RESEARCH

Real Analysis Exchange
Vol. 42(1), 2017, pp. 121–148

Silvestru S. Dragomir, Mathematics, College of Engineering and Science
Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia.
School of Computer Science and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.
email: `sever.dragomir@vu.edu.au`

VARIANCE JENSEN TYPE INEQUALITIES FOR GENERAL LEBESGUE INTEGRAL WITH APPLICATIONS

Abstract

Some inequalities similar to Jensen inequalities for general Lebesgue integral are obtained. Applications for functions of selfadjoint operators and functions of unitary operators on complex Hilbert spaces are provided as well.

1 Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

Assume that $f, g \in L(\Phi, \mu)$ with $fg \in L(\Phi, \mu)$ and consider the *Čebyšev functional*

$$C(f, g) := \int_\Omega fg d\mu - \int_\Omega f d\mu \int_\Omega g d\mu.$$

Mathematical Reviews subject classification: Primary: 26D15, 26D10; Secondary: 47A63
Key words: Jensen's inequality, Selfadjoint operators, Unitary operators
Received by the editors April 21, 2016
Communicated by: Andrei K. Lerner

It is known that if the function f,g are *synchronous*, i.e.

$$(f(t)-f(s))(g(t)-g(s))\geq 0$$

for μ-almost every $t,s\in\Omega$, then we have the *Čebyšev inequality*

$$C(f,g)\geq 0. \tag{1}$$

If there exist constants γ,Γ such that $\infty<\gamma\leq f\leq\Gamma<\infty$ μ-almost everywhere on Ω, then we have the following refinement of Grüss' inequality due to Cerone & Dragomir [2], which was obtained for univariate functions of real variable by Cheng & Sun in [3]:

$$\begin{aligned}|C(f,g)| &\leq \frac{1}{2}(\Gamma-\gamma)\int_\Omega\left|g-\int_\Omega gd\mu\right|d\mu \qquad (2)\\ &\leq \frac{1}{2}(\Gamma-\gamma)\left[\int_\Omega g^2d\mu-\left(\int_\Omega gd\mu\right)^2\right]^{1/2}.\end{aligned}$$

If there exist constants $\infty<\lambda\leq g\leq\Lambda<\infty$ μ-almost everywhere on Ω, then we have the sequence of inequalities

$$\begin{aligned}|C(f,g)| &\leq \frac{1}{2}(\Gamma-\gamma)\int_\Omega\left|g-\int_\Omega gd\mu\right|d\mu \qquad (3)\\ &\leq \frac{1}{2}(\Gamma-\gamma)\left[\int_\Omega g^2d\mu-\left(\int_\Omega gd\mu\right)^2\right]^{1/2}\leq\frac{1}{4}(\Gamma-\gamma)(\Lambda-\lambda).\end{aligned}$$

The inequality between the first and last term in (3) is known in the literature as *Grüss' inequality*.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in 2002 [4] the following result:

Theorem 1. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L(\Omega,\mu)$. Then we have the inequality:*

$$\begin{aligned}0 &\leq \int_\Omega\Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right) \qquad (4)\\ &\leq \int_\Omega(\Phi'\circ f)fd\mu-\int_\Omega\Phi'\circ fd\mu\int_\Omega fd\mu\\ &\leq \frac{1}{2}[\Phi'(M)-\Phi'(m)]\int_\Omega\left|f-\int_\Omega fd\mu\right|d\mu.\end{aligned}$$

In the case of discrete measure, we have:

Corollary 2. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0 \quad (i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \qquad (5) \\
&\leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right| .
\end{aligned}
$$

Remark 1. *We notice that the inequality between the first and the second term in (5) was proved in 1994 by Dragomir & Ionescu, see [34].*

On making use of the results (4) and (3), we can state the following string of reverse inequalities

$$
\begin{aligned}
0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi\left(\int_{\Omega} f d\mu\right) \qquad (6) \\
&\leq \int_{\Omega} (\Phi' \circ f) f d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\Phi'(M) - \Phi'(m)\right](M - m),
\end{aligned}
$$

provided that $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$, with $\int_{\Omega} d\mu = 1$.

The following reverse of the Jensen's inequality also holds [23]:

Theorem 3. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that f, $\Phi \circ f \in L(\Omega,\mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \left(M - \int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right)\frac{\Phi'_-(M) - \Phi'_+(m)}{M-m} \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{7}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

For other reverses of Jensen inequality and applications to divergence measures see [23], [24] and [25].

Motivated by the above results we establish in this paper some new inequalities for convex functions. Applications for functions of selfadjoint operators and functions of unitary operators on complex Hilbert spaces are provided as well.

2 The Results

The following result holds:

Theorem 4. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interior $\mathring{I}$ of I and $[0,\infty) \subset \mathring{I}$, $f : \Omega \to \mathbb{C}$ a μ-measurable function and such that $|f|^2$, $\Phi' \circ |f|^2$ and $\Phi \circ |f|^2 \in L(\Phi,\mu)$. Then*

$$\begin{aligned} &\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\Phi'_+\left(\left|\int_\Omega f d\mu\right|^2\right) \\ &\leq \int_\Omega \Phi\left(|f|^2\right) d\mu - \Phi\left(\left|\int_\Omega f d\mu\right|^2\right) \\ &\leq \int_\Omega \Phi'_-\left(|f|^2\right)\left(|f|^2 - \left|\int_\Omega f d\mu\right|^2\right) d\mu \\ &= C\left(\Phi'_-\left(|f|^2\right), |f|^2\right) + \left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\int_\Omega \Phi'_-\left(|f|^2\right) d\mu \end{aligned} \tag{8}$$

with

$$C\left(\Phi'_-\left(|f|^2\right), |f|^2\right) = \int_\Omega \Phi'_-\left(|f|^2\right)|f|^2 d\mu - \int_\Omega \Phi'_-\left(|f|^2\right) d\mu \int_\Omega |f|^2 d\mu \geq 0.$$

If there exist constants M, m such that

$$\infty > M \geq |f| \geq m \geq 0 \ \mu\text{-almost everywhere,} \tag{9}$$

then

$$\begin{aligned}
&C\left(\Phi'_-\left(|f|^2\right), |f|^2\right) \qquad (10)\\
&\leq \frac{1}{2}\begin{cases} \left(M^2-m^2\right)\int_\Omega \left|\Phi'_-\left(|f|^2\right) - \int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right| d\mu \\ \\ \left[\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right]\int_\Omega \left||f|^2 - \int_\Omega |f|^2 d\mu\right| d\mu \end{cases}\\
&\leq \frac{1}{2}\begin{cases} \left(M^2-m^2\right)\left[\int_\Omega \left[\Phi'_-\left(|f|^2\right)\right]^2 d\mu - \left(\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right)^2\right]^{1/2} \\ \\ \left[\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right]\left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2} \end{cases}\\
&\leq \frac{1}{4}\left(M^2-m^2\right)\left[\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right].
\end{aligned}$$

PROOF. Since $\Phi : I \to \mathbb{R}$ is a convex function on $\mathring{I}$, then

$$\Phi'_-(y)(y-x) \geq \Phi(y) - \Phi(x) \geq \Phi'_+(x)(y-x) \tag{11}$$

for any $x, y \in [0, \infty)$.

