RESEARCH

Real Analysis Exchange
Vol. 41(1), 2016, pp. 101–122

Horst Alzer, Morsbacher Str. 10, 51545 Waldbröl, Germany.
email: `h.alzer@gmx.de`

INEQUALITIES FOR MEAN VALUES IN TWO VARIABLES

Abstract

We present various inequalities for means in two variables. One of our results states that the inequalities

$$0 \leq \frac{1}{M_r} - \frac{1}{M_s} \leq \frac{1}{G} - \frac{1}{A} \quad (r, s \geq 0)$$

hold for all $x, y > 0$ if and only if $0 \leq s - r \leq 1$. Here, $A = A(x,y) = (x+y)/2$, $G = G(x,y) = \sqrt{xy}$ and $M_t = M_t(x,y) = [(x^t + y^t)/2]^{1/t}$ denote the arithmetic, geometric and power mean of x and y, respectively.

1 Introduction

In view of their importance in various parts of mathematics, like, for instance, probability theory, statistics, and the theory of special functions, means and mean value families have attracted the attention of researchers since many years. In this paper we are concerned with certain mean values in two variables. Numerous articles and monographs were published providing remarkable properties of means of two variables. We refer to [10], [19], [20], [21], [23], [26], [27], [28], [30], and the references therein. In particular, we can find many interesting inequalities for these mean values; see [1], [3], [9], [11], [12], [13], [14], [15], [16], [29], [32], [34], [35], [37], [39], [40], [41], [42], [43], [45], [46], [47], [48], [49], [50], [53]. It is the aim of this paper to continue the study of this subject and to present several new inequalities involving the classical arithmetic, geometric and power means as well as the Heinz mean and its complementary.

Mathematical Reviews subject classification: Primary: 26D07; Secondary: 26A48, 26A51
Key words: Inequalities, Mean values, Completely monotonic
Received by the editors January 12, 2015
Communicated by: Paul D. Humke

Throughout, we maintain the notations given in this section. The arithmetic and geometric means,

$$A = A(x,y) = \frac{x+y}{2} \quad \text{and} \quad G = G(x,y) = \sqrt{xy},$$

were already known in the time of Pythagoras around 500 BC. Both are members of the one-parameter family of power means

$$M_t = M_t(x,y) = \Big(\frac{x^t + y^t}{2}\Big)^{1/t} \quad (t \in \mathbb{R} \setminus \{0\}),$$
$$M_0 = M_0(x,y) = \lim_{t \to 0} M_t(x,y) = \sqrt{xy}.$$

The function $t \mapsto M_t(x,y)$ with $x \neq y$ is strictly increasing on $\mathbb{R}$ with

$$\lim_{t \to -\infty} M_t(x,y) = \min\{x,y\} \quad \text{and} \quad \lim_{t \to \infty} M_t(x,y) = \max\{x,y\}.$$

These and other properties of $M_t(x,y)$ can be found, for instance, in [11, chapter III] and [21, chapter II].

The Heinz mean of x and y of order t (introduced by Bhatia [6] in 2006) is defined by

$$H_t(x,y) = \frac{x^t y^{1-t} + x^{1-t} y^t}{2} \quad (0 \leq t \leq 1).$$

We have

$$H_0(x,y) = \frac{x+y}{2}, \quad H_{1/2}(x,y) = \sqrt{xy}, \quad H_t(x,y) = H_{1-t}(x,y).$$

Since $H_t(x,y)$ is decreasing on $[0,1/2]$ with respect to t, we obtain a refinement of the classical arithmetic mean - geometric mean inequality:

$$G(x,y) \leq H_t(x,y) \leq A(x,y).$$

A corresponding result for positive definite matrices was proved by Heinz [22] in 1951.

The following elegant upper and lower bounds for $H_t(x,y)$ and $H_t(x,y)^2$ were published in 2010 and 2011 by Kittaneh and Manasrah [24], [25].

If $x, y > 0$ and $t \in [0,1]$, then

$$A(x,y) + r_0\,(\sqrt{x} - \sqrt{y})^2 \leq H_t(x,y) \leq A(x,y) + R_0\,(\sqrt{x} - \sqrt{y})^2 \tag{1.1}$$

where

$$r_0 = -\max\{t, 1-t\} \quad \text{and} \quad R_0 = -\min\{t, 1-t\}$$

and

$$A(x,y)^2 + r_1\,(x-y)^2 \le H_t(x,y)^2 \le A(x,y)^2 + R_1\,(x-y)^2 \tag{1.2}$$

where

$$r_1 = -\frac{1}{2}\max\{t, 1-t\} \quad \text{and} \quad R_1 = -\frac{1}{2}\min\{t, 1-t\}.$$

Interesting matrix versions of (1.1), (1.2) and numerous related inequalities for positive real numbers can be found in [5], [6], [7], [17], [18], [24], [25].

The weighted arithmetic and geometric means of x and y are given by

$$A_t(x,y) = tx + (1-t)y \quad \text{and} \quad G_t(x,y) = x^t y^{1-t} \quad (0 \le t \le 1).$$

We have the representation

$$H_t(x,y) = A\big(G_t(x,y), G_{1-t}(x,y)\big). \tag{1.3}$$

If we exchange in (1.3) A and G, then we obtain the complementary Heinz mean of x and y of order t:

$$H_t^*(x,y) = G\big(A_t(x,y), A_{1-t}(x,y)\big) = \sqrt{\big(tx + (1-t)y\big)\,\big((1-t)x + ty\big)}.$$

The function $t \mapsto H_t^*(x,y)$ is increasing on $[0,1/2]$ and satisfies $H_t^*(x,y) = H_{1-t}^*(x,y)$. This yields

$$G(x,y) = H_0^*(x,y) \le H_t^*(x,y) \le H_{1/2}^*(x,y) = A(x,y). \tag{1.4}$$

It is the aim of this paper to present various new inequalities for the means G, A, M_t, H_t, and H_t^*. In particular, we obtain several improvements of the arithmetic mean - geometric mean inequality and refinements of (1.1), (1.2) and (1.4). Moreover, we study monotonicity properties of H_t and $1/H_t^*$.

