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SETS OF NON-DIFFERENTIABILITY FOR FUNCTIONS WITH FINITE LOWER SCALED OSCILLATION

Abstract

Up to a set of measure zero we characterize the sets of non-differentiability of functions with everywhere finite lower scaled oscillation.

1 Introduction and statement of results

We are interested in characterizing sets of non-differentiability for real-valued functions satisfying various Lipschitz-like conditions.

We begin by setting notation. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and define $N_f = \{x \in \mathbb{R} \mid f \text{ is not differentiable at } x\}$. What can be said about the set N_f? First, an elementary argument using the continuity of f implies that N_f is a $G_{\delta\sigma}$ set. (A G_δ is a countable intersection of open sets; a $G_{\delta\sigma}$ is a countable union of G_δ's.) By a theorem of Lebesgue, N_f has measure zero for any Lipschitz function f.

Lebesgue's result can be generalized by using the upper scaled oscillation function, Lip f, defined as follows:

$$\text{Lip } f(x) = \limsup_{r \to 0^+} \frac{L_f(x,r)}{r}, \tag{1}$$

where

$$L_f(x,r) = \sup\{|f(x) - f(y)| \colon |x-y| \le r\}.$$

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The Rademacher-Stepanov Theorem (see ([2], Theorem 3.4, or [3]) now says the following:

Theorem 1. *If f is continuous on $\mathbb{R}$, then $N_f \cap \{x \,|\, \mathrm{Lip} f(x) < \infty\}$ is a set of measure zero.*

In the 1940's Zahorski gave sharp conditions characterizing N_f for both continuous and Lipschitz functions defined on $\mathbb{R}$:

Theorem 2. *([5], p.147) $E = N_f$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$ if and only if $E = E_1 \cup E_2$, where E_1 is a G_δ set and E_2 is a $G_{\delta\sigma}$ set of measure 0.*

Theorem 3. *([5], Theorem 3) $E = N_f$ for some Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ if and only if E is a $G_{\delta\sigma}$ set of measure 0.*

We now define Lip $\mathbb{R}$ as the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that Lip $f(x) < \infty$ for all $x \in \mathbb{R}$. Note that every f in Lip $\mathbb{R}$ is continuous on $\mathbb{R}$. Using Theorem 1, we can reformulate Theorem 3 as follows:

Theorem 4. *$E = N_f$ for some $f \in \mathrm{Lip}\ \mathbb{R}$ if and only if E is a $G_{\delta\sigma}$ set and $|E| = 0$.*

We seek to explore the implications of replacing the upper scaled oscillation function Lip f with the *lower* scaled oscillation function lip f, defined as follows:

$$\mathrm{lip}\ f(x) = \liminf_{r \to 0^+} \frac{L_f(x,r)}{r}.$$

We also define lip $\mathbb{R}$ as the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ with lip $f(x) < \infty$ for all $x \in \mathbb{R}$. Again, every function f in lip $\mathbb{R}$ is continuous on $\mathbb{R}$.

As Balogh and Csörnyei showed in ([1]), functions in lip $\mathbb{R}$ can fail to be differentiable a.e. so Theorem 4 fails if we replace the condition $f \in \mathrm{Lip}\ \mathbb{R}$ with $f \in \mathrm{lip}\ \mathbb{R}$. On the other hand, Balogh and Csörnyei also proved the following result (see [1], Lemma 1.1):

Theorem 5. *If $f \in \mathrm{lip}\ \mathbb{R}$, then $|N_f \cap (a,b)| < b - a$ for any open interval (a,b).*

Motivated by this result, we make the following definition:

Definition 6. *A subset E of $\mathbb{R}$ is <u>trim</u> if $|E \cap (a,b)| < b - a$ for all open intervals (a,b).*

Combining Theorems 5 and 2, we see that if f is in lip $\mathbb{R}$, then N_f is the union of a trim G_δ set and a $G_{\delta\sigma}$ set of measure zero. It is now natural to conjecture that a sort of converse holds:

Conjecture 7. *$E = N_f$ for some $f \in \text{lip}\,\mathbb{R}$ if and only if $E = E_1 \cup E_2$, where E_1 is a trim G_δ set and E_2 is a $G_{\delta\sigma}$ set of measure zero.*

Our following results give evidence in favor of the conjecture:

Theorem 8. *For every closed, nowhere dense set E there exists $f \in \text{lip}\,\mathbb{R}$ such that $E = N_f$.*

Note that for closed sets nowhere dense and trim are equivalent.

