$\alpha_1 < \alpha_2 \Rightarrow g_{\alpha_1} < g_{\alpha_2}.$$

Finally, for $x \in E$, let $\alpha(x) = \inf\{\alpha\colon\ x \in P_\alpha\}$ and let g be a function defined by the formula

$$g\,(x) = \begin{cases} g_{\alpha(x)}\,(x)\,, & \text{for } x \in E \\ f\,(x)\,, & \text{for } x \notin E \end{cases}.$$

Now we show that the function g has the three following properties, and thus meets the assertion of Theorem 3:

(1) $g \in lsc$;

(2) $g \in D$;

(3) $g(x) = f(x)$ for $x \in I \setminus E$, and $g(x) < f(x)$ for $x \in E$.

(1) The function $g \in lsc$ iff for every $x_0 \in I$ and for arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$,
$$x_n \to x_0, \Rightarrow \liminf_{x_n \to x_0} g(x_n) \geq g(x_0).$$
Suppose that there exists $\lambda \in \mathbb{R}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \to x_0$, such that
$$g(x_n) < \lambda < g(x_0).$$
Then evidently
$$\lambda < g_\alpha(x_0) < g(x_0)$$
for certain $\alpha \geq 1$. With respect to the continuity of the function g_α, for sufficiently large n, $n > n_0$, the inequality
$$g(x_n) < \lambda < g_\alpha(x_n)$$
holds for every $n > n_0$, which implies $x_n \in P_\alpha$. Then, $x_0 \in P_\alpha$, since the set P_α is closed, and by definition of the function g, we have $g_\alpha(x_0) \geq g(x_0)$. However, this contradicts $\lambda < g_\alpha(x_0) < g(x_0)$.

(2) Since $g \in lsc \subset B_1$, for verifying the property (2) it is sufficient to show (see [1]) that for each $x_0 \in I$ there exist sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $x_n \nearrow x_0$, $y_n \searrow x_0$, (for points 0 and 1 it is required only one of these sequences) such that
$$g(x_0) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(y_n).$$
Now let $x_0 \in I$ be an arbitrary point. Then either $x_0 \in E$, or $x_0 \notin E$. Let us consider both cases.

If $x_0 \in E$, then there exists an integer i_0 and a real number $\alpha_0 \in [0,1)$ such that $x_0 \in P_\alpha$ for every $\alpha > i_0 + \alpha_0$ and $x_0 \notin P_\alpha$ for every $\alpha < i_0 + \alpha_0$. Hence $g(x_0) = g_{i_0+\alpha_0}(x_0)$. Let us assume that a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, satisfies
$$\alpha_n \searrow \alpha_0,\ i_0 \leq i_0 + \alpha_0 < i_0 + \alpha_n < i_0 + 1.$$
Since $P_{i_0+\alpha_0} \subset_c P_{i_0+\alpha_n}$, it is possible to deal with points $x_n \nearrow x_0$ and $y_n \searrow x_0, n = 1, 2, \ldots,$ where $x_n, y_n \in P_{i_0+\alpha_n}$. Thus
$$g(x_n) \leq g_{i_0+\alpha_n}(x_n) \quad \text{and} \quad g(y_n) \leq g_{i_0+\alpha_n}(y_n).$$
The sequence of continuous functions $\{g_{i_0+\alpha_n}\}_{n\in\mathbb{N}}$ uniformly converges to the continuous function $g_{i_0+\alpha_0}$. Therefore
$$\lim_{n\to\infty} g(x_n) \leq \lim_{n\to\infty} g_{i_0+\alpha_n}(x_n) = g(x_0).$$

Since the function $g \in lsc$, the following inequality holds true

$$\lim_{n\to\infty} g(x_n) \geq g(x_0),$$

and it implies

$$\lim_{n\to\infty} g(x_n) = g(x_0).$$

The same is true for the sequence y_n.

If $x_0 \notin E$, then two cases have to be considered: $x_0 \in C_f$, $x_0 \in D_f$.