Now, by taking $y = |f(t)|^2$ and $x = \left|\int_\Omega f d\mu\right|^2$ we get

$$\begin{aligned}
&\Phi'_-\left(|f(t)|^2\right)\left(|f(t)|^2 - \left|\int_\Omega f d\mu\right|^2\right) \qquad (12)\\
&\quad \geq \Phi\left(|f(t)|^2\right) - \Phi\left(\left|\int_\Omega f d\mu\right|^2\right)\\
&\quad \geq \Phi'\left(\left|\int_\Omega f d\mu\right|^2\right)\left(|f(t)|^2 - \left|\int_\Omega f d\mu\right|^2\right)
\end{aligned}$$

for any $t \in \Omega$.

If we integrate over $d\mu(t)$ on Ω the inequality (12) then we get the first part of (8).

Now observe that

$$\int_\Omega \Phi'_-\left(|f|^2\right)\left(|f|^2-\left|\int_\Omega f d\mu\right|^2\right)d\mu \tag{13}$$
$$=\int_\Omega\left[\Phi'_-\left(|f|^2\right)-\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right]\left(|f|^2-\left|\int_\Omega f d\mu\right|^2\right)d\mu$$
$$+\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\left(\left[\int_\Omega |f|^2 d\mu-\left|\int_\Omega f d\mu\right|^2\right]\right).$$

Now,

$$\int_\Omega\left[\Phi'_-\left(|f|^2\right)-\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right]\left(|f|^2-\left|\int_\Omega f d\mu\right|^2\right)d\mu$$
$$=C\left(\Phi'_-\left(|f|^2\right),|f|^2\right).$$

And because the functions $\Phi'_-\left(|f|^2\right),|f|^2$ are synchronous on Ω (due to the fact that Φ'_- is monotonic nondecreasing almost everywhere on $[0,\infty)$), we also have

$$C\left(\Phi'_-\left(|f|^2\right),|f|^2\right)\geq 0.$$

Hence, we obtain the last part of (8).

Utilizing (3) we have either

$$C\left(\Phi'_-\left(|f|^2\right),|f|^2\right)\leq\frac{1}{2}\left(M^2-m^2\right)\int_\Omega\left|\Phi'_-\left(|f|^2\right)-\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right|d\mu$$
$$\leq\frac{1}{2}\left(M^2-m^2\right)\left[\int_\Omega\left[\Phi'_-\left(|f|^2\right)\right]^2 d\mu-\left(\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right)^2\right]^{1/2}$$
$$\leq\frac{1}{4}\left(M^2-m^2\right)\left(\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right),$$

or

$$C\left(\Phi'_-\left(|f|^2\right),|f|^2\right)\leq\frac{1}{2}\left(\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right)\int_\Omega\left||f|^2-\int_\Omega |f|^2 d\mu\right|d\mu$$
$$\leq\frac{1}{2}\left(\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right)\left[\int_\Omega |f|^4 d\mu-\left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2}$$
$$\leq\frac{1}{4}\left(M^2-m^2\right)\left(\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right).$$

This proves the inequality (10). □

Remark 2. *Let $\Phi : [0,\infty) \to \mathbb{R}$ be a convex function on $(0,\infty)$. If $x_i \in \mathbb{C}$ and $w_i \geq 0 \ \ (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then*

$$\left(\sum_{i=1}^{n} w_i \left|x_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i\right|^2\right) \Phi'_+ \left(\left|\sum_{i=1}^{n} w_i x_i\right|^2\right) \tag{14}$$
$$\leq \sum_{i=1}^{n} w_i \Phi\left(\left|x_i\right|^2\right) - \Phi\left(\left|\sum_{i=1}^{n} w_i x_i\right|^2\right)$$
$$\leq \sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right) \left|x_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i\right|^2 \sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right).$$

If $0 \leq m \leq |x_i| \leq M$ for $i = 1, \dots, n$; then

$$\sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right) \left|x_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i\right|^2 \sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right)$$
$$\leq \left(\sum_{i=1}^{n} w_i \left|x_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i\right|^2\right) \sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right)$$
$$+ \frac{1}{2} \begin{cases} \left(M^2 - m^2\right) \sum_{i=1}^{n} w_i \left|\Phi'_-\left(\left|x_i\right|^2\right) - \sum_{j=1}^{n} w_j \Phi'_-\left(\left|x_j\right|^2\right)\right| \\ \\ \left[\Phi'_-\left(M^2\right) - \Phi'_-\left(m^2\right)\right] \sum_{i=1}^{n} w_i \left|\left|x_i\right|^2 - \sum_{j=1}^{n} w_j \left|x_j\right|^2\right| \end{cases}$$
$$\leq \left(\sum_{i=1}^{n} w_i \left|x_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i\right|^2\right) \sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right)$$
$$+ \frac{1}{2} \begin{cases} \left(M^2 - m^2\right) \left[\sum_{i=1}^{n} w_i \left[\Phi'_-\left(\left|x_i\right|^2\right)\right]^2 - \left(\sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right)\right)^2\right]^{1/2} \\ \\ \left[\Phi'_-\left(M^2\right) - \Phi'_-\left(m^2\right)\right] \left[\sum_{i=1}^{n} w_i \left|x_i\right|^4 - \left(\sum_{i=1}^{n} w_i \left|x_i\right|^2\right)^2\right]^{1/2} \end{cases}$$
$$\leq \left(\sum_{i=1}^{n} w_i \left|x_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i\right|^2\right) \sum_{i=1}^{n} w_i \Phi'_-\left(\left|x_i\right|^2\right)$$
$$+ \frac{1}{4} \left(M^2 - m^2\right) \left[\Phi'_-\left(M^2\right) - \Phi'_-\left(m^2\right)\right].$$

We have the following particular cases of interest.

If we take $\Phi(t) = t^r$, $r \geq 1$ and $t \in [0, \infty)$, then we can state the following power inequalities.

Corollary 5. *Let $r \geq 1$. If $f : \Omega \to \mathbb{C}$ is a μ-measurable function and such that $|f|^2$, $|f|^{2(r-1)}$ and $|f|^{2r} \in L(\Phi, \mu)$, then*

$$r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\left|\int_\Omega f d\mu\right|^{2(r-1)} \leq \int_\Omega |f|^{2r} d\mu - \left|\int_\Omega f d\mu\right|^{2r} \leq r\left[\int_\Omega |f|^{2r} d\mu - \int_\Omega |f|^{2(r-1)} d\mu \left|\int_\Omega f d\mu\right|^2 d\mu\right]. \tag{15}$$

If there exist constants M, m such that (9) is valid, then

$$\begin{aligned}
&\int_\Omega |f|^{2r} d\mu - \left|\int_\Omega f d\mu\right|^{2r} \\
&\leq r\left[\int_\Omega |f|^{2r} d\mu - \int_\Omega |f|^{2(r-1)} d\mu \left|\int_\Omega f d\mu\right|^2 d\mu\right] \\
&\leq r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\int_\Omega |f|^{2(r-1)} d\mu \\
&\quad + \frac{1}{2} r \begin{cases} \left(M^2 - m^2\right)\int_\Omega \left||f|^{2(r-1)} - \int_\Omega |f|^{2(r-1)} d\mu\right| d\mu \\ \\ \left[M^{2(r-1)} - m^{2(r-1)}\right]\int_\Omega \left||f|^2 - \int_\Omega |f|^2 d\mu\right| d\mu \end{cases} \\
&\leq r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\int_\Omega |f|^{2(r-1)} d\mu \\
&\quad + \frac{1}{2} r \begin{cases} \left(M^2 - m^2\right)\left[\int_\Omega |f|^{4(r-1)} d\mu - \left(\int_\Omega |f|^{2(r-1)} d\mu\right)^2\right]^{1/2} \\ \\ \left[M^{2(r-1)} - m^{2(r-1)}\right]\left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2} \end{cases}
\end{aligned} \tag{16}$$

$$\leq r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\int_\Omega |f|^{2(r-1)} d\mu + \frac{1}{4} r\left(M^2 - m^2\right)\left[M^{2(r-1)} - m^{2(r-1)}\right].$$

If we take $\Phi(t) = -\ln t$, $t \in (0,\infty)$, then we can state the following logarithmic inequalities.

