RESEARCH

Real Analysis Exchange
Vol. 40(1), 2014/2015, pp. 157–178

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AN INTEGRAL ON A COMPLETE METRIC MEASURE SPACE

Abstract

We study a Henstock-Kurzweil type integral defined on a complete metric measure space X endowed with a Radon measure μ and with a family of "cells" $\mathcal{F}$ that satisfies the Vitali covering theorem with respect to μ. This integral encloses, in particular, the classical Henstock-Kurzweil integral on the real line, the dyadic Henstock-Kurzweil integral, the Mawhin's integral [19], and the s-HK integral [4]. The main result of this paper is the extension of the usual descriptive characterizations of the Henstock-Kurzweil integral on the real line, in terms of ACG^* functions (Main Theorem 1) and in terms of *variational measures* (Main Theorem 2).

1 Introduction

The following descriptive characterizations of the Henstock-Kurzweil integral on the real line are well known:

Mathematical Reviews subject classification: Primary: 26A39, ; Secondary: 28A12
Key words: HK-integral, $ACG^{\triangle}$ function, critical variation
Received by the editors June 25, 2014
Communicated by: Luisa Di Piazza

Theorem A. [13, Theorem 6.12, Theorem 6.13] *A function* $f\colon [a,b] \to \mathbb{R}$ *is Henstock-Kurzweil integrable on* $[a,b]$ *if and only if there exists a function* $F\colon [a,b] \to \mathbb{R}$ *such that* F *is* ACG^* *and* $F'(x) = f(x)$ *almost everywhere on* $[a,b]$.

Theorem B. [2, Theorem 3] *A function* $f\colon [a,b] \to \mathbb{R}$ *is Henstock-Kurzweil integrable on* $[a,b]$ *if and only if there exists a function* $F\colon [a,b] \to \mathbb{R}$ *such that its variational measure is absolutely continuous with respect to the Lebesgue measure and* $F'(x) = f(x)$ *almost everywhere on* $[a,b]$.

Concerning the n-dimensional Henstock-Kurzweil integral, with $n > 1$, theorems of type A were proved by Lee-Leng [14], by Lu-Lee [17], and by Tuo-Yeong [25]. A theorem of type B was proved by Tuo-Yeong [24], [26], [27].

Moreover, in contrast with the one-dimensional case, the n-dimensional Henstock-Kurzweil integral, with $n > 1$, does not integrate all derivatives. This was the reason for several modifications of the definition of the n-dimensional Henstock-Kurzweil integral done by some mathematicians, including Mawhin [19], Jarnik-Kurzweil-Schwabik [12], and Pfeffer [20], [21].

For such above integrals, extensions of theorems of type A and B were done, by others, by Bongiorno-Pfeffer-Thomson [3], by Buczolich-Pfeffer [5], by De Pauw [6], by Di Piazza [7], and by Faure [9].

In the more general setting of a generic metric measure space, it is well known that the biggest difficulty in the definition of a Henstock-Kurzweil type integral is that of finding a suitable family of measurable sets which plays the role of "intervals".

Leng-Yee [16] studied, on a complete metric measure space, the Henstock-Kurzweil integral generated by the family of all finite intersections of sets that are the difference of two closed balls.

Later, a theorem of type A for this integral was proved by Leng [15]. Unfortunately, his characterization requires, on the primitive function F, besides an ACG^*-type notion, some strong additional conditions (see [15, Theorem 19]).

In this paper we prove that, if the family of "intervals", used in the definition of a Henstock-Kurzweil type integral on a complete metric measure space, satisfies, besides the usual conditions, the Vitali covering theorem with respect to the given measure, then it is possible to obtain natural extensions of both Theorems A and B.

2 Preliminaries

We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all natural and real numbers, respectively. Let $X = (X, d)$ be a complete metric space. For each $x \in X$ and $E \subset X$, we denote by χ_E, $\mathrm{diam}(E)$, ∂E, E^o and $\mathrm{d}(x, E)$ the characteristic function of E, the diameter of E, the boundary of E, the interior of E and the distance from x to E, respectively.

Let μ be a non-atomic Radon measure on X, let $\mathcal{G}$ be a family of non-empty closed subsets of X and let $E \subset X$. The family $\mathcal{G}$ is said to be a *fine cover* of E if

$$\inf\{\mathrm{diam}\, Q : Q \in \mathcal{G},\ Q \ni x\} = 0,$$

for each $x \in E$.