Let $x_0 \notin E$ and $x_0 \in C_f$. Let ε be an arbitrary positive real number. Since $g_n(x_0) \nearrow f(x_0) = g(x_0)$, there exists n_0 such that

$$f(x_0) - \frac{\varepsilon}{2} < g_{n_0}(x_0).$$

Because the function g_{n_0} is continuous, the function f is continuous at the point x_0 and $x_0 \notin P_{n_0}$. So there exists a neighbourhood $O(x_0)$ of the point x_0 such that $O(x_0) \cap P_{n_0} = \emptyset$, and for every $x \in O(x_0)$ the following inequalities hold true:

$$\begin{gathered} f\,(x_0) - \frac{\varepsilon}{2} < f\,(x) < f\,(x_0) + \frac{\varepsilon}{2}, \\ g_{n_0}\,(x_0) - \frac{\varepsilon}{2} < g_{n_0}\,(x) < g_{n_0}\,(x_0) + \frac{\varepsilon}{2}, \\ g_{n_0}\,(x) < g\,(x) \leq f\,(x) \end{gathered}$$

Therefore

$$\begin{aligned} g(x_0) - \varepsilon = f(x_0) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} &< g_{n_0}(x_0) - \frac{\varepsilon}{2} < g_{n_0}(x) \\ &< g(x) \leq f(x) < f(x_0) + \varepsilon = g(x_0) + \varepsilon. \end{aligned}$$

Thus for every $x \in O(x_0)$ the inequality

$$|g(x_0) - g(x)| < \varepsilon$$

holds true, which means that the function g is continuous at the point x_0. Therefore

$$g(x_0) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(y_n)$$

for arbitrary sequences $x_n \nearrow x_0$, $y_n \searrow x_0$, $n \in \mathbb{N}$.

Let $x_0 \notin E$ and $x_0 \notin C_f$. Then there exists n_0 such that $x_0 \in D_n$ for every $n \geq n_0$. Because

$$x_0 \notin E = \bigcup_{n=1}^{\infty} P_n,$$

where P_n are perfect sets, there exist sequences of points

$$x_n = \max\{x \in P_n;\ x < x_0\} \wedge y_n = \min\{y \in P_n;\ x_0 < y\}.$$

If $\alpha < n$, then $P_\alpha \subset_c P_n$, and thus $x_n, y_n \notin P_\alpha$ for every $\alpha < n$. Therefore,

$$g(x_n) = g_n(x_n) \quad \text{and} \quad g(y_n) = g_n(y_n)$$

due to the definition of the function g. According to $(**)$, $|x_n - y_n| < \delta_n$, and since x_0 is the bilateral $\mathfrak{c}$-point of the set E, $x_n \nearrow x_0$ and $y_n \searrow x_0$. Moreover, applying $(*)$ the following is true:

$$|g(x_n) - g_n(x_0)| = |g_n(x_n) - g_n(x_0)| < \varepsilon_n.$$

Since $g_n(x_0) \to f(x_0) = g(x_0)$ and $\varepsilon_n \to 0$ for $n \to \infty$, the equality

$$\lim_{n\to\infty} g(x_n) = g(x_0)$$

holds true. Similarly,

$$\lim_{n\to\infty} g(y_n) = g(x_0).$$

Thus $g \in D$.

(3) To verify the property (3) of the function g, it suffices to apply the suitable part of the proof of Theorem 1 in [5].

□

Let us proceed with a useful lemma:

Lemma 4. *Let f^*, $g^* \in lsc$ be functions defined on the interval $[0,1]$ such that the function $|f^* - g^*|$ is bounded by a constant M, and let $E \subset [0,1]$ be a set of type F_σ, bilaterally $\mathfrak{c}$-dense in itself. If $D_{f^*} \cup D_{g^*} \subset_{\mathfrak{c}} E$, then there exist functions f, $g \in Dlsc$ such that the function $|f - g|$ is bounded by the constant $3M$, and*

$$\{x;\ f(x) \neq f^*(x)\} = \{x;\ g(x) \neq g^*(x)\} = E.$$

Proof. According to assumptions $|f^*(x) - g^*(x)| < M$, for every $x \in [0,1]$. Because f^*, $g^* \in lsc$, there exist sequences of continuous functions

$$f_1^* \leq f_2^* \leq f_3^* \leq \ldots \to f^*,$$
$$g_1^* \leq g_2^* \leq g_3^* \leq \ldots \to g^*.$$