2 Inequalities for means

First, we offer a chain of four inequalities which provides improvements of the arithmetic mean - geometric mean inequality.

Theorem 2.1. *Let λ and μ be real numbers. The inequalities*

$$G < \frac{(x+G)(y+G)}{4\,G} < \lambda\big(G + A\big) + \mu\,\frac{G^2 + A^2}{G + A} < \frac{(x+A)(y+A)}{4\,A} < A \tag{2.1}$$

are valid for all $x, y > 0$ with $x \ne y$ if and only if

$$2\lambda + \mu = 1 \quad \textit{and} \quad \frac{1}{2} < \lambda + \mu \le \frac{3}{4}. \tag{2.2}$$

PROOF. Let $x, y > 0$ and $x \neq y$. Then,

$$\frac{(x+G)(y+G)}{4\,G} - G = \frac{A-G}{2} > 0 \quad \text{and}$$
$$A - \frac{(x+A)(y+A)}{4\,A} = \frac{A^2-G^2}{4\,A} > 0.$$

This settles the first and the last inequality in (2.1). We define

$$F(t) = t(G+A) + (1-2t)\frac{G^2+A^2}{G+A}.$$

Since

$$F'(t) = -\frac{(G-A)^2}{G+A} < 0,$$

we obtain

$$F(1/2) < F(t) \leq F(1/4), \quad \text{if} \quad 1/4 \leq t < 1/2.$$

We have

$$F(1/2) = \frac{(x+G)(y+G)}{4\,G} \quad \text{and}$$
$$F(1/4) - \frac{(x+A)(y+A)}{4\,A} = -\frac{G(G-A)^2}{4\,A(G+A)} < 0.$$

Thus, if (2.2) holds, then the second and the third inequality in (2.1) are valid. Next, we assume that (2.1) holds for all $x, y > 0$ with $x \neq y$. We fix x and let y tend to x. Then, (2.1) leads to $2\lambda x + \mu x = x$. Therefore, $2\lambda + \mu = 1$. If $\lambda + \mu = 1/2$, then $\lambda = 1/2$ and $\mu = 0$. Hence,

$$\frac{(x+G)(y+G)}{4\,G} = \lambda\,(G+A) + \mu\,\frac{G^2+A^2}{G+A}.$$

A contradiction. It follows that $\lambda + \mu \neq 1/2$. We let y tend to 0. Then, (2.1) gives

$$\frac{x}{4} \leq \lambda\,\frac{x}{2} + \mu\,\frac{x}{2} \leq \frac{3x}{8}.$$

Thus,

$$\frac{1}{2} < \lambda + \mu \leq \frac{3}{4}.$$

This completes the proof. □

The next theorems provide refinements of $G \leq A$ by using power means.

Theorem 2.2. *Let* r, s *be real numbers. The inequalities*

$$G \le \frac{M_r + M_s}{2} \le A \tag{2.3}$$

hold for all $x, y > 0$ *if and only if* $0 \le r + s \le 2$.

PROOF. We assume that that (2.3) is valid for all $x, y > 0$. Let

$$U_{r,s}(x) = M_r(x,1) + M_s(x,1) - 2\,G(x,1)$$

and

$$V_{r,s}(x) = 2\,A(x,1) - M_r(x,1) - M_s(x,1).$$

Then, for $x > 0$,

$$U_{r,s}(x) \ge 0 = U_{r,s}(1) \quad \text{and} \quad V_{r,s}(x) \ge 0 = V_{r,s}(1).$$

Since $U'_{r,s}(1) = V'_{r,s}(1) = 0$, we obtain

$$U''_{r,s}(1) = \frac{r+s}{4} \ge 0 \quad \text{and} \quad V''_{r,s}(1) = \frac{2-(r+s)}{4} \ge 0.$$

Hence, $0 \le r + s \le 2$.
Next, we suppose that $0 \le r + s \le 2$. Using the identity

$$M_r(x,y)\,M_{-r}(x,y) = G(x,y)^2$$

and the fact that $r \mapsto M_r(x,y)$ is increasing on $\mathbb{R}$, we get

$$\begin{aligned} M_r(x,y) + M_s(x,y) &\ge M_r(x,y) + M_{-r}(x,y) \\ &= 2\,G(x,y) + \frac{\big(M_r(x,y) - G(x,y)\big)^2}{M_r(x,y)} \ge 2\,G(x,y). \end{aligned}$$

This settles the left-hand side of (2.3). We set $r = 1 + t$. Then, $s \le 1 - t$ and

$$M_r(x,y) + M_s(x,y) \le M_{1+t}(x,y) + M_{1-t}(x,y).$$

Thus, it remains to show that

$$M_{1+t}(x,y) + M_{1-t}(x,y) \le 2\,A(x,y)$$

is valid for $x, y, t > 0$. Since

$$M_r(x,y) = y\,M_r(x/y,1),$$

we may assume that $y = 1$ and $x \geq 1$. Let

$$P_t(x) = 2\,A(x,1) - M_{1-t}(x,1) - M_{1+t}(x,1).$$

We have

$$P_t(1) = P_1'(1) = 0 \tag{2.4}$$

and

$$\begin{aligned}&\frac{x^{1+t}}{t}(x^{1+t}+1)^2(x^{1-t}+1)^2 P_t''(x)\\ &= (x^{1+t}+1)^2 M_{1-t}(x,1) - x^{2t}(x^{1-t}+1)^2 M_{1+t}(x,1).\end{aligned} \tag{2.5}$$

Let

$$Q_t(x) = \log\big[(x^{1+t}+1)^2 M_{1-t}(x,1)\big] - \log\big[x^{2t}(x^{1-t}+1)^2 M_{1+t}(x,1)\big].$$

Since $Q_t(1) = 0$ and

$$x(x^{1+t}+1)(x^{1-t}+1)Q_t'(x) = x^{1-t}(x^{2t}-1) + 2t(x^2-1) \geq 0,$$

we get $Q_t(x) \geq 0$ for $x \geq 1$ and $t > 0$. From (2.5) we conclude that $P_t''(x) \geq 0$, so that (2.4) implies that P_t is non-negative on $[1,\infty)$. This proves the right-hand inequality of (2.3). $\square$

The following companion of (2.1) holds.