Corollary 6. *If $f : \Omega \to \mathbb{C}$ a μ-measurable function and such that $|f|^2$, $|f|^{-2}$, $\ln|f|^2 \in L(\Phi,\mu)$ and $\int_\Omega f d\mu \neq 0$, then*

$$\frac{\int_\Omega |f|^2 d\mu}{\left|\int_\Omega f d\mu\right|^2} - 1 \geq \int_\Omega \ln\left(|f|^2\right) d\mu - \ln\left(\left|\int_\Omega f d\mu\right|^2\right) \geq 1 - \left|\int_\Omega f d\mu\right|^2 \int_\Omega |f|^{-2} d\mu. \tag{17}$$

Finally, if we take $\Phi(t) = \exp(t)$, $t \in \mathbb{R}$, then we can state the following exponential inequalities.

Corollary 7. *If $f : \Omega \to \mathbb{C}$ is a μ-measurable function and such that $|f|^2$, $\exp\left(|f|^2\right)$, $|f|^2 \exp\left(|f|^2\right) \in L(\Phi,\mu)$, then*

$$\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\exp\left(\left|\int_\Omega f d\mu\right|^2\right) \leq \int_\Omega \exp\left(|f|^2\right) d\mu - \exp\left(\left|\int_\Omega f d\mu\right|^2\right) \leq \int_\Omega |f|^2 \exp\left(|f|^2\right) d\mu - \left|\int_\Omega f d\mu\right|^2 d\mu \int_\Omega \exp\left(|f|^2\right) d\mu. \tag{18}$$

If there exists the constants M, m such that (9) is valid, then

$$\begin{aligned}
&\int_\Omega \exp\left(|f|^2\right) d\mu - \exp\left(\left|\int_\Omega f d\mu\right|^2\right) \qquad (19)\\
&\quad\le \int_\Omega |f|^2 \exp\left(|f|^2\right) d\mu - \left|\int_\Omega f d\mu\right|^2 d\mu \int_\Omega \exp\left(|f|^2\right) d\mu\\
&\quad\le \left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right) \int_\Omega \exp\left(|f|^2\right) d\mu\\
&\qquad + \frac{1}{2}\begin{cases} \left(M^2 - m^2\right) \int_\Omega \left|\exp\left(|f|^2\right) - \int_\Omega \exp\left(|f|^2\right) d\mu\right| d\mu \\ \\ \left[\exp\left(M^2\right) - \exp\left(m^2\right)\right] \int_\Omega \left||f|^2 - \int_\Omega |f|^2 d\mu\right| d\mu \end{cases}\\
&\quad\le \left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right) \int_\Omega \exp\left(|f|^2\right) d\mu\\
&\qquad + \frac{1}{2}\begin{cases} \left(M^2 - m^2\right) \left[\int_\Omega \exp\left(2|f|^2\right) d\mu - \left(\int_\Omega \exp\left(|f|^2\right) d\mu\right)^2\right]^{1/2} \\ \\ \left[\exp\left(M^2\right) - \exp\left(m^2\right)\right] \left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2} \end{cases}\\
&\quad\le \left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right) \int_\Omega \exp\left(|f|^2\right) d\mu\\
&\qquad + \frac{1}{4}\left(M^2 - m^2\right)\left[\exp\left(M^2\right) - \exp\left(m^2\right)\right].
\end{aligned}$$

We also have:

Theorem 8. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interior $\mathring{I}$ of I and $[0,\infty) \subset \mathring{I}$, $f : \Omega \to \mathbb{C}$ a μ-measurable function such that $|f|^2$, $\Phi' \circ |f|^2$ and $\Phi \circ |f|^2 \in L(\Phi, \mu)$. Assume that $\int_\Omega \Phi'_+\left(|f|^2\right) d\mu \neq 0$ and*

$$\frac{\int_\Omega |f|^2 \Phi'_+\left(|f|^2\right) d\mu}{\int_\Omega \Phi'_+\left(|f|^2\right) d\mu} \ge 0, \qquad (20)$$

then

$$0 \leq \Phi\left(\frac{\int_\Omega |f|^2 \Phi'_+\left(|f|^2\right) d\mu}{\int_\Omega \Phi'_+\left(|f|^2\right) d\mu}\right) - \int_\Omega \Phi\left(|f|^2\right) d\mu \tag{21}$$
$$\leq \frac{1}{\int_\Omega \Phi'_+\left(|f|^2\right) d\mu} \Phi'_-\left(\frac{\int_\Omega |f|^2 \Phi'_+\left(|f|^2\right) d\mu}{\int_\Omega \Phi'_+\left(|f|^2\right) d\mu}\right) C\left(\Phi'_+\left(|f|^2\right), |f|^2\right),$$

where

$$C\left(\Phi'_+\left(|f|^2\right), |f|^2\right) = \int_\Omega \Phi'_+\left(|f|^2\right) |f|^2 d\mu - \int_\Omega \Phi'_+\left(|f|^2\right) d\mu \int_\Omega |f|^2 d\mu \geq 0.$$

If there exist constants M, m such that (9) is valid, then

$$C\left(\Phi'_+\left(|f|^2\right), |f|^2\right) \tag{22}$$
$$\leq \frac{1}{2}\begin{cases} \left(M^2 - m^2\right) \int_\Omega \left|\Phi'_+\left(|f|^2\right) - \int_\Omega \Phi'_+\left(|f|^2\right) d\mu\right| d\mu \\ \\ \left[\Phi'_+\left(M^2\right) - \Phi_+\left(m^2\right)\right] \int_\Omega \left||f|^2 - \int_\Omega |f|^2 d\mu\right| d\mu \end{cases}$$
$$\leq \frac{1}{2}\begin{cases} \left(M^2 - m^2\right) \left[\int_\Omega \left[\Phi'_+\left(|f|^2\right)\right]^2 d\mu - \left(\int_\Omega \Phi'_+\left(|f|^2\right) d\mu\right)^2\right]^{1/2} \\ \\ \left[\Phi'_+\left(M^2\right) - \Phi'_+\left(m^2\right)\right] \left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4}\left(M^2 - m^2\right)\left[\Phi'_+\left(M^2\right) - \Phi'_+\left(m^2\right)\right].$$

Proof. Let $t \in \Omega$. By taking $x = |f(t)|^2$ in (11) we have

$$\Phi'_-(y)\left(y - |f(t)|^2\right) \geq \Phi(y) - \Phi\left(|f(t)|^2\right) \tag{23}$$
$$\geq \Phi'_+\left(|f(t)|^2\right)\left(y - |f(t)|^2\right)$$

for any $y \in [0, \infty)$ and $t \in \Omega$.