Theorem 9. *Suppose that E is a trim G_δ set. Then there exists a function $f \in \text{lip}\,\mathbb{R}$ such that $|E \triangle N_f| = 0$.*

2 Tools for the proofs of Theorems 8 and 9

We begin by establishing some elementary facts about perfect, nowhere dense sets which will be used in the proofs of both theorems. Throughout the rest of this paper S will be the set of dyadic rationals in the interval $(0,1)$. More precisely:

$$S = \{\frac{m}{2^n} \,|\, 1 \le m \le 2^n - 1,\ n \ge 1\}.$$

Definition 10. *Suppose that*

$$F \text{ is perfect, nowhere dense and } \{a,b\} \subset F \subset [a,b]. \tag{2}$$

Let $b_0 = a$ and $a_1 = b$. Suppose that $\{I_s\}_{s\in S} = \{(a_s,b_s)\}_{s\in S}$ satisfies:

$$\cup_{s\in S} I_s = [a,b]\backslash F \tag{3}$$

$$s < t \Rightarrow b_s < a_t. \tag{4}$$

Then we say that $\{I_s\}_{s\in S}$ is a dyadic decomposition of $[a,b]\backslash F$.

A simple induction proof shows that if (2) holds, then a dyadic decomposition of $[a,b]\backslash F$ exists. Given a dyadic decomposition $\{I_s\}_{s\in S}$ as defined above, for $s = \frac{2i-1}{2^n}$ where $1 \le i \le 2^{n-1}$ and $n \ge 1$, we define $\tilde{I}_s = [b_r, a_t]$ where $r = \frac{i-1}{2^{n-1}}$ and $t = \frac{i}{2^{n-1}}$. Note that $\overline{I}_s \subset \tilde{I}_s$ for all $s \in S$ and

$$\{x\} = \cap_{x\in \tilde{I}_s} \tilde{I}_s \text{ for all } x \in F. \tag{5}$$

We will need the following lemma, which follows easily from the fact that the complement of F is open and dense in $[a,b]$.

Lemma 11. *Suppose that (2) holds. Then there exists a dyadic decomposition* $\mathcal{I} = \{I_s\}_{s\in S}$ *of* $[a,b]\backslash F$ *satisfying:*

$$I_s \cap \frac{1}{4}\tilde{I}_s \neq \emptyset \text{ for all } s \in S. \tag{6}$$

(Here, and elsewhere in the paper, we use the convention that if I is an (open, closed) interval centered at x_0 and $C > 0$, then CI is the interval with length $C|I|$ centered at x_0.)

For the remainder of this paper whenever we have a set F satisfying (2) we will assume that a dyadic decomposition satisfying (6) has been chosen as well. Furthermore, we will also assume that for each $s \in S$ we have chosen c_s, d_s, m_s and h_s satisfying:

$$h_s = \frac{1}{6}|\tilde{I}_s| \tag{7}$$

$$a_s < c_s < m_s < d_s < b_s \tag{8}$$

$$I'_s = [c_s, d_s] \subset I_s \cap \frac{1}{3}\tilde{I}_s. \tag{9}$$

Definition 12. *Given a set* F *satisfying (2) and a dyadic decomposition* $\mathcal{I} = \{(a_s, b_s)\}_{s\in S}$ *of* $[a,b]\backslash F$ *and* $\{c_s, m_s, d_s\}_{s\in S}$ *satisfying (8) and (9) , we define*

$$T_{F,[a,b]} = \{c_s, m_s, d_s\}_{s\in S},$$

$$\tilde{\mathcal{I}}_{F,[a,b]} = \cup_{s\in S}\{(a_s, b_s)\}$$

and

$$\mathcal{I}_{F,[a,b]} = \cup_{s\in S}\{(a_s, c_s), (c_s, m_s), (m_s, d_s), (d_s, b_s)\}.$$

The remainder of this section will be useful for proving Theorem 9.