Let us define sequences of continuous functions $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$, in the following way:

$$f_n(x) = \max\left\{g_n^*(x) - \frac{1}{n} - M, f_n^*(x) - \frac{1}{n}\right\},$$
$$g_n(x) = \max\left\{f_n^*(x) - \frac{1}{n} - M, g_n^*(x) - \frac{1}{n}\right\}.$$

The inequality

$$f_n(x) = \max\left\{g_n^*(x) - \frac{1}{n} - M, f_n^*(x) - \frac{1}{n}\right\} < \max\{g^*(x) - M, f_n^*(x)\} \le f^*(x)$$

implies $f_n^* - \frac{1}{n} \le f_n < f^*$, so that $f_n \to f^*$. The inequality

$$\max\left\{g_n^*(x) - \frac{1}{n} - M, f_n^*(x) - \frac{1}{n}\right\} < \max\left\{g_{n+1}^*(x) - \frac{1}{n+1} - M, f_{n+1}^*(x) - \frac{1}{n+1}\right\}$$

implies $f_n < f_{n+1}$ for $n \in \mathbb{N}$. The same holds true for the sequence $\{g_n\}_{n\in\mathbb{N}}$; therefore

$$f_1 < f_2 < f_3 < \ldots \to f^*,$$
$$g_1 < g_2 < g_3 < \ldots \to g^*.$$

Because

$$\max\left\{f_n^*(x) - \frac{1}{n} - M, g_n^*(x) - \frac{1}{n}\right\} - M \le \max\left\{g_n^*(x) - \frac{1}{n} - M, f_n^*(x) - \frac{1}{n}\right\}$$
$$\le \max\left\{f_n^*(x) - \frac{1}{n} - M, g_n^*(x) - \frac{1}{n}\right\} + M,$$

obviously,

$$g_n(x) - M \le f_n(x) \le g_n(x) + M;$$

i.e.,

$$|f_n(x) - g_n(x)| \le M \text{ for every } x \in [0,1],\ n \in \mathbb{N}.$$

To construct functions f and g of required properties we use the approach of Theorem 3.

The set $D_{f^*} \cup D_{g^*}$ is of type F_σ, and thus let

$$D_{f^*} \cup D_{g^*} = \bigcup_{n=1}^{\infty} D_n,$$

where $\{D_n\}_{n\in\mathbb{N}}$, is an increasing sequence of closed nowhere dense sets. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. The functions f_n and g_n are uniformly continuous on $[0,1]$. Thus the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ determines a sequence of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ such that for every $x_1, x_2 \in [0,1]$

$$|x_1 - x_2| < \delta_n \Rightarrow \begin{array}{l} |f_n(x_1) - f_n(x_2)| < \varepsilon_n \\ |g_n(x_1) - g_n(x_2)| < \varepsilon_n \end{array} \tag{$*$}$$

holds. The set E is of type F_σ, bilaterally $\mathfrak{c}$-dense in itself, and therefore there exists a family of closed sets $\{P_\alpha;\ \alpha \geq 1\}$ satisfying the following conditions:

(4) $\displaystyle\bigcup_{\alpha\geq 1} P_\alpha = E;$

(5) for every $x \in D_n$ there exist $a, b \in P_n$ such that $a < x < b \wedge\ b - a < \delta_n;$

(6) $P_{\alpha_1} \subset_{\mathfrak{c}} P_{\alpha_2},\ $ for $\alpha_1 < \alpha_2$.

If $n \leq \alpha < n+1$, then the set P_α is said to be associated to the pair of functions

$$f_\alpha = f_n + (\alpha - n)(f_{n+1} - f_n)$$
$$g_\alpha = g_n + (\alpha - n)(g_{n+1} - g_n)\,.$$

Now, define the function f by

$$f(x) = \begin{cases} f_{\alpha(x)}(x)\,, & \text{for } x \in E,\ \alpha(x) = \inf\{\alpha, x \in P_\alpha\} \\ f^*(x)\,, & \text{for } x \in [0,1] \setminus E \end{cases}$$

and, analogously, the function g. The construction of f and g coincides with the fitting function of Theorem 3. Thus

$$f, g \in Dlsc,\ f \leq f^*,\ g \leq g^*,$$
$$\{x, f(x) \neq f^*(x)\} = \{x, g(x) \neq g^*(x)\} = E.$$

Now we show that the function $|f(x) - g(x)|$ is bounded by the constant $3M$.