If we integrate over $d\mu(t)$ on Ω the inequality (23) we have

$$\Phi'_-(y)\left(y - \int_\Omega |f|^2 d\mu\right) \geq \Phi(y) - \int_\Omega \Phi\left(|f|^2\right) d\mu \tag{24}$$
$$\geq y \int_\Omega \Phi'_+\left(|f|^2\right) d\mu - \int_\Omega |f|^2 \Phi'_+\left(|f|^2\right) d\mu$$

for any $y \in [0, \infty)$.

If we take

$$y = \frac{\int_\Omega |f|^2 \Phi'_+ \left(|f|^2\right) d\mu}{\int_\Omega \Phi'_+ \left(|f|^2\right) d\mu} \in [0, \infty)$$

in (24) then we get the first and the second inequalities in (23).

The last part of the proof follows in a similar way with the proof of Theorem 4 by replacing Φ'_- with Φ'_+. We omit the details. □

Remark 3. *Let $\Phi : [0, \infty) \to \mathbb{R}$ be a convex function on $(0, \infty)$, $x_i \in \mathbb{C}$ and $w_i \geq 0 \ \ (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$. Assume that $\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right) \neq 0$ and*

$$\frac{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right) |x_i|^2}{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right)} \geq 0,$$

then

$$\begin{aligned} 0 &\leq \Phi \left(\frac{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right) |x_i|^2}{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right)} \right) - \sum_{i=1}^n w_i \Phi \left(|x_i|^2\right) \qquad (25) \\ &\leq \frac{1}{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right)} \Phi'_- \left(\frac{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right) |x_i|^2}{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right)} \right) \\ &\quad \times \left[\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right) |x_i|^2 - \sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right) \sum_{i=1}^n w_i |x_i|^2 \right] \\ &\leq \frac{1}{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right)} \Phi'_- \left(\frac{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right) |x_i|^2}{\sum_{i=1}^n w_i \Phi'_+ \left(|x_i|^2\right)} \right) \\ &\quad \times \frac{1}{2} \begin{cases} \left(M^2 - m^2\right) \sum_{i=1}^n w_i \left| \Phi'_+ \left(|x_i|^2\right) - \sum_{j=1}^n w_j \Phi'_+ \left(|x_j|^2\right) \right| \\ \\ \left[\Phi'_+ \left(M^2\right) - \Phi_+ \left(m^2\right)\right] \sum_{i=1}^n w_i \left| |x_i|^2 - \sum_{j=1}^n w_j |x_j|^2 \right| \end{cases} \end{aligned}$$

$$\leq \frac{1}{2\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)} \Phi'_-\left(\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)}\right)$$
$$\times \begin{cases} \left(M^2-m^2\right)\left[\sum_{i=1}^n w_i \left[\Phi'_+\left(|x_i|^2\right)\right]^2 - \left(\sum_{i=1}^n w_i \left(|x_i|^2\right)\right)^2\right]^{1/2} \\ \\ \left[\Phi'_+\left(M^2\right)-\Phi'_+\left(m^2\right)\right]\left[\sum_{i=1}^n w_i |x_i|^4 - \left(\sum_{i=1}^n w_i |x_i|^2\right)^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)} \Phi'_-\left(\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)}\right)$$
$$\times \left(M^2-m^2\right)\left[\Phi'_+\left(M^2\right)-\Phi'_+\left(m^2\right)\right].$$

We have the following particular case of interest.

If we take $\Phi(t)=t^r$, $r\geq 1$ and $t\in[0,\infty)$, then we can state the following power inequalities.

Corollary 9. *Let $r\geq 1$. If $f:\Omega\to\mathbb{C}$ is a μ-measurable function such that $|f|^2$, $|f|^{2(r-1)}$ and $|f|^{2r}\in L(\Phi,\mu)$, then*

$$0 \leq \left(\frac{\int_\Omega |f|^{2r}\,d\mu}{\int_\Omega |f|^{2(r-1)}\,d\mu}\right)^r - \int_\Omega |f|^{2r}\,d\mu \tag{26}$$
$$\leq r\left(\frac{\int_\Omega |f|^{2r}\,d\mu}{\int_\Omega |f|^{2(r-1)}\,d\mu}\right)^{r-1}\left[\frac{\int_\Omega |f|^{2r}\,d\mu}{\int_\Omega |f|^{2(r-1)}\,d\mu} - \int_\Omega |f|^2\,d\mu\right].$$

If there exist constants M, m such that (9) is valid, then

$$0 \leq \left(\frac{\int_\Omega |f|^{2r}\,d\mu}{\int_\Omega |f|^{2(r-1)}\,d\mu}\right)^r - \int_\Omega |f|^{2r}\,d\mu \tag{27}$$
$$\leq \frac{r}{\int_\Omega |f|^{2(r-1)}\,d\mu}\left(\frac{\int_\Omega |f|^{2r}\,d\mu}{\int_\Omega |f|^{2(r-1)}\,d\mu}\right)^{r-1}$$
$$\times\left[\int_\Omega |f|^{2r}\,d\mu - \int_\Omega |f|^{2(r-1)}\,d\mu \int_\Omega |f|^2\,d\mu\right]$$

$$\leq \frac{r}{2\int_{\Omega}|f|^{2(r-1)}d\mu}\left(\frac{\int_{\Omega}|f|^{2r}d\mu}{\int_{\Omega}|f|^{2(r-1)}d\mu}\right)^{r-1}$$
$$\times\begin{cases}\left(M^2-m^2\right)\int_{\Omega}\left||f|^{2(r-1)}-\int_{\Omega}|f|^{2(r-1)}d\mu\right|d\mu\\ \\ \left[M^{2(r-1)}-m^{2(r-1)}\right]\int_{\Omega}\left||f|^2-\int_{\Omega}|f|^2d\mu\right|d\mu\end{cases}$$
$$\leq \frac{r}{2\int_{\Omega}|f|^{2(r-1)}d\mu}\left(\frac{\int_{\Omega}|f|^{2r}d\mu}{\int_{\Omega}|f|^{2(r-1)}d\mu}\right)^{r-1}$$
$$\times\begin{cases}\left(M^2-m^2\right)\left[\int_{\Omega}|f|^{4(r-1)}d\mu-\left(\int_{\Omega}|f|^{2(r-1)}d\mu\right)^2\right]^{1/2}\\ \\ \left[M^{2(r-1)}-m^{2(r-1)}\right]\left[\int_{\Omega}|f|^4d\mu-\left(\int_{\Omega}|f|^2d\mu\right)^2\right]^{1/2}\end{cases}$$
$$\leq \frac{r}{4\int_{\Omega}|f|^{2(r-1)}d\mu}\left(\frac{\int_{\Omega}|f|^{2r}d\mu}{\int_{\Omega}|f|^{2(r-1)}d\mu}\right)^{r-1}\times\left(M^2-m^2\right)\left[M^{2(r-1)}-m^{2(r-1)}\right].$$

If we take $\Phi(t)=-\ln t$, $t\in(0,\infty)$, then we can state the following logarithmic inequalities.