Lemma 13. *Suppose that* E *is a trim* G_δ *set. Then we can decompose* E *into sets* $E_0, E_1, E_2, ...$ *such that*

$$E = \cup_{n=0}^{\infty} E_n \tag{10}$$

$$E_j \cap E_k = \emptyset \text{ for } j \neq k \tag{11}$$

$$E_0 \text{ is a } G_\delta \text{ set of measure } 0 \tag{12}$$

$$\text{for each } n \geq 1 \text{ the set } E_n \text{ is perfect and nowhere dense.} \tag{13}$$

PROOF. We assume without loss of generality that E is a bounded, trim G_δ set. We note first of all that, according to the Cantor-Bendixson Theorem, every closed set F is the union of a perfect set and a countable set and thus given any measurable set G and $\epsilon > 0$, we can always find a perfect set F such that $F \subset G$ and $|G \backslash F| < \epsilon$. We begin by choosing E_1 to be a perfect set such that $E_1 \subset E$ and $|E \backslash E_1| < \frac{1}{2}$. Proceeding inductively, assuming that we have chosen a collection of pairwise disjoint perfect sets $\{E_1, E_2, ..., E_n\}$ such that $\cup_{i=1}^n E_i \subset E$ and $|E \backslash (\cup_{i=1}^n E_i)| < (\frac{1}{2})^n$, we choose E_{n+1} to be a perfect subset of $E \backslash (\cup_{i=1}^n E_i)$ such that $|E \backslash (\cup_{i=1}^{n+1} E_i)| = |E \backslash (\cup_{i=1}^n E_i) \backslash E_{n+1}| < (\frac{1}{2})^{n+1}$. Defining $E_0 = E \backslash (\cup_{n=1}^\infty E_n)$, we see that $\{E_0, E_1, E_2, ...\}$ satisfies the conclusion of the lemma. □

Lemma 14. *Suppose that $\{F_1, F_2, ...\}$ is a collection of pairwise disjoint perfect subsets of $\mathbb{R}$. Given $k \in \mathbb{N}$, we define $\mathcal{F}_k = \{F_n\}_{n=k}^\infty$. Suppose that $I = (a, b)$ with $\cup_{n=k}^\infty F_n \subset I$ and $\sum_{n=k}^\infty |F_n| = \delta < b - a$ and let $\epsilon > 0$. Then for each $n \geq k$ we can find a collection $\mathcal{C}_n = \mathcal{C}_n(\mathcal{F}, (a, b))$, such that each $\mathcal{C}_n$ is a finite collection of pairwise disjoint, closed subintervals of (a, b) and such that letting $K_n = \cup_{J \in \mathcal{C}_n} J$ and $K = \cup_{n=k}^\infty K_n$, we have for each $n, m \geq k$:*

$$K_n \cap K_m = \emptyset \textit{ if } n \neq m \tag{14}$$

$$\cup_{j=k}^n F_j \subset \cup_{j=k}^n K_j \tag{15}$$

$$\textit{for each } J = [c, d] \in \mathcal{C}_n, \textit{ we have } \{c, d\} \subset F_n \tag{16}$$

$$|K| = \sum_{n=k}^\infty |K_n| = \gamma < \min\{\delta + \epsilon, b - a\}. \tag{17}$$

Moreover, given any $c, d \in \mathbb{R}$, there exists a continuous, monotonic function $\beta = \beta_{I,\mathcal{F},c,d}$ which maps $[a, b]$ onto $[\min\{c, d\}, \max\{c, d\}]$ and satisfies the following:

$$\beta(a) = c \textit{ and } \beta(b) = d \tag{18}$$

$$\beta \textit{ is constant on each } J \in \cup_{n=k}^\infty \mathcal{C}_n \tag{19}$$

$$\beta \textit{ is Lipschitz on } (a, b). \tag{20}$$

For future reference, if h is a function defined on $I = [a,b]$, we define

$$\beta_{I,\mathcal{F},h} = \beta_{I,\mathcal{F},h(a),h(b)}. \tag{21}$$

Note that (15) and (16) imply that $\mathcal{C}_k$ is a finite covering of F_k with closed intervals whose endpoints are in F_k.

Proof. We assume without loss of generality that $k = 1$. Suppose that $\{F_n\}_{n=1}^{\infty}$ satisfies the hypotheses of the lemma. Choose $\{\alpha_n\}$ such that $|F_n| < \alpha_n$ and $\sum_{n=1}^{\infty} \alpha_n < \min\{\delta + \epsilon, b - a\}$. Using the compactness of F_1, we can find a finite collection of pairwise disjoint, open intervals which cover F_1 and have total length less than α_1. Then using the fact that F_1 is perfect, we can shrink each of these intervals down to a closed interval whose endpoints are in F_1. This gives us $\mathcal{C}_1$. Proceeding inductively, assume that the collections $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_r$ have been chosen to satisfy equations (14), (15) and (16) for $n, m \leq r$. Noting that (because of (16)) $F_{r+1} \backslash (\cup_{j=1}^{r} K_j)$ is a perfect set, we choose $\mathcal{C}_{r+1}$ to be a collection of pairwise disjoint, closed intervals (with endpoints in F_{r+1}) covering $F_{r+1} \backslash (\cup_{j=1}^{r} K_j)$ whose total length is less than α_{r+1}. This establishes (14) - (17). (Note that it may happen that $F_{r+1} \backslash (\cup_{j=1}^{r} K_j) = \emptyset$, in which case $\mathcal{C}_{r+1}$ is an empty collection.)