A family $\mathcal{F}$ of non-empty closed subsets of X is said to be a μ-*Vitali* family if it satisfies the following Vitali covering theorem:

Theorem 2.1. *For each subset E of X and for each subfamily $\mathcal{G}$ of $\mathcal{F}$ that is a fine cover of E, there exists a countable system $\{Q_1, Q_2, \cdots, Q_j, \cdots\} \subset \mathcal{G}$ such that Q_i and Q_j are non-overlapping (i.e. the interiors of Q_i and Q_j are disjoint), for each $i \neq j$, and such that $\mu(E \setminus \bigcup Q_j) = 0$.*

A μ-Vitali family $\mathcal{F}$ is said to be a family of μ-*cells* if it satisfies the following conditions:

(a) Given $Q \in \mathcal{F}$ and a constant $\delta > 0$, there exist $Q_1, Q_2, \cdots, Q_m$, subcells of Q, such that Q_i and Q_j are non-overlapping for each $i \neq j$, $\cup_{i=1}^m Q_i = Q$, and $\mathrm{diam}(Q_i) < \delta$, for $i = 1, \cdots, m$;

(b) Given $A, Q \in \mathcal{F}$ with $A \subset Q$, there exist $Q_1, Q_2, \cdots, Q_m$, subcells of Q, such that Q_i and Q_j are non-overlapping for each $i \neq j$, and $A = Q_1$;

(c) $\mu(\partial Q) = 0$ for each $Q \in \mathcal{F}$.

Example 2.1. Let X be the interval $[0, 1]$ of the real line endowed with the Euclidean distance in $\mathbb{R}$ and with the one-dimensional Lebesgue measure $\mathcal{L}$. The system $\mathcal{F}$ of all non-empty closed subintervals of X is the simplest example of a family of $\mathcal{L}$-cells in $[0, 1]$.

In fact, $\mathcal{F}$ is a $\mathcal{L}$-Vitali family by the well known Vitali covering theorem on the real line (see [23, Chapter IV, § 3]), and conditions (a), (b), and (c) are trivially satisfied.

Example 2.2. Let X be the interval $[0, 1]$ of the real line endowed with the Euclidean distance in $\mathbb{R}$ and with the one-dimensional Lebesgue measure. It is easy to see that the system $\mathcal{F}_d$ of all non-empty closed dyadic subintervals of $[0, 1]$ is also a family of $\mathcal{L}$-cells in $[0, 1]$.

Example 2.3. Let $n > 1$ and let X be the unit cube $[0,1]^n$ of $\mathbb{R}^n$ endowed with the Euclidean distance in $\mathbb{R}^n$ and with the n-dimensional Lebesgue measure $\mathcal{L}^n$. For a fixed $\alpha \in (0,1]$, the system $\mathcal{F}_\alpha$ of all non-empty closed subintervals Q of $[0,1]^n$ such that $\mathcal{L}^n(Q) \geq \alpha\,\mathcal{L}^n(B)$, for some ball B containing Q, is a family of $\mathcal{L}^n$-cells.

In fact, $\mathcal{F}_\alpha$ is a $\mathcal{L}^n$-Vitali family by [23, Chapter IV, § 3], and conditions (a), (b) and (c) are trivially satisfied.

Example 2.4. Let X be the interval $[0,1]$ of the real line endowed with the Euclidean distance in $\mathbb{R}$, and let $K \subset [0,1]$ be an s-set; i.e., a closed fractal subset of $[0,1]$ of positive s-Hausdorff measure $\mathcal{H}^s$, with $0 < s < 1$. The system $\mathcal{F}_K$ of all non-empty closed subintervals of $[0,1]$ is a family of cells with respect to the measure $\mu_K(\cdot) = \mathcal{H}^s(\cdot \cap K)$.

In fact, the measure μ_K is Radon by [18, Theorem 1.9 (2) and Corollary 1.11], $\mathcal{F}_K$ is a μ_K-Vitali family by [18, Theorem 2.8], and conditions (a), (b) and (c) are trivially satisfied.