Corollary 2.3. *For all $x, y > 0$ with $x \neq y$ and all real numbers t we have*

$$G \leq \frac{G^2 + M_t^2}{4\,M_t} + \frac{G}{2} < \frac{G + A}{2} \leq \frac{G^2 + M_t^2}{4\,M_t} + \frac{A}{2} < A.$$

The sign of equality holds if and only if $t = 0$.

Proof. The first and the third inequality are equivalent to $(M_t - G)^2 \geq 0$. Using (2.3) with $r = t$ and $s = -t$ we obtain

$$\frac{A}{2} - \frac{G^2 + M_t^2}{4\,M_t} = \frac{1}{2}\Big(A - \frac{M_t + M_{-t}}{2}\Big) \geq 0.$$

Since $x \neq y$, strict inequality holds. This settles the second and the fourth inequality. $\square$

Theorem 2.4. *Let r, s be real numbers. The inequalities*

$$G \le \sqrt{M_r\, M_s} \le A \tag{2.6}$$

hold for all $x, y > 0$ if and only if $0 \le r+s \le 2$.

PROOF. We assume that (2.6) is valid for all $x, y > 0$. Then we have for $x > 0$:

$$Y_{r,s}(x) = M_r(x,1)\, M_s(x,1) - G(x,1)^2 \ge 0 = Y_{r,s}(1)$$

and

$$Z_{r,s}(x) = A(x,1)^2 - M_r(x,1)\, M_s(x,1) \ge 0 = Z_{r,s}(1).$$

Since

$$Y'_{r,s}(1) = 0, \quad Y''_{r,s}(1) = \frac{r+s}{4} \ge 0$$

and

$$Z'_{r,s}(1) = 0, \quad Z''_{r,s}(1) = \frac{2-(r+s)}{4} \ge 0,$$

we obtain $0 \le r+s \le 2$.
Conversely, if $0 \le r+s \le 2$, then

$$\begin{aligned} G(x,y) &= \sqrt{M_r(x,y)\, M_{-r}(x,y)} \le \sqrt{M_r(x,y)\, M_s(x,y)} \\ &\le \frac{M_r(x,y) + M_s(x,y)}{2} \le A(x,y), \end{aligned}$$

where the right-hand inequality follows from Theorem 2.2. □

Remark 2.5. The referee pointed out that (2.6) can be refined. For all $x, y > 0$ and $r, s \ge 0$ with $r+s \le 2$ we have

$$G \le \sqrt{M_r\, M_s} \le M_{(r+s)/2} \le A.$$

These inequalities follow from the monotonicity of $t \mapsto M_t(x,y)$ and the concavity of $t \mapsto \log M_t(x,y)$ on $[0,\infty)$. See [4], [11, pp. 168-169], [33], [44].

In order to prove the next theorem we need a functional inequality for convex functions which was published by Petrović [36] in 1932; see also [31, pp. 22-23].

Lemma 2.6. *If the function f is convex on $[0,\infty)$, then we have for $x, y \ge 0$:*

$$f(x) + f(y) \le f(x+y) + f(0).$$

Theorem 2.7. *Let r, s be nonnegative real numbers. The inequalities*

$$0 \le \frac{1}{M_r} - \frac{1}{M_s} \le \frac{1}{G} - \frac{1}{A} \tag{2.7}$$

hold for all $x, y > 0$ if and only if $0 \le s - r \le 1$.

Proof. Since $t \mapsto 1/M_t(x,y)$ $(x \ne y)$ is strictly decreasing on $\mathbb{R}$, we conclude from the left-hand side of (2.7) that $r \le s$. Let

$$B_{r,s}(x) = \frac{1}{G(x,1)} - \frac{1}{A(x,1)} - \frac{1}{M_r(x,1)} + \frac{1}{M_s(x,1)}.$$

We assume that the right-hand of (2.7) is valid for all $x, y > 0$. Then, for $x > 0$,

$$B_{r,s}(x) \ge 0 = B_{r,s}(1).$$

Since $B'_{r,s}(1) = 0$, we obtain

$$B''_{r,s}(1) = \frac{r - s + 1}{4} \ge 0.$$

Thus, $s \le r + 1$.

Next, let $r \le s \le r + 1$. Then, the first inequality in (2.7) holds for all $x, y > 0$. Moreover, we get

$$\frac{1}{M_r(x,y)} - \frac{1}{M_s(x,y)} \le \frac{1}{M_r(x,y)} - \frac{1}{M_{r+1}(x,y)}.$$

Therefore, to prove the second inequality in (2.7) it suffices to show that if $x \ge y > 0$ and $r > 0$, then

$$\frac{1}{M_r(x,y)} - \frac{1}{M_{r+1}(x,y)} \le \frac{1}{G(x,y)} - \frac{1}{A(x,y)}. \tag{2.8}$$

Let $t > 0$. We define

$$C_r(t) = \frac{1}{G(e^t, e^{-t})} - \frac{1}{A(e^t, e^{-t})} - \frac{1}{M_r(e^t, e^{-t})} + \frac{1}{M_{r+1}(e^t, e^{-t})}$$

and

$$D_t(r) = \big(\cosh(tr)\big)^{-1/r} \quad (r \ne 0), \quad D_t(0) = \lim_{r \to 0} D_t(r) = 1.$$