We now construct β. Let $\cup_{n=1}^{\infty} \mathcal{C}_n = \{I_j\}_{j=1}^{\infty} = \{[a_j, b_j]\}_{j=1}^{\infty}$ and assume without loss of generality that $c = 0$ and $d = 1$. Furthermore, we let $E = \cup_{n=1}^{\infty} I_n = \cup_{n=1}^{\infty} [a_n, b_n]$ and define $I_j < I_k$ if $a_j < a_k$. For each $n \in \mathbb{N}$ let $\delta_n = a_n - a - |\cup_{I_k < I_n} I_k|$ and $h_n = \frac{\delta_n}{b-a-\gamma}$ and define $\beta(x) = h_n$ if $x \in I_n$. Also define $\beta(a) = 0$ and $\beta(b) = 1$. We extend β to $\overline{E}$ by continuity. Then $[0,1] \backslash \overline{E}$ is a (possibly empty) disjoint union of open intervals. On each of these intervals extend β linearly. It is a straightforward exercise to show that β is Lipschitz on (a,b) with Lipschitz constant $\frac{1}{b-a-\gamma}$ and clearly (19) is satisfied. This completes the proof of the lemma. $\square$

Definition 15. *Given a closed interval* $J = [a,b]$ *and* $n \in \mathbb{N}$, *we define* $\phi_J \colon J \to [0, \frac{b-a}{2}]$ *as follows:*

$$\phi_J(x) = \begin{cases} x - a & \text{if } a \leq x \leq \frac{a+b}{2} \\ b - x & \text{if } \frac{a+b}{2} \leq x \leq b. \end{cases} \tag{22}$$

Definition 16. *Suppose that* F *satisfies (2). Let* $\mathcal{I} = \{I_s\}_{s \in S} = \{(a_s, b_s)\}_{s \in S}$ *be a dyadic decomposition of* $[a,b] \backslash F$ *with* $b_0 = a$ *and* $a_1 = b$ *and assume that (6)-(9) hold. For each* $s \in S$ *define* $\gamma = \gamma_s \colon [a_s, b_s] \to [0, \infty)$ *to be the unique function which is linear on the intervals* $[a_s, c_s]$, $[c_s, m_s]$, $[m_s, d_s]$,

$[d_s, b_s]$ *with* $\gamma(a_s) = \gamma(c_s) = \gamma(d_s) = \gamma(b_s) = 0$ *and* $\gamma(m_s) = h_s$. *We define* $\alpha_F\colon [a,b] \to [0,\infty)$ *as follows:*

$$\alpha_F(x) = \begin{cases} \gamma_s(x) & \text{if } x \in I_s \\ 0 & \text{if } x \notin \cup_{s\in S} I_s. \end{cases} \tag{23}$$

Note that technically the definition of α_F depends not only on F, but also on the dyadic decomposition $\mathcal{I}$, so we should really use $\alpha_{F,\mathcal{I}}$ in place of α_F. In the interest of avoiding notational overload we use the deliberately sloppy, but more streamlined notation.

Lemma 17. *Assume that* F *satisfies (2) and* $\mathcal{I} = \{I_s\}_{s\in S} = \{(a_s, b_s)\}_{s\in S}$ *is a dyadic decomposition of* $[a,b]\backslash F$ *with* $b_0 = a$ *and* $a_1 = b$. *Then for each* $s \in S$ *we have*

$$h_s \le \frac{1}{2}\phi_{\tilde{I}_s}(x) \text{ for all } x \in I'_s, \tag{24}$$

and

$$\alpha_F(x) \le \frac{1}{2}\phi_{\tilde{I}_s}(x) \text{ for all } x \in \tilde{I}_s. \tag{25}$$