If $x \notin E$, then

$$|f(x) - g(x)| = |f^*(x) - g^*(x)| < M.$$

If $x \in E$, then there exists a real number $\alpha(x) \geq 1$ such that $x \in P_\alpha$ for $\alpha > \alpha(x)$, and $x \notin P_\alpha$ for $\alpha < \alpha(x)$. Let us assume that $n \leq \alpha(x) < n+1$.

Then

$$\begin{aligned}|f(x)-g(x)| &= |f_n(x)+(\alpha(x)-n)(f_{n+1}(x)-f_n(x))-g_n(x)\\ &\qquad -(\alpha(x)-n)(g_{n+1}(x)-g_n(x))|\\ &\le (\alpha(x)-(n-1))|f_n(x)-g_n(x)|+(\alpha(x)-n)|f_{n+1}(x)-g_{n+1}(x)|\le 3M.\end{aligned}$$

It means that

$$|f(x)-g(x)|\le 3M \text{ for each } x\in[0,1].$$

□

In what follows we will show the existence of the function answering Maliszewski's questions.

Let F be a perfect, nowhere dense subset of the interval $[0,1]$, $\lambda(F)>0$, such that

$$F=[0,1]\setminus\bigcup_{n=1}^{\infty} I_n,$$

where I_n are open contiguous intervals of the set F. Let

$$\sum_{k=1}^{\infty}\alpha_k$$

be any convergent series of positive real numbers. Let $\{k_n\}_{n=1}^{\infty}$ be an increasing sequence of natural numbers such that the sequence

$$\sigma_{k_n}=\sum_{k=n}^{k_n}\Big(\frac{1}{k}-\alpha_k\Big),\ n\in\mathbb{N},$$

diverges to ∞. For every $n\in\mathbb{N}$, in the open interval I_n, define a finite sequence of perfect sets

$$P_1^n\subset_{\mathfrak{c}} P_2^n\subset_{\mathfrak{c}}\cdots\subset_{\mathfrak{c}} P_{k_n+1}^n,$$

such that the Lebesgue measure $\lambda(P_{k_n+1}^n)=0$. Next, we will deal with functions f_1^* and f_2^* defined as follows:

For $x\in I_n,\ n\in\mathbb{N}$,

$$f_1^*(x)=\begin{cases}1, & \text{for } x\in P_1^n\\ k+\frac{1+(-1)^k}{2(n+k-1)}, & \text{for } x\in P_k^n\setminus P_{k-1}^n,\ k=2,3,...,k_n+1\\ k_n+2, & \text{for } x\in I_n\setminus P_{k_n+1}\end{cases}$$

$$f_2^*(x)=\begin{cases}1+\frac{1}{n}, & \text{for } x\in P_1^n\\ k+\frac{1-(-1)^k}{2(n+k-1)}, & \text{for } x\in P_k^n\setminus P_{k-1}^n,\ k=2,3,...,k_n+1\\ k_n+2, & \text{for } x\in I_n\setminus P_{k_n+1}\end{cases}$$

and

$$f_1^*(x) = f_2^*(x) = 0, \text{ for } x \in F.$$

Obviously, $f_1^*,\ f_2^* \in lsc$,

$$D_{f_1^*} = D_{f_2^*} = F \cup P, \text{ where } P = \bigcup_{n=1}^{\infty} P_{k_n+1}^n,$$

and

$$f_1^*(x) - f_2^*(x) = \begin{cases} -\frac{1}{n}, & \text{for } x \in P_1^n \\ \frac{(-1)^k}{n+k-1}, & \text{for } x \in P_k^n \setminus P_{k-1}^n,\ k = 2, ..., k_n+1,\ n \in \mathbb{N}, \\ 0, & \text{for } x \in I_n \setminus P_k^n \end{cases}$$

$$f_1^*(x) - f_2^*(x) = 0, \quad \text{for } x \in F.$$

Thus

$$|f_1^*(x) - f_2^*(x)| \le \frac{1}{n} \quad \text{for every } x \in I_n.$$

Now let E be a set of type F_σ such that $\lambda(E) = 0$, E is bilaterally $\mathfrak{c}$-dense in $F \cup P$, and