Corollary 10. *If $f:\Omega\to\mathbb{C}$ is a μ-measurable function such that $|f|^2$, $|f|^{-2}$, $\ln|f|^2\in L(\Phi,\mu)$ and $\int_{\Omega}fd\mu\neq 0$, then*

$$0\leq\int_{\Omega}\ln\left(|f|^2\right)d\mu-\ln\left(\frac{1}{\int_{\Omega}|f|^{-2}d\mu}\right)\leq\int_{\Omega}|f|^{-2}d\mu\int_{\Omega}|f|^2d\mu-1. \quad (28)$$

Finally, if we take $\Phi(t)=\exp(t)$, $t\in\mathbb{R}$, then we can state the following exponential inequalities.

Corollary 11. *If $f:\Omega\to\mathbb{C}$ is a μ-measurable function such that $|f|^2$, $\exp\left(|f|^2\right)$, $|f|^2\exp\left(|f|^2\right)\in L(\Phi,\mu)$, then*

$$0\leq\exp\left(\frac{\int_{\Omega}|f|^2\exp\left(|f|^2\right)d\mu}{\int_{\Omega}\exp\left(|f|^2\right)d\mu}\right)-\int_{\Omega}\exp\left(|f|^2\right)d\mu \quad (29)$$
$$\leq\exp\left(\frac{\int_{\Omega}|f|^2\exp\left(|f|^2\right)d\mu}{\int_{\Omega}\exp\left(|f|^2\right)d\mu}\right)\left[\frac{\int_{\Omega}\exp\left(|f|^2\right)|f|^2d\mu}{\int_{\Omega}\exp\left(|f|^2\right)d\mu}-\int_{\Omega}|f|^2d\mu\right].$$

If there exist constants M, m such that (9) is valid, then

$$
\begin{aligned}
0 &\le \exp\left(\frac{\int_\Omega |f|^2 \exp\left(|f|^2\right) d\mu}{\int_\Omega \exp\left(|f|^2\right) d\mu}\right) - \int_\Omega \exp\left(|f|^2\right) d\mu \qquad (30)\\
&\le \frac{1}{\int_\Omega \exp\left(|f|^2\right) d\mu} \exp\left(\frac{\int_\Omega |f|^2 \exp\left(|f|^2\right) d\mu}{\int_\Omega \exp\left(|f|^2\right) d\mu}\right)\\
&\quad \times \left[\int_\Omega \exp\left(|f|^2\right) |f|^2 d\mu - \int_\Omega \exp\left(|f|^2\right) d\mu \int_\Omega |f|^2 d\mu\right]\\
&\le \frac{1}{2\int_\Omega \exp\left(|f|^2\right) d\mu} \exp\left(\frac{\int_\Omega |f|^2 \exp\left(|f|^2\right) d\mu}{\int_\Omega \exp\left(|f|^2\right) d\mu}\right)\\
&\quad \times \begin{cases} \left(M^2 - m^2\right) \int_\Omega \left|\exp\left(|f|^2\right) - \int_\Omega \exp\left(|f|^2\right) d\mu\right| d\mu \\ \\ \left[\exp\left(M^2\right) - \exp\left(m^2\right)\right] \int_\Omega \left||f|^2 - \int_\Omega |f|^2 d\mu\right| d\mu \end{cases}\\
&\le \frac{1}{2\int_\Omega \exp\left(|f|^2\right) d\mu} \exp\left(\frac{\int_\Omega |f|^2 \exp\left(|f|^2\right) d\mu}{\int_\Omega \exp\left(|f|^2\right) d\mu}\right)\\
&\quad \times \begin{cases} \left(M^2 - m^2\right) \left[\int_\Omega \exp\left(2|f|^2\right) d\mu - \left(\int_\Omega \exp\left(|f|^2\right) d\mu\right)^2\right]^{1/2} \\ \\ \left[\exp\left(M^2\right) - \exp\left(m^2\right)\right] \left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2} \end{cases}\\
&\le \frac{1}{4\int_\Omega \exp\left(|f|^2\right) d\mu} \exp\left(\frac{\int_\Omega |f|^2 \exp\left(|f|^2\right) d\mu}{\int_\Omega \exp\left(|f|^2\right) d\mu}\right)\\
&\quad \times \left(M^2 - m^2\right) \left[\exp\left(M^2\right) - \exp\left(m^2\right)\right].
\end{aligned}
$$

3 Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{31}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [35, p. 256]:

Theorem 12 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;

b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;

c) We have the representation

$$A = \int_{m-0}^{M} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{32}$$

where the integral is of Riemann-Stieltjes type.

Corollary 13. *With the assumptions of Theorem 12 for* A, E_λ *and* φ *we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{33}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

For a bounded linear operator B on H we denote $|B| := \sqrt{B^*B}$. We can state the following result for functions of selfadjoint operators.

Theorem 14. *Let* A *be a bounded selfadjoint operator on the Hilbert space* H *and let* $m = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ *and* $M = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. *Assume that* $\Phi : I \to \mathbb{R}$ *is a differentiable convex function on the interior* $\mathring{I}$ *of* I *with* $[0,\infty) \subset \mathring{I}$, *and the derivative* Φ' *is continuous on* $\mathring{I}$. *If* $f : J \to \mathbb{C}$ *is a continuous function on* J *with* $[m, M] \subset \mathring{J}$, *then we have the inequalities*

$$\begin{aligned}
&\left(\|f(A)x\|^2 - |\langle f(A)x, x\rangle|^2\right)\Phi'\left(|\langle f(A)x, x\rangle|^2\right) \\
&\quad \le \left\langle \Phi\left(|f(A)|^2\right)x, x\right\rangle - \Phi\left(|\langle f(A)x, x\rangle|^2\right) \\
&\quad \le \left\langle \Phi'\left(|f(A)|^2\right)|f(A)|^2 x, x\right\rangle - |\langle f(A)x, x\rangle|^2 \left\langle \Phi'\left(|f(A)|^2\right)x, x\right\rangle
\end{aligned} \tag{34}$$

for any $x \in H$, $\|x\| = 1$. *If* $\gamma := \min_{t\in[m,M]} |f(t)|$ *and* $\Gamma = \max_{t\in[m,M]} |f(t)|$, *then*

$$\begin{aligned}
&\left\langle \Phi'\left(|f(A)|^2\right)|f(A)|^2 x, x\right\rangle - |\langle f(A)x, x\rangle|^2 \left\langle \Phi'\left(|f(A)|^2\right)x, x\right\rangle \\
&\quad \le \left(\|f(A)x\|^2 - |\langle f(A)x, x\rangle|^2\right)\left\langle \Phi'\left(|f(A)|^2\right)x, x\right\rangle \\
&\qquad + \frac{1}{4}\left(\Gamma^2 - \gamma^2\right)\left[\Phi'\left(\Gamma^2\right) - \Phi'\left(\gamma^2\right)\right]
\end{aligned} \tag{35}$$

for any $x \in H$, $\|x\| = 1$. In particular, for $f(t) = t$, and for any $x \in H$ with $\|x\| = 1$.we have

$$\left(\|Ax\|^2 - |\langle Ax, x\rangle|^2\right) \Phi'\left(|\langle Ax, x\rangle|^2\right) \tag{36}$$
$$\leq \left\langle \Phi\left(A^2\right) x, x\right\rangle - \Phi\left(|\langle Ax, x\rangle|^2\right)$$
$$\leq \left\langle \Phi'\left(A^2\right) A^2 x, x\right\rangle - |\langle Ax, x\rangle|^2 \left\langle \Phi'\left(A^2\right) x, x\right\rangle .$$