Proof. Inequalities (24) and (25) follow easily from (7), (8), (9), the definition of γ and the fact that $J \subset K$ implies $\phi_J(x) \le \phi_K(x)$ for all $x \in J$. Note that, taking $s = 1/2$ in (25), we get $\alpha_F(x) \le \frac{1}{2}\phi_{[a,b]}(x)$ for all $x \in [a,b]$. □

3 Proof of Theorem 8

Let E be a closed, nowhere dense set. We assume without loss of generality that E is bounded, and we normalize E so that $\{0,1\} \subset E \subset [0,1]$. We first note that we may assume that E has no isolated points. To see this, we use the Cantor-Bendixson Theorem to write E as the disjoint union of E_1 and E_2, where E_1 is perfect and E_2 is countable. Suppose that we can find a function f satisfying the conclusion of Theorem 8 with E_1 in place of E. Then using Theorem 3, we find a Lipschitz function g such that $N_g = E_2$ and we see that $f + g$ satisfies the conclusion of Theorem 8.

Let $\mathcal{I} = \{I_s\}_{s\in S} = \{(a_s, b_s)\}s \in S$ be a dyadic decomposition of E and define $a_1 = 1$, $b_0 = 0$. For each $s \in S$ we define $\delta_s\colon [a_s, b_s] \to [0,\infty)$ to satisfy the following:

$$\delta_s(a_s) = \delta_s(b_s) = 0 \tag{26}$$

$$\delta_s \text{ is differentiable on } (a_s, b_s) \tag{27}$$

$$0 < \delta_s(x) \le \delta_s(m_s) = h_s \text{ for all } x \in (a_s, b_s) \tag{28}$$

$$\delta_s \text{ is linear on } [a_s, a_s + \epsilon_s] \text{ and } [b_s - \epsilon_s, b_s] \text{ for some } \epsilon_s > 0 \tag{29}$$

$$\delta_s(x) \le \phi_{\tilde{I}_s}(x) \text{ for all } x \in \tilde{I}_s. \tag{30}$$

Now define f as follows:

$$f(x) = \begin{cases} \delta_s(x) & \text{if } x \in I_s \\ 0 & \text{if } x \in E \\ -x & \text{if } x < 0 \\ x - 1 & \text{if } x > 1 \end{cases} \tag{31}$$

Note first of all that f is clearly differentiable on $\mathbb{R}\backslash E$ and therefore

$$\text{lip } f(x) = |f'(x)| < \infty \text{ for all } x \in \mathbb{R}\backslash E. \tag{32}$$

Now assume that $x \in E$. It remains to show that

$$\text{lip } f(x) < \infty \tag{33}$$

and

$$f \text{ is not differentiable at } x. \tag{34}$$

Since $x \in E$, by (5) there is a sequence $\{s_i\}$ in S such that $\{x\} = \cap_{i=1}^{\infty} \tilde{I}_{s_i}$.

Suppose first of all that x lies in the interior of each $\tilde{I}_{s_i} = [c_i, d_i]$. Then from (30) we see that $0 \le f(y) \le \phi_{\tilde{I}_{s_i}}(y)$ for all $y \in \tilde{I}_{s_i}$ and it follows that $L_f(x, r_i) \le 2r_i$, where $r_i = \min\{x - c_i, d_i - x\}$. Since $r_i \to 0$, it follows that lip $f(x) \le 2$.

On the other hand, if x is not in the interior of each $\tilde{I}_{s_i}$, then we have $x \in \cup_{s \in S}\{a_s, b_s\}$. Suppose that $x = a_s$. Then by (29) for some $\epsilon > 0$, f is linear on $[a_s, a_s + \epsilon]$. Moreover, $a_s = d_i$ for some $i \in \mathbb{N}$ and therefore it follows from (30) and the definition of f, that $0 \le f(y) \le \phi_{\tilde{I}_{s_i}}(y) \le x - y$ for all $y \in \tilde{I}_{s_i}$. It follows that (33) holds in this case and a similar argument shows that (33) holds when $x = b_s$ also.

To complete the proof we need to show that f is not differentiable at x. Note, first of all, that $f = 0$ on E and since E is perfect, it follows that $\liminf_{y \to x} \dfrac{|f(y) - f(x)|}{|y - x|} = 0$. On the other hand, letting $h_i = h_{s_i}$ and $m_i = m_{s_i}$, and using the fact that $|x - m_i| < |\tilde{I}_{s_i}|$ along with (24), we get that $\dfrac{|f(m_i)-f(x)|}{|m_i-x|} \geq \dfrac{h_i}{|\tilde{I}_{s_i}|} = \dfrac{1}{6}$ for all $i \in \mathbb{N}$, and it follows that $\limsup_{y\to x} \dfrac{|f(y)-f(x)|}{|y-x|} \geq \dfrac{1}{6}$, and therefore we get (34).