In the next definition of the HK-integral on X, a family of μ-cells will takes the role of the usual "intervals" in the classical definition of the Henstock-Kurzweil integral on the real line.

3 The HK-Integral

Throughout this paper, $X = (X, d)$ is a fixed complete metric space endowed with a non-atomic Radon measure μ and with a family $\mathcal{F}$ of μ-cells. For simplicity, in the rest of this paper, we use the name cell instead of the name of μ-cell each time there is no ambiguity.

A *gauge* on a cell Q is any positive real function δ defined on Q. Let $Q \in \mathcal{F}$, let $E \subset Q$ and let δ be a gauge on Q. A collection $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$ of ordered pairs (points-cells) is said to be

- a *partition* of Q, if $Q_1, Q_2, \cdots, Q_m$ are pairwise non-overlapping elements of $\mathcal{F}$ such that $\cup_{i=1}^m Q_i = Q$ and $x_i \in Q_i$ for $i = 1, \cdots, m$;
- a *partial partition* of Q, if $Q_1, Q_2, \cdots, Q_m$ are pairwise non-overlapping elements of $\mathcal{F}$ such that $\cup_{i=1}^m Q_i \subset Q$ and $x_i \in Q_i$ for $i = 1, \cdots, m$;
- *δ-fine*, if $\operatorname{diam}(Q_i) < \delta(x_i)$ for $i = 1, \cdots, m$;
- *E-anchored*, if the points $x_1, \cdots, x_m$ belong to E.

The following Cousin's type lemma addresses the existence of δ-fine partitions of a given cell Q.

Lemma 3.1. *If δ is a gauge on a cell Q, then there exists a δ-fine partition of Q.*

Proof. Let us observe that if $Q = \cup_i^m Q_i$, with $Q_i \in \mathcal{F}$, and if $\mathcal{P}_1, ..., \mathcal{P}_m$ are δ-fine partitions of cells $Q_1, Q_2, \cdots, Q_m$, respectively, then $\bigcup_{i=1}^m \mathcal{P}_i$ is a δ-fine partition of Q. Using this observation we proceed by contradiction.

By condition (a) there exist $Q_1, Q_2, \cdots, Q_m$ subcells of Q such that $\cup_i^m Q_i = Q$ and $\text{diam}(Q_i) < \text{diam}(Q)/2$. Let us suppose that Q does not have a δ-fine partition. Then, there exists an index $i \in \{1, 2, \cdots, m\}$ such that Q_i does not have a δ-fine partition.

Let us say $i = 1$. By indefinitely repeating this argument we obtain a sequence of nested cells:

$$Q \supset Q_1 \supset \cdots \supset Q_k \supset \cdots$$

such that $\text{diam}(Q_k) \leq \text{diam}(Q)/2^k$ and Q_k does not have a δ-fine partition. Since $\text{diam}(Q_k) \to 0$, and the cells are closed sets, then there exists a point $\xi \in Q$ such that

$$\bigcap_{k=1}^{\infty} Q_k = \{\xi\}.$$

So, by $\delta(\xi) > 0$, we can find a natural k such that $\text{diam}(Q_k) < \delta(\xi)$. Thus, $\{(\xi, Q_k)\}_k$ is a δ-fine partition of Q_k, contrary to our assumption. □

Given a partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$ of a cell Q and a function $f : Q \to \mathbb{R}$ we set

$$S(f, \mathcal{P}) = \sum_{i=1}^{m} f(x_i)\mu(Q_i).$$

Definition 3.1. We say that a function $f : Q \to \mathbb{R}$ is HK-*integrable* on a cell Q (with respect to μ) if there exists a number I such that for each $\varepsilon > 0$ there is a gauge δ on Q with

$$|S(f, \mathcal{P}) - I| < \varepsilon,$$

for each δ-fine partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$ of Q. The number I is called the HK-*integral of* f on Q (with respect to μ), and we write

$$I = \int_Q f \, d\mu.$$

The collection of all HK-integrable functions on Q (with respect to μ) will be denoted by μ-HK(Q), or simply by HK(Q) if it is clear that μ is our fixed non-atomic Radon measure.

Remark 3.1. If X, μ, and $\mathcal{F}$ are defined as in the Example 2.1, then the μ-HK integral is the classical Henstock-Kurzweil integral on $[0,1]$.