If we denote $n := \min_{t\in[m,M]} |t|$ and $N := \max_{t\in[m,M]} |t|$, then

$$\left\langle \Phi'\left(A^2\right) A^2 x, x\right\rangle - |\langle Ax, x\rangle|^2 \left\langle \Phi'\left(A^2\right) x, x\right\rangle \tag{37}$$
$$\leq \left(\|Ax\|^2 - |\langle Ax, x\rangle|^2\right) \left\langle \Phi'\left(A^2\right) x, x\right\rangle$$
$$+ \frac{1}{4}\left(N^2 - n^2\right)\left[\Phi'\left(N^2\right) - \Phi'\left(n^2\right)\right]$$

for any $x \in H$, $\|x\| = 1$.

PROOF. Let $x \in H$, $\|x\| = 1$, $\varepsilon > 0$ and $f : [m-\varepsilon, M] \subset J \to \mathbb{C}$, which is continuous on $[m-\varepsilon, M]$. If we use the inequality (8) for the measure $d\mu = dg$, where $g : [m-\varepsilon, M] \to \mathbb{R}$ is the monotonic nondecreasing function $g(t) := \langle E_t x, x\rangle$, and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, is the spectral family of A, then we have the inequality

$$\left(\int_{m-\varepsilon}^{M} |f(t)|^2 d\langle E_t x, x\rangle - \left|\int_{m-\varepsilon}^{M} f(t)\, d\langle E_t x, x\rangle\right|^2\right) \tag{38}$$
$$\times \Phi'\left(\left|\int_{m-\varepsilon}^{M} f(t)\, d\langle E_t x, x\rangle\right|^2\right)$$
$$\leq \int_{m-\varepsilon}^{M} \Phi\left(|f(t)|^2\right) d\langle E_t x, x\rangle - \Phi\left(\left|\int_{m-\varepsilon}^{M} f(t)\, d\langle E_t x, x\rangle\right|^2\right)$$
$$\leq \int_{m-\varepsilon}^{M} \Phi'\left(|f(t)|^2\right) |f(t)|^2 d\langle E_t x, x\rangle$$
$$- \left|\int_{m-\varepsilon}^{M} f(t)\, d\langle E_t x, x\rangle\right|^2 \int_{m-\varepsilon}^{M} \Phi'\left(|f(t)|^2\right) d\langle E_t x, x\rangle .$$

Taking the limit over $\varepsilon \to 0+$ in (38) and utilizing the Spectral Representation Theorem for selfadjoint operators we get the desired inequality (34).

The inequality (35) follows in a similar manner by making use of (10). The details are omitted.

For $f(t)=t$ we have $|f(A)|^2=|A|^2=A^*A=A^2$ and the inequalities (36) and (37) follow from (34) and (35). □

We have the following power inequalities.

Corollary 15. *Let A be a bounded selfadjoint operator on the Hilbert space H and $0\le mI\le A\le MI$. Then for $r\ge 1$ we have*

$$\begin{aligned} & r\left(\|Ax\|^2-|\langle Ax,x\rangle|^2\right)|\langle Ax,x\rangle|^{2(r-1)} \\ & \le \left\langle A^{2r}x,x\right\rangle-|\langle Ax,x\rangle|^{2r} \\ & \le r\left[\left\langle A^{2r}x,x\right\rangle-|\langle Ax,x\rangle|^2\left\langle A^{2(r-1)}x,x\right\rangle\right] \\ & \le r\left(\|Ax\|^2-|\langle Ax,x\rangle|^2\right)\left\langle A^{2(r-1)}x,x\right\rangle \\ & \quad +\frac{1}{4}r\left(M^2-m^2\right)\left[M^{2(r-1)}-m^{2(r-1)}\right] \end{aligned} \tag{39}$$

for any $x\in H$, $\|x\|=1$.

We have the following logarithmic inequalities.

Corollary 16. *Let A be a positive definite operator on the Hilbert space H. Then*

$$\begin{aligned} \frac{\|Ax\|^2}{|\langle Ax,x\rangle|^2}-1 & \ge \left\langle \ln\left(A^2\right)x,x\right\rangle-\ln\left(|\langle Ax,x\rangle|^2\right) \\ & \ge 1-|\langle Ax,x\rangle|^2\left\langle A^{-2}x,x\right\rangle \end{aligned} \tag{40}$$

for any $x\in H$, $\|x\|=1$.

We have the following exponential inequalities.

Corollary 17. *Let A be a bounded selfadjoint operator on the Hilbert space H and $mI\le A\le MI$. Then we have*

$$\begin{aligned} & \left(\|Ax\|^2-|\langle Ax,x\rangle|^2\right)\exp\left(|\langle Ax,x\rangle|^2\right) \\ & \le \left\langle \exp\left(A^2\right)x,x\right\rangle-\exp\left(|\langle Ax,x\rangle|^2\right) \\ & \le \left\langle \exp\left(A^2\right)A^2x,x\right\rangle-|\langle Ax,x\rangle|^2\left\langle \exp\left(A^2\right)x,x\right\rangle \end{aligned} \tag{41}$$

for any $x\in H$, $\|x\|=1$.

If we denote $n := \min_{t\in[m,M]} |t|$ *and* $N := \max_{t\in[m,M]} |t|$, *then*

$$\begin{aligned}\left\langle \exp\left(A^2\right) A^2 x, x\right\rangle - |\langle Ax, x\rangle|^2 \left\langle \exp\left(A^2\right) x, x\right\rangle \\ \leq \left(\|Ax\|^2 - |\langle Ax, x\rangle|^2\right) \left\langle \exp\left(A^2\right) x, x\right\rangle \\ + \frac{1}{4}\left(N^2 - n^2\right)\left[\exp\left(N^2\right) - \exp\left(n^2\right)\right]\end{aligned} \tag{42}$$

for any $x \in H$, $\|x\| = 1$.

For recent inequalities for functions of selfadjoint operators on Hilbert spaces, see [5]-[20] and the monographs [21] and [22].

4 Applications for Unitary Operators

A *unitary operator* is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying

$$U^*U = UU^* = 1_H$$

where U^* is the adjoint of U, and $1_H : H \to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle \cdot, \cdot \rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux, Uy\rangle = \langle x, y\rangle$ and

(ii) U is surjective.