4 Proof of Theorem 9

Assume E is a trim G_δ set. We may clearly assume that E is bounded, and we normalize so that $E \subset [0,1]$. Using Lemma 13, we choose sets E_0, E_1, E_2, ... satisfying (10) - (13). We may assume without loss of generality that $\{0,1\} \subset E_1$.

Note that in order to prove the theorem it suffices to construct a continuous function f such that $\cup_{n=1}^{\infty} E_n \subset N_f$ and $|N_f \backslash (\cup_{n=1}^{\infty} E_n)| = 0$.

Given $I = (a,b)$ such that $\{a,b\} \cap E_i = \emptyset$ for $i = k, k+1, ...$, we define

$$\mathcal{F}_{k,I} = \{E_i \cap I\}_{i=k}^{\infty}.$$

It follows from the fact that E is trim and (10), (11), and (13), that $\mathcal{F}_{k,I}$ satisfies the hypotheses of Lemma 14 with $\mathcal{F}_k = \mathcal{F}_{k,I}$ and $F_j = E_j \cap I$. Using the notation from Lemma 14, for each $n \geq k$ we define

$$\mathcal{C}_{n,k,I} = \mathcal{C}_n(\mathcal{F}_{k,I}, I),$$

and note that $\mathcal{C}_I = \cup_{n=k}^{\infty} \mathcal{C}_{n,k,I}$ is a collection of pairwise disjoint, closed intervals which covers $(\cup_{j=k}^{\infty} E_j) \cap I$.

We next construct collections of pairwise disjoint, closed intervals $\mathcal{J}_n$ and collections of pairwise disjoint, open intervals $\mathcal{I}_n$. Define $\mathcal{J}_1 = \{[0,1]\}$ and (using the notation from Definition 12), $\mathcal{I}_1 = \mathcal{I}_{E_1,[0,1]}$. For $n > 1$, we define $\mathcal{J}_n$ and $\mathcal{I}_n$ recursively as follows:

$$\mathcal{J}_{n,k} = \cup_{I \in \mathcal{I}_{n-1}} \mathcal{C}_{k,n,I} \text{ for } k \geq n \tag{35}$$

$$\mathcal{J}_n = \cup_{k=n}^{\infty} \mathcal{J}_{n,k} \tag{36}$$

$$\mathcal{I}_{n,k} = \cup_{J \in \mathcal{J}_{n,k}} \mathcal{I}_{E_k \cap J, J} \text{ for } k \geq n \tag{37}$$

$$\tilde{\mathcal{I}}_{n,k} = \cup_{J \in \mathcal{J}_{n,k}} \tilde{\mathcal{I}}_{E_k \cap J, J} \text{ for } k \geq n \tag{38}$$

$$\mathcal{I}_n = \cup_{k=n}^{\infty} \mathcal{I}_{n,k} \tag{39}$$

$$\tilde{\mathcal{I}}_n = \cup_{k=n}^{\infty} \tilde{\mathcal{I}}_{n,k}. \tag{40}$$

Using the fact that $[0,1] \backslash E$ is dense in $[0,1]$, we also assume that for each $J \in \mathcal{J}_{n,k}$ we have $T_{E_k \cap J, J} \cap E = \emptyset$. Furthermore, for each $n \in \mathbb{N}$ define

$$H_n = \cup_{J \in \mathcal{J}_n} J \tag{41}$$

$$G_n = \cup_{I \in \mathcal{I}_n} I \tag{42}$$

$$G'_n = \cup_{I \in \tilde{\mathcal{I}}_n} I', \tag{43}$$

where I' in (43) is defined as in (9). Note that we have

$$H_{n+1} \subset G_n \text{ and } G'_n \subset H_n \text{ for every } n \in \mathbb{N}. \tag{44}$$

Moreover, for each $n \geq 2$ and for every $I \in \mathcal{I}_n$, we have $I \cap E_j = \emptyset$ for $j = 1, 2, ..., n$.