Remark 3.2. If X, μ, and $\mathcal{F}$ are defined as in the Example 2.2, then the μ-HK integral is the dyadic Henstock-Kurzweil integral on $[0,1]$.

Remark 3.3. If X, μ, and $\mathcal{F}$ are defined as in the Example 2.3, then the μ-HK integral is the Mawhin's integral on $[0,1]^n$.

Remark 3.4. If X, μ, and $\mathcal{F}$ are defined as in the Example 2.4, then the μ-HK integral is the s-HK integral on a s-set studied in [4].

4 Some Properties of the HK-Integral

It is easy to see that the HK-integral is uniquely determined and that for each cell Q the space $\mathrm{HK}(Q)$ is closed under addition and scalar multiplication. Furthermore, by condition (b), it follows that if $f \in \mathrm{HK}(Q)$, and if A is a subcell of Q, then $f \in \mathrm{HK}(A)$ and

$$\int_A f\,d\mu = \int_Q f\,\chi_A\,d\mu.$$

Moreover, if $f \in \mathrm{HK}(Q)$ and if $Q_1, Q_2, \cdots, Q_m$ are non-overlapping subcells of Q such that $Q = \bigcup_i Q_i$, then

$$\int_Q f\,d\mu = \sum_{i=1}^{m} \int_{Q_i} f\,d\mu.$$

The map

$$F : A \rightsquigarrow \int_A f\,d\mu,$$

defined on each subcell A of Q, is called the *indefinite* HK-*integral* of f on Q. Obviously, the indefinite HK-integral is an additive function of cells.

It is useful to remark that each Lebesgue integrable function on a cell Q is also HK-integrable on Q and the two integrals coincide.

Theorem 4.1. *Let Q be a cell and let $f\colon Q \to \mathbb{R}$. If f is Lebesgue integrable on Q with respect to μ, then f is HK-integrable on Q and*

$$(L)\int_Q f\ d\mu = \int_Q f\ d\mu,$$

where by $(L)\int_Q f\ d\mu$ we denote the Lebesgue integral of f on Q with respect to μ.

Proof. By the Vitali-Carathéodory Theorem (see [22, Theorem 2.25]), given $\varepsilon > 0$ there exist functions u and v on Q that are upper and lower semicontinuos respectively, such that $-\infty \leq u \leq f \leq v \leq +\infty$ and $(L)\int_Q (v-u)\ d\mu < \varepsilon$. Define on Q a gauge δ so that

$$u(t) \leq f(x) + \varepsilon \quad \text{and} \quad v(t) \geq f(x) - \varepsilon,$$

for each $t \in Q$ with $d(x,t) < \delta(x)$.

Let $\mathcal{P} = \{(x_1, Q_1), (x_2, Q_2), \cdots, (x_m, Q_m)\}$ be a δ-fine partition of Q. Then, for each $i \in \{1, 2, \cdots, p\}$, we have

$$(L)\int_{Q_i} u\ d\mu \leq (L)\int_{Q_i} f\ d\mu \leq (L)\int_{Q_i} v\ d\mu. \tag{1}$$

Moreover, by $u(t) \leq f(x_i) + \varepsilon$, for each $t \in Q_i$, it follows that

$$(L)\int_{Q_i} (u-\varepsilon)\ d\mu \leq (L)\int_{Q_i} f(x_i)\ d\mu,$$

and therefore,

$$(L)\int_{Q_i} u\ d\mu - \varepsilon\, \mu(Q_i) \leq f(x_i)\, \mu(Q_i).$$

Similarly, by $v(t) \geq f(x_i) - \varepsilon$, for each $t \in Q_i$, it follows that

$$f(x_i)\, \mu(Q_i) \leq (L)\int_{Q_i} v\ d\mu + \varepsilon\, \mu(Q_i).$$

So, for $i = 1, 2, \cdots, p$, we have

$$(L)\int_{Q_i} u\ d\mu - \varepsilon\, \mu(Q_i) \leq f(x_i)\, \mu(Q_i) \leq (L)\int_{Q_i} v\ d\mu + \varepsilon\, \mu(Q_i).$$