The following result is well known [35, p. 275 - p. 276]:

Theorem 18 (Spectral Representation Theorem)**.** *Let* U *be a unitary operator on the Hilbert space* H*. Then there exists a family of projections* $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, *called the spectral family of* U*, with the following properties:*

a) $P_\lambda \leq P_{\lambda'}$ *for* $\lambda \leq \lambda'$.

b) $P_0 = 0, P_{2\pi} = I$ *and* $P_{\lambda+0} = P_\lambda$ *for all* $\lambda \in [0, 2\pi)$.

c) We have the representation

$$U = \int_0^{2\pi} \exp(i\lambda)\, dP_\lambda.$$

More generally, for every continuous complex-valued function f defined on the unit circle $\mathcal{C}(0,1)$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| f(U) - \sum_{k=1}^{n} f(\exp(i\lambda_k')) \left[P_{\lambda_k} - P_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$f(U) = \int_0^{2\pi} f(\exp(i\lambda))\, dP_\lambda, \tag{43}$$

where the integral is of Riemann-Stieltjes type.

Corollary 19. *With the assumptions of Theorem 18 for U, P_λ and f we have the representations*

$$f(U)x = \int_0^{2\pi} f(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{44}$$

In particular,

$$\langle f(U)x, x\rangle = \int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|f(U)x\|^2 = \int_0^{2\pi} |f(\exp(i\lambda))|^2\, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

The following result holds.

Theorem 20. *Let U be a unitary operator on the Hilbert space H. Assume that $\Phi : [0, 2\pi] \to \mathbb{R}$ is a differentiable convex function on $(0, 2\pi)$ and the* *derivative* Φ' *is continuous on* $(0,2\pi)$. *If* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *is a continuous function on the unit circle* $\mathcal{C}(0,1)$, *then we have the inequalities*

$$\begin{aligned}
&\left(\|f(U)x\|^2 - |\langle f(U)x,x\rangle|^2\right)\Phi'\left(|\langle f(U)x,x\rangle|^2\right) \\
&\quad\le \left\langle \Phi\left(|f(U)|^2\right)x,x\right\rangle - \Phi\left(|\langle f(U)x,x\rangle|^2\right) \\
&\quad\le \left\langle \Phi'\left(|f(U)|^2\right)|f(U)|^2 x,x\right\rangle - |\langle f(U)x,x\rangle|^2\left\langle \Phi'\left(|f(U)|^2\right)x,x\right\rangle
\end{aligned} \tag{45}$$

for any $x \in H$, $\|x\| = 1$.

If $k = \min_{z\in\mathcal{C}(0,1)} |f(z)|$ *and* $K = \max_{z\in\mathcal{C}(0,1)} |f(z)|$, *then*

$$\begin{aligned}
&\left\langle \Phi'\left(|f(U)|^2\right)|f(U)|^2 x,x\right\rangle - |\langle f(U)x,x\rangle|^2\left\langle \Phi'\left(|f(U)|^2\right)x,x\right\rangle \\
&\quad\le \left(\|f(U)x\|^2 - |\langle f(U)x,x\rangle|^2\right)\left\langle \Phi'\left(|f(U)|^2\right)x,x\right\rangle \\
&\qquad + \frac{1}{4}\left(K^2-k^2\right)\left[\Phi'\left(K^2\right)-\Phi'\left(k^2\right)\right]
\end{aligned} \tag{46}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Let $x \in H$ with $\|x\| = 1$ and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U. Utilising the inequality (8) we have

$$\begin{aligned}
&\left(\int_0^{2\pi} |f(\exp(i\lambda))|^2 d\langle P_\lambda x,x\rangle - \left|\int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x,x\rangle\right|^2\right) \\
&\qquad \times \Phi'\left(\left|\int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x,x\rangle\right|^2\right) \\
&\quad\le \int_0^{2\pi} \Phi\left(|f(\exp(i\lambda))|^2\right) d\langle P_\lambda x,x\rangle - \Phi\left(\left|\int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x,x\rangle\right|^2\right) \\
&\quad\le \int_0^{2\pi} \Phi'\left(|f(\exp(i\lambda))|^2\right)|f(\exp(i\lambda))|^2 d\langle P_\lambda x,x\rangle \\
&\qquad - \left|\int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x,x\rangle\right|^2 \int_0^{2\pi} \Phi'\left(|f(\exp(i\lambda))|^2\right) d\langle P_\lambda x,x\rangle .
\end{aligned} \tag{47}$$

By making use of the Spectral Representation Theorem for unitary operators we get from (47) the desired result (45). □

Corollary 21. *Let U be a unitary operator on the Hilbert space H. Assume that $\Phi : [0, 2\pi] \to \mathbb{R}$ is a differentiable convex function on $(0, 2\pi)$ and the derivative Φ' is continuous on $(0, 2\pi)$. Then we have the inequalities*

$$\begin{aligned}
&\left(\langle \exp[2\operatorname{Re}(U)]x, x\rangle - |\langle \exp(U)x, x\rangle|^2\right)\Phi'\left(|\langle \exp(U)x, x\rangle|^2\right) \qquad (48)\\
&\quad\le \langle \Phi(\exp[2\operatorname{Re}(U)])x, x\rangle - \Phi\left(|\langle \exp(U)x, x\rangle|^2\right)\\
&\quad\le \langle \Phi'(\exp[2\operatorname{Re}(U)])\exp[2\operatorname{Re}(U)]x, x\rangle\\
&\qquad - |\langle \exp(U)x, x\rangle|^2 \langle \Phi'(\exp[2\operatorname{Re}(U)])x, x\rangle
\end{aligned}$$

for any $x \in H$, $\|x\| = 1$, where $\operatorname{Re}(U) = \frac{1}{2}(U^* + U)$.

PROOF. If we take in (45) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \exp(z)$, then

$$\begin{aligned}
|f(U)|^2 &= |\exp(U)|^2 = [\exp(U)]^* \exp(U)\\
&= \exp(U^*)\exp(U) = \exp(U^* + U) = \exp[2\operatorname{Re}(U)]
\end{aligned}$$

since $U^*U = UU^* = I$.

This proves the inequality (48). □

Remark 4. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ is a continuous function on the unit circle $\mathcal{C}(0,1)$. If we take $\Phi(t) = t^r$, $r \ge 1$, then we have from (45) and (46) that*

$$\begin{aligned}
&r\left(\|f(U)x\|^2 - |\langle f(U)x, x\rangle|^2\right)|\langle f(U)x, x\rangle|^{2(r-1)} \qquad (49)\\
&\quad\le \left\langle |f(U)|^{2r} x, x\right\rangle - |\langle f(U)x, x\rangle|^{2r}\\
&\quad\le r\left\langle |f(U)|^{2r} x, x\right\rangle - |\langle f(U)x, x\rangle|^2 \left\langle |f(U)|^{2(r-1)} x, x\right\rangle
\end{aligned}$$

for any $x \in H$, $\|x\| = 1$. If $k = \min_{z\in\mathcal{C}(0,1)} |f(z)|$ and $K = \max_{z\in\mathcal{C}(0,1)} |f(z)|$, then

$$\begin{aligned}
&\left\langle |f(U)|^{2r} x, x\right\rangle - |\langle f(U)x, x\rangle|^2 \left\langle |f(U)|^{2(r-1)} x, x\right\rangle \qquad (50)\\
&\quad\le \left(\|f(U)x\|^2 - |\langle f(U)x, x\rangle|^2\right)\left\langle |f(U)|^{2(r-1)} x, x\right\rangle\\
&\qquad + \frac{1}{4}\left(K^2 - k^2\right)\left[K^{2(r-1)} - k^{2(r-1)}\right]
\end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

Assume that $f(z) \neq 0$ *for any* $z \in \mathcal{C}(0,1)$. *By taking* $\Phi(t) = -\ln t$ *in (45) and (46), we get*

$$\frac{\|f(U)x\|^2}{|\langle f(U)x,x\rangle|^2} - 1 \geq \left\langle \ln\left(|f(U)|^2\right)x,x\right\rangle - \ln\left(|\langle f(U)x,x\rangle|^2\right) \geq 1 - |\langle f(U)x,x\rangle|^2 \left\langle |f(U)|^{-2}x,x\right\rangle \tag{51}$$

for any $x \in H$, $\|x\| = 1$.