Then,

$$C_r(t) = D_t(0) - D_t(1) - D_t(r) + D_t(r+1). \tag{2.9}$$

We obtain for $r > 0$:

$$r^3 \frac{d^2}{dr^2} \log D_t(r) = E(rt), \tag{2.10}$$

where

$$E(x) = 2x \tanh(x) - \Big(\frac{x}{\cosh(x)}\Big)^2 - 2\log(\cosh(x)).$$

Since

$$E(0) = 0 \quad \text{and} \quad E'(x) = 2\Big(\frac{x}{\cosh(x)}\Big)^2 \tanh(x) \geq 0 \quad \text{for} \quad x \geq 0,$$

we conclude that E is non-negative on $[0, \infty)$. From (2.10) we obtain that $D_t(r)$ is log-convex on $[0, \infty)$ with respect to r. It follows that D_t is convex on $[0, \infty)$. Applying Lemma 2.6 yields

$$D_t(r) + D_t(1) \leq D_t(r + 1) + D_t(0). \tag{2.11}$$

From (2.9) and (2.11) we conclude that $C_r(t) \geq 0$. We set $t = (1/2) \log(x/y)$. Then,

$$0 \leq \frac{1}{\sqrt{xy}} C_r\Big(\frac{1}{2} \log \frac{x}{y}\Big) = \frac{1}{G(x,y)} - \frac{1}{A(x,y)} - \frac{1}{M_r(x,y)} + \frac{1}{M_{r+1}(x,y)}.$$

This settles (2.8). □

The next lemma plays an important role in the proof of following two theorems; see [21, p. 106].

Lemma 2.8. *Let f and g be functions which are continuous on $[0,1]$ and differentiable on $(0,1)$. Moreover, let $f(1) = g(1) = 0$ and $g' \neq 0$ on $(0,1)$. If f'/g' is increasing on $(0,1)$, then f/g is also increasing on $(0,1)$.*

We are now in a position to show that in (1.1) the given factors r_0 and R_0 can be replaced by better constants.

Theorem 2.9. *Let $t \in (0,1)$. For all $x, y > 0$ we have*

$$A(x,y) + \delta_t (\sqrt{x} - \sqrt{y})^2 \leq H_t(x,y) \leq A(x,y) + \Delta_t (\sqrt{x} - \sqrt{y})^2 \tag{2.12}$$

with the best possible factors

$$\delta_t = -\frac{1}{2} \quad \textit{and} \quad \Delta_t = -2t(1-t). \tag{2.13}$$

Proof. It suffices to prove (2.12) for $x \in (0,1)$ and $y=1$. We define

$$R(x) = H_t(x^2,1) - A(x^2,1) \quad \text{and} \quad S(x) = (x-1)^2.$$

Then,

$$R(1) = R'(1) = 0, \quad S(1) = S'(1) = 0 \quad \text{and} \quad S' \neq 0 \neq S'' \quad \text{on} \quad (0,1).$$

Let

$$T(x) = \frac{R''(x)}{S''(x)} = \frac{1}{2}\Big(t(2t-1)x^{2t-2} + (1-t)(1-2t)x^{-2t} - 1\Big).$$

We have

$$T'(x) = t(1-t)(2t-1)x^{-2t-1}\big(1 - x^{2(2t-1)}\big) \geq 0.$$

Applying Lemma 2.8 reveals that R'/S' is increasing on $(0,1)$. Applying Lemma 2.8 again gives that R/S is also increasing on $(0,1)$. It follows that

$$W_t(x) = \frac{H_t(x,1) - A(x,1)}{(\sqrt{x}-1)^2}$$

is increasing on $(0,1)$. Since

$$\lim_{x\to 0} W_t(x) = -\frac{1}{2} \quad \text{and} \quad \lim_{x \to 1} W_t(x) = -2t(1-t),$$

we conclude that (2.12) holds and that the factors given in (2.13) are sharp. □

The following improvement of double-inequality (1.2) is valid.

Theorem 2.10. *Let $t \in (0,1)$. For all $x, y > 0$ we have*

$$A(x,y)^2 + \theta_t\,(x-y)^2 \leq H_t(x,y)^2 \leq A(x,y)^2 + \Theta_t\,(x-y)^2 \tag{2.14}$$

with the best possible factors

$$\theta_t = -\frac{1}{4} \quad \textit{and} \quad \Theta_t = -t(1-t). \tag{2.15}$$

Proof. In order to prove (2.14) for $x \in (0,1)$ and $y=1$ we apply Lemma 2.8. Let

$$I(x) = H_t(x,1)^2 - A(x,1)^2 \quad \text{and} \quad J(x) = (x-1)^2.$$

Then,

$$I(1) = I'(1) = 0, \quad J(1) = J'(1) = 0, \quad J' \neq 0 \neq J'' \quad \text{on} \quad (0,1)$$

and

$$-4\,\frac{I''(x)}{J''(x)} = t(1-2t)x^{2t-2} - (1-t)(1-2t)x^{-2t} + 1 = K(x), \quad \text{say.}$$

Since

$$K'(x) = 2t(1-t)(1-2t)x^{-2t-1}\big(1 - x^{2(2t-1)}\big) \le 0,$$

we conclude that I''/J'' is increasing on $(0,1)$. This implies that I'/J' and I/J are also increasing on $(0,1)$. We have

$$\lim_{x\to 0} \frac{I(x)}{J(x)} = -\frac{1}{4} \quad \text{and} \quad \lim_{x\to 1} \frac{I(x)}{J(x)} = -t(1-t).$$

This proves (2.14) and reveals that the factors in (2.15) are best possible. □

The logarithmic mean

$$L = L(x,y) = \frac{x-y}{\log x - \log y} \quad (x, y > 0;\ x \ne y)$$

has interesting applications in physics, chemistry and economics. A known result states that L separates the geometric and arithmetic means,

$$G < L < A; \tag{2.16}$$

see [31, pp. 272-274]. For more information on this mean value we refer to [38]. The logarithmic mean plays a role in the proof of the next theorem which offers sharp upper and lower bounds for the ratio of two mean value differences.

Theorem 2.11. *Let t and λ be real numbers with $t \in (0,1)$, $t \ne 1/2$ and $\lambda \ge 1$. For all positive real numbers x, y with $x \ne y$ we have*

$$4t(1-t) < \frac{A(x,y)^\lambda - H_t(x,y)^\lambda}{A(x,y)^\lambda - G(x,y)^\lambda} < 1. \tag{2.17}$$

Both bounds are sharp.