$$E \bigcap (F \bigcup P) = \emptyset.$$

(With respect to Lemma 7 in [4], it is possible to require that the set E is bilaterally $\mathfrak{c}$-dense in itself.) According to Lemma 4, there exist functions $f_1,\ f_2 \in Dlsc$ such that

$$\{x;\ f_1(x) \ne f_1^*(x)\} = \{x;\ f_2(x) \ne f_2^*(x)\} = E,$$

and

$$(7) \quad |f_1(x) - f_2(x)| \le \frac{3}{n}, \text{ for every } x \in I_n,\ n \in \mathbb{N},$$

$$(8) \quad f_1(x) = f_2(x) = 0, \text{ for every } x \in F.$$

The function $f = f_1 - f_2 \in Dlsc - Dlsc$, and, due to Proposition 3 in [3], the function f has the Darboux property. If a point $x \in C_{f_1^*}$ and $x \notin E$, then according to the part (2) in the proof of Theorem 3, $x \in C_{f_1}$. Therefore

$$D_{f_1} \subset E \cup F \cup P,$$

and the same holds true for the set D_{f_2}. Let $I_n = (a_n, b_n)$, $n \in \mathbb{N}$, and let $a_n^1, b_n^1 \notin E$ be such that

$$a_n < a_n^1 < \min P_{k_n+1}^n < \max P_{k_n+1}^n < b_n^1 < b_n.$$

Since

$$f_1^*(x) = f_2^*(x), \text{ for every } x \in \left[a_n, a_n^1\right] \cup \left[b_n^1, b_n\right],$$

it is possible to require that

$$f_1(x) = f_2(x), \text{ for every } x \in \left[a_n, a_n^1\right] \cup \left[b_n^1, b_n\right];$$

that is,

$$f_1(x) - f_2(x) = 0, \text{ for every } x \in \left[a_n, a_n^1\right] \cup \left[b_n^1, b_n\right], \, n \in \mathbb{N}.$$

From the foregoing, together with (7) and (8) it follows that

$$\lim_{x \to x_0} f_1(x) - f_2(x) = 0 = f_1(x_0) - f_2(x_0) \text{ for every } x_0 \in F.$$

Obviously, the set $F \subset C_f$, so that the set of discontinuity points of the function f is a subset of $E \bigcup P$. Since $\lambda(E \bigcup P) = 0$, the function f is bounded and a.e. continuous. Because $f \in lsc - lsc$ there are infinitely many pairs of lower semicontinuous functions such that the function f equals their difference. Let l, $d \in lsc$ be any such pair; i.e. $f = l - d$, or $l = f + d$, respectively. We will proceed similarly to the proof of the Proposition 2 in [6]. Since a function is bounded or a.e. continuous iff the sum of the function and constant is bounded or a.e. continuous, we may assume $l \geq 0$, $d \geq 0$. Choose a point $x_1 \in P_1^n \subset I_n$. Then

$$f(x_1) = f_1(x_1) - f_2(x_1) = f_1^*(x_1) - f_2^*(x_1) = -\frac{1}{n}.$$

Since

$$l(x_1) = f(x_1) + d(x_1),$$

and by assumption $l(x_1) \geq 0$, we obtain $d(x_1) \geq \frac{1}{n}$. The function d is lower semicontinuous, and thus there exists a neighbourhood $U_0^n \subset I_n$ of the point x_1 such that $d(x) \geq \frac{1}{n} - \alpha_n$, for every $x \in U_0^n$. We will use the notation

$$d(U_0^n) \geq \frac{1}{n} - \alpha_n. \tag{1}$$

Since $P_1 \subset_c P_2$, subsequently choose the point $x_2 \in P_2^n \cap U_0^n$. Then

$$f(x_2) = f_1(x_2) - f_2(x_2) = f_1^*(x_2) - f_2^*(x_2) = \frac{1}{n+1},$$

and, from (1), it follows

$$l(x_2) = f(x_2) + d(x_2) \geq \frac{1}{n} - \alpha_n + \frac{1}{n+1}.$$

The function l is lower semicontinuous. Therefore there exists a neighbourhood $U_1^n \subset U_0^n \subset I_n$ of the point x_2 such that