For recent inequalities for functions of unitary operators, see [26]-[33].

References

[1] R. P. Agarwal and S S. Dragomir, *A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces.* Comput. Math. Appl., **59(12)** (2010), 3785–3812.

[2] P. Cerone and S. S. Dragomir, *A refinement of the Grüss inequality and applications,* Tamkang J. Math., **38(1)** (2007), 37–49. Preprint RGMIA Res. Rep. Coll., **5(2)** (2002), Art. 14.

[3] X. L. Cheng and J. Sun, *Note on the perturbed trapezoid inequality*, J. Inequal. Pure & Appl. Math., **3(2)** (2002), Art. 21.

[4] S. S. Dragomir, *A Grüss type inequality for isotonic linear functionals and applications.* Demonstratio Math. **36(3)** (2003), 551–562. Preprint RGMIA Res. Rep. Coll., **5**(2002), Suplement, Art. 12. `[Online http://rgmia.org/v5(E).php]`.

[5] S. S. Dragomir, *Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces.* Aust. J. Math. Anal. Appl., **6(1)** (2009), Art. 7, 58 pp.

[6] S. S. Dragomir, *Some inequalities for power series of selfadjoint operators in Hilbert spaces via reverses of the Schwarz inequality.* Integral Transforms Spec. Funct., **20(9-10)** (2009), 757–767.

[7] S. S. Dragomir, *Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces.* J. Inequal. Appl., **2010**, Art. ID 496821, 15 pp.

[8] S. S. Dragomir, *Some new Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces.* Sarajevo J. Math., **6(1)**(18) (2010), 89–107.

[9] S. S. Dragomir, *Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces.* Linear Multilinear Algebra, **58(7-8)** (2010), 805–814.

[10] S. S. Dragomir, *Vector and operator trapezoidal type inequalities for continuous functions of selfadjoint operators in Hilbert spaces.* Electron. J. Linear Algebra, **22** (2011), 161–178.

[11] S. S. Dragomir, *Some trapezoidal vector inequalities for continuous functions of selfadjoint operators in Hilbert spaces.* Abstr. Appl. Anal., **2011**, Art. ID 941286, 13 pp.

[12] S. S. Dragomir, *Ostrowski's type inequalities for continuous functions of selfadjoint operators on Hilbert spaces: a survey of recent results.* Ann. Funct. Anal. **2(1)** (2011), 139–205.

[13] S. S. Dragomir, *New Jensen's type inequalities for differentiable log-convex functions of selfadjoint operators in Hilbert spaces.* Sarajevo J. Math., **7(1)**(19) (2011), 67–80.

[14] S. S. Dragomir, *Some Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces.* Bull. Malays. Math. Sci. Soc., (2) **34(3)** (2011), 445–454.

[15] S. S. Dragomir, *Some Slater type inequalities for convex functions of selfadjoint operators in Hilbert spaces.* Rev. Un. Mat. Argentina, **52(1)** (2011), 109–120.

[16] S. S. Dragomir, *Ostrowski's type inequalities for some classes of continuous functions of selfadjoint operators in Hilbert spaces.* Comput. Math. Appl., **62(12)** (2011), 4439–4448.

[17] S. S. Dragomir, *Refinements of the Cauchy-Bunyakovsky-Schwarz inequality for functions of selfadjoint operators in Hilbert spaces.* Linear Multilinear Algebra, **59(7)** (2011), 711–717.

[18] S. S. Dragomir, *Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces.* Ital. J. Pure Appl. Math., **28** (2011), 207–224.

[19] S. S. Dragomir, *Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces.* Linear Algebra Appl., **436(5)** (2012), 1503–1515.

[20] S. S. Dragomir, *Some Jensen type inequalities for square-convex functions of selfadjoint operators in Hilbert spaces.* Commun. Math. Anal., **14(1)** (2013), 42–58.

[21] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type*. Springer Briefs in Mathematics, Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[22] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type*. Springer Briefs in Mathematics, Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

[23] S. S. Dragomir, *Reverses of the Jensen inequality in terms of the first derivative and applications,* Acta Math. Vietnam., **38(3)** (2013), 429–446. Preprint RGMIA Res. Rep. Coll., **14** (2011), Art. 71 [http://rgmia.org/papers/v14/v14a71.pdf].

[24] S. S. Dragomir, *Some reverses of the Jensen inequality with applications,* Bull. Aust. Math. Soc., **87(2)** (2013), 177–194. Preprint RGMIA Res. Rep. Coll., **14** (2011), Art. 72. [http://rgmia.org/papers/v14/v14a72.pdf].

[25] S. S. Dragomir, *A refinement and a divided difference reverse of Jensen's inequality with applications*, Preprint RGMIA Res. Rep. Coll., **14** (2011), Art. 74. [http://rgmia.org/papers/v14/v14a74.pdf].

[26] S. S. Dragomir, *Jensen type weighted inequalities for functions of selfadjoint and unitary operators,* Ital. J. Pure Appl. Math., **32** (2014), 247–264. Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 1.

[27] S. S. Dragomir, *A generalized Cebysev functional for the Riemann-Stieltjes integral with applications for selfadjoint and unitary operators,* Math. Inequal. Appl., **18(3)** (2015), 959–973. Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 3.

[28] S. S. Dragomir, *Some inequalities of Jensen type for arg-square convex functions of unitary operators in Hilbert spaces*, Bull. Aust. Math. Soc., **90(1)** (2014), 65–73. Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 5.

[29] S. S. Dragomir, *Ostrowski's type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces,* Arch. Math. (Brno), **51(4)** (2015), 233–254. Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 6.

[30] S. S. Dragomir, *Trapezoid type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces*, Georgian Math. J., **23(2)** (2016), 199–210. Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 7.

[31] S. S. Dragomir, *Quasi Gruss type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces*, Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 8.

[32] S. S. Dragomir, *Generalized trapezoid type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces*, Mediterr. J. Math., **12(3)** (2015), 573–591. Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 9.

[33] S. S. Dragomir, *Gruss type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces*, Preprint RGMIA Res. Rep. Coll., **16** (2013), Art. 10.

[34] S. S. Dragomir and N. M. Ionescu, *Some converse of Jensen's inequality and applications.* Rev. Anal. Numér. Théor. Approx., **23(1)** (1994), 71–78.

[35] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. New York, 1969.