Proof. From

$$G(x,y) < H_t(x,y) < A(x,y) \quad (x \ne y;\ 0 < t < 1,\ t \ne 1/2)$$

we conclude that the second inequality in (2.17) is valid. Next, we show that if $t \in (0,1)$, $t \ne 1/2$, $\lambda \ge 1$ and $0 < x \ne 1$, then the function

$$\Phi(t) = \Phi(t;\lambda,x) = A(x,1)^\lambda - H_t(x,1)^\lambda - 4t(1-t)\big(A(x,1)^\lambda - G(x,1)^\lambda\big)$$

is positive. In view of $\Phi(t) = \Phi(1-t)$, we may assume that $t \in (0,1/2)$. Differentiation yields

$$\Phi'(t) = -\lambda(\log x)\frac{x^t - x^{1-t}}{x^t + x^{1-t}} H_t(x,1)^\lambda + 4(2t-1)\big(A(x,1)^\lambda - G(x,1)^\lambda\big).$$

Applying

$$q - 1 \le \frac{q^\lambda - 1}{\lambda} \quad (0 < q < 1)$$

with $q = G(x,1)/A(x,1)$ yields

$$\frac{1}{4\lambda A(x,1)^\lambda}\Phi'(0) = (\log x)\frac{x-1}{4(x+1)} + \frac{\big(G(x,1)/A(x,1)\big)^\lambda - 1}{\lambda}$$

$$\ge (\log x)\frac{x-1}{4x(x+1)} + \frac{G(x,1)}{A(x,1)} - 1 = \frac{(\log x)(\sqrt{x}-1)}{2(x+1)}\big(A(\sqrt{x},1) - L(\sqrt{x},1)\big),$$

where L denotes the logarithmic mean. Using the right-hand side of (2.16) we get $\Phi'(0) > 0$. Since $\Phi(0) = 0$, we conclude that Φ attains positive values in the neighbourhood of 0.

We assume (for a contradiction) that Φ' has two zeros on $(0,1/2)$. Since $\Phi'(1/2) = 0$, it follows that Φ' has three zeros on $(0,1/2]$. Then, Φ'' has two zeros on $(0,1/2)$ and Φ''' has at least one zero on $(0,1/2)$. We obtain

$$\Phi'''(t) = -\lambda(\log x)^3 \frac{x^t - x^{1-t}}{(x^t + x^{1-t})^3}\, H_t(x,1)^\lambda\, \chi(t;\lambda,x)$$

with

$$\chi(t;\lambda,x) = \lambda^2(x^t - x^{1-t})^2 + 4(3\lambda - 2)x > 0.$$

Using $(\log x)(x^t - x^{1-t}) < 0$ gives $\Phi'''(t) > 0$ for $t \in (0,1/2)$. This contradiction reveals that Φ' has at most one zero on $(0,1/2)$. We have $\Phi(0) = \Phi(1/2) = 0$. This implies that Φ' has precisely one zero on $(0,1/2)$ and that Φ has no zero on $(0,1/2)$. Since Φ attains positive values in the neighbourhood of 0, we conclude that Φ is positive on $(0,1/2)$. Thus,

$$0 < \frac{y^\lambda\, \Phi(t;\lambda,x/y)}{A(x,y)^\lambda - G(x,y)^\lambda} = \frac{A(x,y)^\lambda - H_t(x,y)^\lambda}{A(x,y)^\lambda - G(x,y)^\lambda} - 4t(1-t).$$

This settles the left-hand side of (2.17).

The limit relations

$$\lim_{x\to 1}\frac{A(x,1)^\lambda - H_t(x,1)^\lambda}{A(x,1)^\lambda - G(x,1)^\lambda} = 4t(1-t) \quad \text{and} \quad \lim_{x\to 0}\frac{A(x,1)^\lambda - H_t(x,1)^\lambda}{A(x,1)^\lambda - G(x,1)^\lambda} = 1$$

reveal that the lower and upper bounds given in (2.17) are sharp. □

The next theorem is a counterpart of Theorem 2.9. It offers upper and lower bounds for $H_t^*(x,y)$.

Theorem 2.12. *Let* $t \in (0,1)$*. For all* $x, y > 0$ *we have*

$$G(x,y) + \kappa_t \left(\sqrt{x} - \sqrt{y}\right)^2 \le H_t^*(x,y) \le G(x,y) + K_t \left(\sqrt{x} - \sqrt{y}\right)^2 \tag{2.18}$$

with the best possible factors

$$\kappa_t = 2t(1-t) \quad \textit{and} \quad K_t = \sqrt{t(1-t)}. \tag{2.19}$$

PROOF. It suffices to prove (2.18) for $x \in (0,1)$ and $y = 1$. Let

$$\Psi(t,x) = H_t^*(x,1)^2 - \left[\sqrt{x} + 2t(1-t)(\sqrt{x}-1)^2\right]^2$$

and

$$\Omega(t,x) = \left[\sqrt{x} + \sqrt{t(1-t)}(\sqrt{x}-1)^2\right]^2 - H_t^*(x,1)^2.$$

We have to show that

$$\Psi(t,x) \ge 0 \quad \text{and} \quad \Omega(t,x) \ge 0. \tag{2.20}$$

Since $\Psi(t,x) = \Psi(1-t,x)$ and $\Omega(t,x) = \Omega(1-t,x)$, we may assume that $0 < t \le 1/2$. Partial differentiation gives

$$\frac{\partial}{\partial t}\Psi(t,x) = 8(t-t_1)(t-t_2)(1-2t)(\sqrt{x}-1)^4,$$

where

$$t_1 = \frac{2-\sqrt{2}}{4} = 0.14... \quad \text{and} \quad t_2 = \frac{2+\sqrt{2}}{4} = 0.85....$$

It follows that

$$\frac{\partial}{\partial t}\Psi(t,x) \ge 0, \quad \text{if} \quad 0 < t \le t_1$$

and

$$\frac{\partial}{\partial t}\Psi(t,x) \le 0, \quad \text{if} \quad t_1 \le t \le 1/2.$$

This implies that

$$\Psi(t,x) \ge \min\{\Psi(0,x), \Psi(1/2,x)\} = 0.$$

We have

$$\frac{\partial}{\partial t}\Omega(t,x) = 16\,\frac{(t-t_3)(t-t_4)(1-2t)}{\left(4\sqrt{t(1-t)}+1\right)\sqrt{t(1-t)}}\sqrt{x}(\sqrt{x}-1)^2,$$

where

$$t_3 = \frac{2-\sqrt{3}}{4} = 0.06... \quad \text{and} \quad t_4 = \frac{2+\sqrt{3}}{4} = 0.93....$$

This gives

$$\frac{\partial}{\partial t}\Omega(t,x) \geq 0, \quad \text{if} \quad 0 < t \leq t_3$$

and

$$\frac{\partial}{\partial t}\Omega(t,x) \leq 0, \quad \text{if} \quad t_3 \leq t \leq 1/2.$$

It follows that

$$\Omega(t,x) \geq \min\{\Omega(0,x), \Omega(1/2,x)\} = 0.$$

Thus, (2.20) is proved.