Hence,

$$(L)\int_{Q} u\ d\mu - \varepsilon\, \mu(Q) \leq S(f, \mathcal{P}) \leq (L)\int_{Q} v\ d\mu + \varepsilon\, \mu(Q),$$

and, by (1),

$$(L)\int_{Q} u\ d\mu \leq (L)\int_{Q} f\ d\mu \leq (L)\int_{Q} v\ d\mu.$$

Thus,

$$\left| S(f, \mathcal{P}) - (L)\int_{Q} f\ d\mu \right| \leq (L)\int_{Q} (v-u)\ d\mu + 2\varepsilon\, \mu(Q) < \varepsilon + 2\varepsilon\, \mu(Q),$$

and the theorem is proved. □

In the sequel, we need the following Saks-Henstock type Lemma, whose proof is identical to that used in the case $X = [0,1]$. Therefore, it will be omitted.

Lemma 4.2. *A function $f\colon Q \to \mathbb{R}$ is HK-integrable on a cell Q if and only if there exists an additive cell function π defined on the family of all subcells of Q such that, for each $\varepsilon > 0$, there exists a gauge δ on Q with*

$$\sum_{(x_i,Q_i)\in\mathcal{P}} \Big|\pi(Q_i) - f(x_i)\mu(Q_i)\Big| < \varepsilon$$

for each δ-fine partial partition $\mathcal{P}$ of Q. In this situation, π is the indefinite HK-integral of f on Q.

5 Absolutely HK-integrable Functions

Let Q be a cell. We recall that a function $f\colon Q \to \mathbb{R}$ is said to be absolutely HK-integrable on Q if $|f|$ is HK-integrable on Q. In this section we study the absolutely HK-integrable functions. In particular, we prove that these functions are Lebesgue integrable and that their primitives are differentiable μ-almost everywhere.

Given a cell function F defined on $\mathcal{F}$ and given $x \in X$, we remind the reader that the *upper derivative* of F at x, with respect to μ, is defined as follows

$$\overline{\mathrm{D}}F(x) = \limsup_{\mathcal{F}\ni B\to x} \frac{F(B)}{\mu(B)},$$

where $B \to x$ means $\mu(B) \neq 0$, $\mathrm{diam}(B) \to 0$, and $x \in B$.

Analogously, *lower derivative* of F at x is defined, and it is denoted by $\underline{\mathrm{D}}F(x)$. Whenever $\overline{\mathrm{D}}F(x) = \underline{\mathrm{D}}F(x) \neq \infty$, then F is said to be *differentiable* at x and their common value is called the *derivative* of F at x and it is denoted by $F'(x)$.

Theorem 5.1. *If f is a non-negative HK-integrable function on a cell Q and if F is its indefinite HK-integral, then F is differentiable μ- almost everywhere on Q and $F' = f$.*

Proof. To prove that $F' = f$ μ-almost everywhere on Q, it is enough to show that $\overline{\mathrm{D}}F \leq f \leq \underline{\mathrm{D}}F$ μ-almost everywhere on Q, since $\underline{\mathrm{D}}F \leq \overline{\mathrm{D}}F$ everywhere.

To this end, we consider positive rational numbers p, q such that $q > p$ and we set

$$A_{p,q} = \{x \in Q : \overline{\mathrm{D}}F(x) > q > p > f(x)\}.$$

If we prove that $\mu(A_{p,q}) = 0$ for each p and q, then $\overline{\mathrm{D}}F(x) \leq f(x)$ μ-almost everywhere on Q. Similarly, we can prove that $\underline{\mathrm{D}}F(x) \geq f(x)$ μ-almost everywhere on Q.

Given $\varepsilon > 0$, by Lemma 4.2 there exists a gauge δ on Q such that

$$\sum_{j=1}^{m} |F(Q_j) - f(x_j)\mu(Q_j)| < \varepsilon,$$

for each δ-fine partial partition $\{(x_j, Q_j)\}_{j=1}^m$ of Q.