We now begin the construction of f. We start by setting

$$f_1 = \alpha_{E_1}.$$

Proceeding recursively, (and recalling the notation in (21)), for every $n \in \mathbb{N}$ we define

$$\tilde{f}_n(x) = \begin{cases} \beta_{I, \mathcal{F}_{n+1,I}, f_n}(x) & \text{if } x \in I \in \mathcal{I}_n \\ f_n(x) & \text{if } x \notin G_n \end{cases} \tag{45}$$

$$g_n(x) = \begin{cases} \tilde{f}_n(x) + \phi_J(x) & \text{if } x \in J \in \mathcal{J}_{n+1} \\ \tilde{f}_n(x) & \text{if } x \notin H_{n+1} \end{cases} \tag{46}$$

and

$$f_{n+1}(x) = \begin{cases} \tilde{f}_n(x) + \alpha_{E_k \cap J}(x) & \text{if } x \in J \in \mathcal{J}_{n+1,k} \subset \mathcal{J}_{n+1} \\ \tilde{f}_n(x) & \text{if } x \notin H_{n+1}. \end{cases} \tag{47}$$

Note that if $I_s = (a_s, b_s) \in \tilde{\mathcal{I}}_{n+1}$, then $\gamma_s \equiv 0$ on $[a_s, c_s] \cup [d_s, b_s]$, and it follows that

$$\tilde{f}_{n+1}(x) = f_{n+1}(x) = \tilde{f}_n(x) \text{ for all } x \notin G'_{n+1}. \tag{48}$$

We claim that for any $x \in [0,1]$ we have

$$g_{n+1}(x) \leq g_n(x). \tag{49}$$

In order to prove the claim, first of all note that if $x \notin H_{n+1}$, then $g_{n+1}(x) = g_n(x)$ so we may as well assume that $x \in H_{n+1}$. Choose $J = [a, b]$ such that $x \in J \in \mathcal{J}_{n+1,k} \subset \mathcal{J}_{n+1}$. Note that $\tilde{f}_n$ is constant on J and therefore we have

$$g_n(x) = \tilde{f}_n(x) + \phi_J(x) = \tilde{f}_n(a) + \phi_J(x). \tag{50}$$

We next show that

$$\tilde{f}_{n+1}(x) \leq \tilde{f}_n(a) + \frac{1}{2}\phi_J(x). \tag{51}$$

Note that if $x \notin G'_{n+1}$, (51) follows from (48) and the fact that $\tilde{f}_n$ is constant on J. On the other hand, suppose that $x \in G'_{n+1}$. In this case, $x \in I'_s$, where $I_s \in \tilde{I}_{n+1,k}$ and it follows from (24), (45), and the fact that $I'_s \subset \tilde{I}_s \subset J$, that we have

$$\tilde{f}_{n+1}(x) \leq \tilde{f}_n(a) + h_s \leq \tilde{f}_n(a) + \frac{1}{2}\phi_{\tilde{I}_s}(x) \leq \tilde{f}_n(a) + \frac{1}{2}\phi_J(x), \tag{52}$$

which gives us (51) again.

Now if $x \notin H_{n+2}$, we have $g_{n+1}(x) = \tilde{f}_{n+1}(x)$ and (49) follows from (50) and (51) so suppose that $x \in H_{n+2}$. Then $x \in K \subset J$, where $K \in \mathcal{J}_{n+2}$ and $g_{n+1}(x) = \tilde{f}_{n+1}(x) + \phi_K(x)$. Assume, first of all, that $x \notin G'_{n+1}$. Then (49) follows from (48), (50) and $K \subset J$. On the other hand, if $x \in G'_{n+1}$, then $x \in I'$, where $I \in \tilde{\mathcal{I}}_{n+1}$ and we have $x \in K \subset I' \subset I \subset J$ and it follows from (9) that $K \subset \frac{1}{3}J$. Thus, $\phi_K(x) \leq \frac{1}{2}\phi_J(x)$ and (49) follows from (50) and (51) once again and we are done proving the claim.

Note that we have the following inequalities, which hold for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$:

$$\tilde{f}_n(x) \leq \tilde{f}_{n+1}(x) \leq f_{n+2}(x) \leq g_{n+1}(x) \leq g_n(x) \tag{53}$$

$$0 \leq g_n(x) - \tilde{f}_n(x) \leq \sup_{J \in \mathcal{J}_{n+1}} \frac{|J|}{2}. \tag{54}$$

It follows easily that the sequence f_n converges uniformly on $[0, 1]$ to a continuous function f and for each $n \in \mathbb{N}$ we have

$$\tilde{f}_n(x) \leq f(x) \leq g_n(x) \text{ for all } x \in [0, 1]. \tag{55}$$

We extend f to all of $\mathbb{R}$ by defining $f(x) = 0$ if $x \notin [0, 1]$.