If (2.18) is valid for all $x, y > 0$, then we get for $x \neq 1$:

$$\kappa_t \leq \frac{H_t^*(x,1) - \sqrt{x}}{(\sqrt{x}-1)^2} \leq K_t.$$

Since

$$\lim_{x \to 0} \frac{H_t^*(x,1) - \sqrt{x}}{(\sqrt{x}-1)^2} = \sqrt{t(1-t)} \quad \text{and} \quad \lim_{x \to 1} \frac{H_t^*(x,1) - \sqrt{x}}{(\sqrt{x}-1)^2} = 2t(1-t),$$

we conclude that the factors given in (2.19) are sharp. □

In order to prove the following theorems we need convexity and concavity properties of $H_t(x,y)$ and $H_t^*(x,y)$.

Lemma 2.13. *Let $x, y > 0$ with $x \neq y$. Then, $t \mapsto H_t(x,y)$ is strictly log-convex on $[0,1]$ and $t \mapsto H_t^*(x,y)$ is strictly concave on $[0,1]$.*

Proof. We have

$$\frac{\partial^2}{\partial t^2} \log H_t(x,y) = \frac{xy(\log x - \log y)^2}{H_t(x,y)^2} > 0$$

and

$$\frac{\partial^2}{\partial t^2} H_t^*(x,y) = \frac{-(x^2-y^2)^2}{4\, H_t^*(x,y)^3} < 0.$$

□

Next, we present refinements of the inequalities $G^2/A \leq A$ and $G \leq 2A - G$ by using the Heinz mean and its complementary mean.

Theorem 2.14. *Let $x, y > 0$ with $x \neq y$ and $s, t \geq 0$ with $s + t \leq 1$. Then,*

$$\frac{G(x,y)^2}{A(x,y)} \leq \frac{H_s(x,y)\, H_t(x,y)}{H_{s+t}(x,y)} \leq A(x,y). \tag{2.21}$$

Equality holds on the left-hand side if and only if $s = t = 1/2$ and on the right-hand side if and only if $s = 0$ or $t = 0$.

PROOF. Let $s \in [0,1]$ be a fixed number. If $s = 0$, then the left-hand side of (2.21) holds with "<", whereas equality is valid on the right-hand side. Next, let $0 < s \leq 1$. We assume that $s \leq t$. Then, $0 < s \leq t \leq 1 - s < 1$. Let

$$\sigma(t) = \sigma(t; x, y) = \log H_t(x,y)$$

and

$$\eta(t) = \eta(t; x, y) = \sigma(s) + \sigma(t) - \sigma(s+t).$$

Applying Lemma 2.13 yields

$$\eta'(t) = \sigma'(t) - \sigma'(s+t) < 0.$$

This leads to

$$\eta(t) \leq \eta(s) \tag{2.22}$$

and

$$\eta(1-s) \leq \eta(t), \tag{2.23}$$

where the sign of equality is valid in (2.23) if and only if $t = 1 - s$. We have $0 < s \leq 1/2$. Since σ is strictly convex on $[0,1]$, we have

$$\sigma(s) < \frac{\sigma(0) + \sigma(2s)}{2}.$$

Thus,

$$\eta(s) = 2\sigma(s) - \sigma(2s) < \sigma(0). \tag{2.24}$$

From (2.22) and (2.24) we obtain the right-hand side of (2.21) with "<". Since σ is strictly decreasing on $[0, 1/2]$, we find

$$\eta(1-s) = \sigma(s) + \sigma(1-s) - \sigma(1) = 2\sigma(s) - \sigma(1) \geq 2\sigma(1/2) - \sigma(1) \tag{2.25}$$

with equality if and only if $s = 1/2$. Combining (2.23) and (2.25) gives the left-hand side of (2.21), where the sign of equality holds if and only if $t = 1 - s = 1/2$. □

Theorem 2.15. *Let $x, y > 0$ with $x \neq y$ and $s, t \geq 0$ with $s + t \leq 1$. Then,*

$$G(x,y) \leq H_s^*(x,y) + H_t^*(x,y) - H_{s+t}^*(x,y) \leq 2\,A(x,y) - G(x,y). \tag{2.26}$$

Equality holds on the left-hand side if and only if $s = 0$ or $t = 0$ and on the right-hand side if and only if $s = t = 1/2$.

Proof. The proof is similar to that of Theorem 2.14. Therefore, we only offer a proof sketch. Let $0 \leq s \leq t \leq 1 - s \leq 1$ and

$$\zeta(s,t) = \zeta(s,t;x,y) = H_s^*(x,y) + H_t^*(x,y) - H_{s+t}^*(x,y).$$

Since $t \mapsto \zeta(s,t)$ is strictly increasing on $[s, 1-s]$ we obtain

$$\zeta(s,s) \leq \zeta(s,t) \leq \zeta(s,1-s). \tag{2.27}$$

We have

$$\zeta(0,0) \leq \zeta(s,s) \quad \text{and} \quad \zeta(s,1-s) \leq \zeta(1/2,1/2). \tag{2.28}$$

From (2.27) and (2.28) we conclude that (2.26) is valid. □

The following lemma is due to Wright [52].

Lemma 2.16. *Let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is positive, monotone or convex, then, for $x, y, z \in I$,*

$$0 < (x-y)(x-z)f(x) + (y-x)(y-z)f(y) + (z-x)(z-y)f(z), \tag{2.29}$$

unless $x = y = z$.