$$l(U_1^n) \geq \frac{1}{n} - \alpha_n + \frac{1}{n+1} - \alpha_{n+1}. \tag{2}$$

Let us repeat the algorithm: Choose a point $x_3 \in P_3^n \cap U_1^n$. Then

$$f(x_3) = -\frac{1}{n+2}.$$

From (2) follows

$$l(x_3) \geq \frac{1}{n} - \alpha_n + \frac{1}{n+1} - \alpha_{n+1},$$

and, consequently,

$$d(x_3) = l(x_3) - f(x_3) \geq \frac{1}{n} - \alpha_n + \frac{1}{n+1} - \alpha_{n+1} + \frac{1}{n+2}.$$

Thus there exists a neighbourhood $U_2^n \subset U_1^n \subset U_0^n \subset I_n$ of the point x_3 such that

$$d(U_2^n) \geq \frac{1}{n} - \alpha_n + \frac{1}{n+1} - \alpha_{n+1} + \frac{1}{n+2} - \alpha_{n+2}.$$

Applying the same algorithm we find a neighbourhood $U_{k_n}^n \subset I_n$ of a point $x_{k_n+1} \in P_{k_n+1}^n$ such that

$$d(U_{k_n}^n) \geq \frac{1}{n} - \alpha_n + \frac{1}{n+1} - \alpha_{n+1} + \cdots + \frac{1}{n+k_n} - \alpha_{k_n} = \sigma_{k_n}, \text{ if } k_n \text{ is even,}$$

or

$$l(U_{k_n}^n) \geq \frac{1}{n} - \alpha_n + \frac{1}{n+1} - \alpha_{n+1} + \cdots + \frac{1}{n+k_n} - \alpha_{k_n} = \sigma_{k_n}, \text{ if } k_n \text{ is odd.}$$

Consequently, it follows that, for every $x_0 \in F$, there exists a sequence of points $x_{n_i} \in I_{n_i},\ i = 1, 2, \ldots,\ x_{n_i} \to x_0$ such that

$$\lim_{i\to\infty} d(x_{n_i}) \geq \lim_{i\to\infty} \sigma_{k_{n_i}} = \infty$$

or

$$\lim_{i\to\infty} l(x_{n_i}) \geq \lim_{i\to\infty} \sigma_{k_{n_i}} = \infty.$$

Because the function f is bounded and $f = l - d$, the equalities

$$\lim_{i\to\infty} d(x_{n_i}) = \lim_{i\to\infty} l(x_{n_i}) = \infty$$

hold true. Hence the functions l and d are necessarily unbounded on $[0,1]$ and discontinuous at each point of the set F with positive Lebesgue measure.

Thus, finally, in $lsc - lsc$ there exists a bounded, Darboux, a.e. continuous function such

a) it cannot be written as the difference of two bounded lower semicontinuous functions,

b) and it cannot be written as the difference of two a.e. continuous lower semicontinuous functions.

References

[1] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics, **659**, Springer, Berlin, 1978.

[2] A. Maliszewski, *Darboux Property and Quasi-continuity: A Uniform Approach*, Wydaw. Uczelniane WSP, 1996.

[3] A. Maliszewski, *On the differences of Darboux upper semicontinuous functions*, Real Anal. Exchange., **21(1)**, (1995–1996), 258–263.

[4] R. Menkyna and Ľ. Mydielka, *Approximations by Darboux functions in the Baire one class*, Tatra Mt. Math. Publ., **55** (2013), 57–66.

[5] R. Menkyna, *On approximations of semicontinuous Functions by Darboux semicontinuous Functions*, Real Anal. Exchange, **35(2)** (2009–2010), 423–430.

[6] R. Menkyna, *On representations of Baire one functions as the sum of lower and upper semicontinuous functions*, Real. Anal. Exchange, **38(1)**, (2012–2013), 169–176.

[7] M. Morayne, *Sierpiński's hierarchy and locally Lipschitz functions*, Fund. Math., **147(1)** (1995), 73–82.

[8] W. Sierpiński, *Sur les fonctions développables en séries absolument convergentes de fonctions continues*, Fund. Math., **2** (1921), 15–27.