We now show that lip $f(x) < \infty$ for all $x \in [0, 1]$. Let $x \in [0, 1]$. First assume that $x \in \cap_{n=1}^{\infty} H_n$. In this case we can find a sequence of intervals

$\{J_n\}$ such that each $J_n \in \mathcal{J}_n$ and x is in the interior of each J_n. Note that if $x \in J = [a,b] \in \mathcal{J}_n$, then we have $f(a) \le f(y) \le f(a) + \phi_J(y)$ for all $y \in J$ and it follows that $L_f(x,r) \le 2r$, where $r = \min\{x-a, b-x\}$. This implies that $\mathrm{lip}\, f(x) \le 2$.

Now suppose that $x \notin \cap_{n=1}^{\infty} H_n$. Let n be the largest integer such that $x \in H_n$ and choose $J \in \mathcal{J}_n$ such that $x \in J$. Since $x \notin H_{n+1}$, it follows that $f(x) = \tilde{f}_n(x) = g_n(x)$. Then using (55) and the fact that $\tilde{f}_n$ and g_n are locally Lipschitz, it follows that $\mathrm{Lip}\, f(x) < \infty$ so trivially $\mathrm{lip}\, f(x) < \infty$.

We next show that $\cup_{n=1}^{\infty} E_n \subset N_f$. Let $x \in E_n$ for some $n \in \mathbb{N}$. Since $x \in E_n$, it follows that $x \in [a,b] = J \in \mathcal{J}_k$ for some $k \le n$, where $\{a,b\} \subset E_n$. From the construction of f it follows that $f(x) = f(y)$ for all $y \in E_n \cap J$ and therefore $\liminf_{t \to x} \dfrac{|f(t) - f(x)|}{|t-x|} = 0$. Moreover, if $\{I_s\}_{s \in S} = \mathcal{I}_{E_n \cap J, J}$, then by (5) we can choose a sequence $\{s_i\}$ in S such that $\{x\} = \cap_{i=1}^{\infty} \tilde{I}_{s_i}$. It follows that $m_{s_i} \to x$. Furthermore,

$$f(m_{s_i}) = f_k(m_{s_i}) = f_k(x) + \alpha_{E_n \cap J, J}(m_{s_i}) = f_k(x) + \frac{1}{6}|\tilde{I}_{s_i}|.$$

Thus, we get $\frac{|f(m_{s_i}) - f(x)|}{|m_{s_i} - x|} \ge \frac{1}{6}$ and therefore $\limsup_{t \to x} \frac{|f(t)-f(x)|}{|t-x|} \ge \frac{1}{6}$. Hence, f is not differentiable at x so $x \in N_f$.

It remains to show that f is differentiable a.e. on $[0,1] \backslash E$. For $J \in \mathcal{J}_{n,k}$ we let $T_J = T_{E_k \cap J, J}$ (where we use the notation from Definition 12) and we define $T_n = \cup_{J \in \mathcal{J}_n} T_J$ and note that $T_n \cap E = \emptyset$ for all $n \in \mathbb{N}$. We also observe that $f_k(x) = f_n(x)$ for all $x \in T_n$ and for all $k \ge n$ and hence $f(x) = f_n(x)$ for all $x \in T_n$. Finally, we define $T = \cup_{n=1}^{\infty} T_n$. Note that T is countable and therefore $|T| = 0$. We also note that, using (17) with $\epsilon = (b-a)/k$, it follows that $|H_n| < \sum_{k=n}^{\infty} |E_k| + \frac{1}{n}$ and therefore $|\cap_{n=1}^{\infty} H_n| = 0$. Thus it suffices to show that f is differentiable a.e. on $[0,1] \backslash (E \cup T \cup H)$, where $H = \cap_{n=1}^{\infty} H_n$. Suppose that $x \in [0,1] \backslash (E \cup T \cup H)$. Then there exists $n \in \mathbb{N}$ and $I \in \mathcal{I}_n$, such that $x \in I$ and $x \notin H_{n+1}$ so $\tilde{f}_n(x) = g_n(x) = f(x)$. Since $\tilde{f}_n \le f \le g_n$, it follows that f is differentiable at x if $\tilde{f}_n$ and g_n are both differentiable at x. But $\tilde{f}_n$ and g_n are both Lipschitz on I and therefore differentiable a.e. on I, which gives us the result we need.

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