This lemma extends a result of Schur, who proved (2.29) for the special case $f(x) = x^\mu$ ($\mu \geq 0$). We conclude this section with two Schur-type inequalities involving $H_t(x,y)$ and $H_t^*(x,y)$.

Theorem 2.17. *Let x and y be positive real numbers.*

(i) If $xy > 1$, then, for $r, s, t \in [0,1]$,

$$1 \leq H_r(x,y)^{(r-s)(r-t)}\, H_s(x,y)^{(s-r)(s-t)}\, H_t(x,y)^{(t-r)(t-s)}. \tag{2.30}$$

(ii) If $x + y < 2$, then, for $r, s, t, \in [0,1]$,

$$H_r^*(x,y)^{(r-s)(r-t)}\, H_s^*(x,y)^{(s-r)(s-t)}\, H_t^*(x,y)^{(t-r)(t-s)} \leq 1. \tag{2.31}$$

The sign of equality holds in (2.30) and (2.31) if and only if $r = s = t$.

PROOF. If $xy > 1$, then

$$\log H_t(x,y) \geq \log H_{1/2}(x,y) = \frac{1}{2}\log(xy) > 0. \tag{2.32}$$

Using Lemma 2.13 and (2.32) we obtain that $t \mapsto \log H_t(x,y)$ is positive and convex on $[0,1]$.

Since $t \mapsto H_t^*(x,y)$ is concave on $[0,1]$, we conclude that $t \mapsto -\log H_t^*(x,y)$ is convex on $[0,1]$. Moreover, if $x+y<2$, then

$$-\log H_t^*(x,y) \geq -\log H_{1/2}^*(x,y) = -\log\frac{x+y}{2} > 0.$$

Applying Lemma 2.16 with $f(t) = \log H_t(x,y)$ and $f(t) = -\log H_t^*(x,y)$, respectively, leads to (2.30) and (2.31). □

3 Complete monotonicity

In Section 1, we pointed out that $H_t(x,y)$ and $1/H_t^*(x,y)$ are decreasing on $[0,1/2]$ with respect to t. In the final part of this paper we show that these monotonicity properties can be substantially extended.

A function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (n = 0,1,2,...;\ x \in I).$$

These functions play an important role in probability theory and they have applications in potential theory, numerical analysis and other branches. The basic properties of completely monotonic functions are collected in [51, chapter IV]. In several recently published articles it was proved that certain functions which are defined in terms of gamma, polygamma and other classical functions are completely monotonic; see [2] and the references therein. A helpful tool for proving the complete monotonicity of a function is

Lemma 3.1. *Let $I \subset \mathbb{R}$ be an interval. The function $\exp\bigl(-f(x)\bigr)$ is completely monotonic on I, if f' is completely montonic on I.*

This can be proved by using induction and the Leibniz rule for differentiation; see also [8, p. 83].

Theorem 3.2. *Let $x,y > 0$. The functions $t \mapsto H_t(x,y)$ and $t \mapsto 1/H_t^*(x,y)$ are completely monotonic on $[0,1/2]$.*

Proof. (i) We may assume that $x \geq y$. Let $z = x/y \geq 1$. Since

$$(-1)^n z^{2t} + z \geq -z^{2t} + z \geq -z + z = 0 \quad (n = 0, 1, 2, ...; \, 0 \leq t \leq 1/2),$$

we obtain

$$(-1)^n \frac{\partial^n}{\partial t^n} H_t(x, y) = \frac{y\,(\log z)^n}{2\,z^t} [(-1)^n z^{2t} + z] \geq 0.$$

(ii) Let $x \geq y$ and $t \in [0, 1/2]$. We apply Lemma 3.1 with

$$f(t) = -\log \frac{1}{H_t^*(x, y)} = \frac{1}{2} \log\bigl(tx + (1-t)y\bigr) + \frac{1}{2} \log\bigl((1-t)x + ty\bigr).$$

Then, for $n \geq 0$,

$$(-1)^n f^{(n+1)}(t) = \frac{n!}{2}(x-y)^{n+1} \left[\frac{1}{\bigl(tx + (1-t)y\bigr)^{n+1}} + \frac{(-1)^{n+1}}{\bigl((1-t)x + ty\bigr)^{n+1}} \right].$$

If $n+1$ is even, then $(-1)^n f^{(n+1)}(t) \geq 0$, and if $n+1$ is odd, then we conclude from

$$\frac{1}{tx + (1-t)y} - \frac{1}{(1-t)x + ty} = \frac{(x-y)(1-2t)}{\bigl(tx + (1-t)y\bigr)\bigl((1-t)x + ty\bigr)} \geq 0$$

that $(-1)^n f^{(n+1)}(t) \geq 0$. If follows that $t \mapsto 1/H_t^*(x, y)$ is completely monotonic on $[0, 1/2]$. □

Acknowledgment. I thank P. Haukkanen, J.K. Merikoski and the referee for helpful comments.

References

[1] H. Alzer, *Ungleichungen für Mittelwerte*, Arch. Math., **47** (1986), 422–426.

[2] H. Alzer and C. Berg, *Some classes of completely monotonic functions II*, Ramanujan J., **11** (2006), 225–248.

[3] H. Alzer and S.-L. Qiu, *Inequalities for means in two variables*, Arch. Math., **80** (2003), 201–215.

[4] A. Bege, J. Bukor and J. T. Tóth, *On (log-) convexity of power mean*, Ann. Math. Inf., **42** (2013), 3–7.

[5] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[6] R. Bhatia, *Interpolating the arithmetic-geometric inequality and its operator version*, Linear Alg. Appl., **413** (2006), 355–363.

[7] R. Bhatia, *Positive Definite Matrices*, Princeton Univ. Press, Princeton, 2007.

[8] S. Bochner, *Harmonic Analysis and the Theory of Probability*, Univ. of California Press, Berkeley-Los Angeles, 1960.

[9] P. Bracken, *An arithmetic-geometric mean inequality*, Expos. Math., **19** (2001), 273–279.

[10] J. L. Brenner and M. E. Mays, *Some reproducing identities for families of mean values*, Aequat. Math., **33** (1987), 106–113.

[11] P. S. Bullen, D. S. Mitrinović and P.M. Vasić, *Means and Their Inequalities*, Reidel, Dordrecht, 1988.

[12] Y.-M. Chu and S.-W. Hou, *Sharp bounds for Seiffert mean in terms of contraharmonic mean*, Abstr. Appl. Anal., (2012), article ID 425175, 6 pp.

[13] Y.-M. Chu, S.-W. Hou and Z.-H. Shen, *Sharp bounds for Seiffert mean in terms of root mean square*, J. Inequal. Appl., (2012), article 11, 6 pp.

[14] Y.-M. Chu, M.-K.Wang and Z.-K. Wang, *A sharp double inequality between harmonic and identric means*, Abstr. Appl. Anal., (2011), article ID 657935, 7 pp.

[15] P. Czinder and Z. Páles, *Minkowski-type inequalities for two variable Stolarsky means*, Acta Sci. Math. (Szeged), **69** (2003), 27–47.

[16] P. Czinder and Z. Páles, *Some comparison inequalities for Gini and Stolarsky means*, Math. Ineq. Appl., **9** (2006), 607–616.

[17] D. Drissi, *Sharp inequalities for some operator means*, SIAM J. Matrix Anal. Appl., **28** (2006), 822–828.

[18] S. Furuichi and N. Minculete, *Alternative reverse inequalities for Young's inequality*, J. Math. Ineq., **5** (2011), 595–600.

[19] H. W. Gould and M. E. Mays, *Series expansion of means*, J. Math. Anal. Appl., **101** (1984), 611–621.

[20] M. Grossman and R. Katz, *A new approach to means of two positive numbers*, Int. J. Math. Educ. Sci. Technol., **17** (1986), 205–208.

[21] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.

[22] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438.

[23] P. Kahlig and J. Matkowski, *Functional equations involving the logarithmic mean*, Z. angew. Math. Mech., **76** (1996), 385–390.

[24] F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., **361** (2010), 262–269.

[25] F. Kittaneh and Y. Manasrah, *Reverse Young and Heinz inequalities for matrices*, Lin. Multilin. Alg., **59** (2011), 1031–1037.

[26] E. B. Leach and M. C. Sholander, *Extended mean values*, Amer. Math. Monthly, **85** (1978), 84–90.

[27] E. B. Leach and M. C. Sholander, *Extended mean values II*, J. Math. Anal. Appl., **92** (1983), 207–223.

[28] D. H. Lehmer, *On the compounding of certain means*, J. Math. Anal. Appl., **36** (1971), 183–200.

[29] L. Losonczi and Z. Páles, *Minkowski's inequality for two variable difference means*, Proc. Amer. Math. Soc., **126** (1998), 779–789.

[30] M. E. Mays, *Functions which parameterize means*, Amer. Math. Monthly, **90** (1983), 677–683.

[31] D. S. Mitrinović, *Analytic Inequalities*, Spinger, New York, 1970.

[32] E. Neuman and J. Sándor, *On certain means of two arguments and their extensions*, Int. J. Math. Math. Sci., **16** (2003), 981–993.

[33] E. Neuman and J. Sándor, *Inequalities involving Stolarsky and Gini means*, Math. Pann., **14** (2003), 29–44.

[34] E. Neuman and J. Sándor, *Companion inequalities for certain bivariate means*, Appl. Anal. Discr. Math., **3** (2009), 46–51.

[35] Z. Páles, *Comparison of two variable homogeneous means*, in: General Inequalties 6 (W. Walter, ed.), Birkhäuser, Basel, (1992), 59–70.

[36] M. Petrović, *Sur une fonctionnelle*, Publ. Math. Univ. Belgrade, **1** (1932), 149–156.

[37] A. O. Pittenger, *The symmetric, logarithmic and power means*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **678-715** (1980), 19–23.

[38] A. O. Pittenger, *The logarithmic mean in n variables*, Amer. Math. Monthly, **92** (1985), 99–104.

[39] J. Sándor, *On the identric and logarithmic means*, Aequat. Math., **40** (1990), 261–270.

[40] J. Sándor, *A note on some inequalities for means*, Arch. Math., **56** (1991), 471–473.

[41] J. Sándor, *On certain inequalities for means*, J. Math. Anal. Appl., **189** (1995), 602–606.

[42] J. Sándor, *On certain inequalities for means II*, J. Math. Anal. Appl., **199** (1996), 629–635.

[43] J. Sándor, *On certain inequalities for means III*, Arch. Math., **76** (2001), 34-40.

[44] J. Sándor, *Logarithmic convexity of t-modification of a mean*, Octogon Math. Mag., **9** (2001), 737–739.

[45] J. Sándor, *On certain inequalities for means in two variables*, J. Inequal. Pure Appl. Math., **10** (2006), issue 2, article 47, 3 pp.

[46] J. Sándor and I. Raşa, *Inequalities for certain means in two arguments*, Nieuw Arch. Wisk. (Ser. 4), **15** (1997), 51–55.

[47] J. Sándor and Gh. Toader, *Some general means*, Czech. Math. J., **49** (1999), 53-62.

[48] H.-J. Seiffert, *Ungleichungen für elementare Mittelwerte*, Arch. Math., **64** (1995), 129–131.

[49] H.-J. Seiffert, *Ungleichungen für einen Mittelwert*, Nieuw Arch. Wisk. (Ser. 4), **13** (1995), 195–198.

[50] G. Toader, *Seiffert type means*, Nieuw Arch. Wisk. (Ser. 4), **17** (1999), 379–382.

[51] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, NJ, 1941.

[52] E. M. Wright, *A generalisation of Schur's inequality*, Math. Gaz., **40** (1956), 217.

[53] T.-H. Zhao, Y.-M. Chu, Y.-L. Jiang and Y.-M. Li, *Best possible bounds for Neuman-Sándor mean by the identric, quadratic and contraharmonic means*, Abstr. Appl. Anal., (2013), article ID 348326, 12 pp.