Let $\mathcal{V}$ be the system of all cells $B \subset Q$ such that $F(B) \geq q\,\mu(B)$ and that there exists $x \in B \cap A_{p,q}$ with $\mathrm{diam}(B) < \delta(x)$. It is easy to see that this system $\mathcal{V}$ is a fine cover of $A_{p,q}$. Therefore, ($\mathcal{F}$ being a μ-Vitali family) there exists a system of pairwise non-overlapping cells $\{B_j\}_{j=1}^m \subset \mathcal{V}$ such that

$$\mu(A_{p,q}) \leq \sum_{j=1}^{m} \mu(B_j) + \varepsilon. \tag{2}$$

For $j = 1, 2, \cdots, m$, let $x_j \in B_j \cap A_{p,q}$ such that $\mathrm{diam}(B_j) < \delta(x_j)$. Since $\{(x_j, B_j)\}_{j=1}^m$ is a δ-fine partial partition of Q, we get

$$\begin{aligned} q\sum_{j=1}^{m} \mu(B_j) &\leq \sum_{j=1}^{m} F(B_j) \\ &\leq \sum_{j=1}^{m} |F(B_j) - f(x_j)\mu(B_j)| + \sum_{j=1}^{m} f(x_j)\mu(B_j) \\ &< \varepsilon + p\sum_{j=1}^{m} \mu(B_j). \end{aligned}$$

Therefore $(q-p)\sum_{j=1}^m \mu(B_j) < \varepsilon$.

So, by (2) and by the arbitrariness of ε we obtain $\mu(A_{p,q}) = 0$. □

Now, we prove that each absolutely HK-integrable function is Lebesgue integrable. To this end, we need the following Monotone Convergence type Theorem.

Theorem 5.2. *Let $\{f_k\}_k$ be an non-decreasing sequence of HK-integrable functions on a cell Q and let $f = \lim_k f_k$. If*

$$\lim_{k\to\infty} \int_Q f_k \, d\mu < \infty,$$

then f is HK-integrable on Q and

$$\int_Q f \, d\mu = \lim_{k \to \infty} \int_Q f_k \, d\mu.$$

The proof is similar to that for the classical HK-integral on the real line, and it is omitted.

Theorem 5.3. *If f is a non-negative HK-integrable function on a cell Q and if F is its indefinite HK-integral, then f is μ-measurable.*

Proof. For $k \in \mathbb{N}$, let $\mathcal{P}_k$ be a $1/k$-fine partial partition of Q, and let f_k be the simple function defined as follows

$$f_k(x) = \sum_{(x,B) \in \mathcal{P}_k} \frac{F(B)}{\mu(B)}.$$

We set $C = \bigcup_{k=1}^{\infty} \bigcup_{B \in \mathcal{P}_k} \partial B$ and

$$D = \{x \in Q : F'(x) \text{ does not exist, or } F'(x) \text{ exists and } F'(x) \neq f(x)\}.$$

By condition (c) and by Theorem 5.1, the set $E = C \cup D$ is μ-null.

Now, let $x \in Q \setminus E$. For each $k \in \mathbb{N}$ there exists $Q_{k,x} \in \mathcal{F}$ such that $(x, Q_{k,x}) \in \mathcal{P}_k$, $\operatorname{diam}(Q_{k,x}) < 1/k$ and $f_k(x) = F(Q_{k,x})/\mu(Q_{k,x})$. Then, by $F'(x) = f(x)$, we obtain $f_k(x) \to f(x)$. Thus, the claim follows by the μ-measurability of f_k, for each $k \in \mathbb{N}$. □

Theorem 5.4. *If f is absolutely HK-integrable on a cell Q, then f is Lebesgue integrable on Q.*

Proof. For $k \in \mathbb{N}$, let $f_k(x) = \min\{|f(x)|, k\}$, for each $x \in Q$. By Theorem 5.3, $|f|$ is Lebesgue measurable. Therefore, if f_k is Lebesgue measurable and bounded, then it is Lebesgue integrable on Q. Thus, by Theorem 4.1, f_k is HK-integrable on Q. Hence, since $\{f_k\}_k$ is an non-decreasing sequence of non-negative functions convergent to $|f|$, by Theorem 5.2 we have

$$(L)\int_Q |f| \, d\mu = (L)\lim_{k \to \infty} \int_Q f_k \, d\mu = \lim_{k \to \infty} \int_Q f_k \, d\mu = \int_Q |f| \, d\mu < \infty,$$

and the proof is complete. □

6 Characterization of the indefinite HK-integral

Hereafter, we denote by π a fixed additive function defined on the family of all subcells of Q. Given $E \subset Q$ and a gauge δ on E, we set

$$V^\delta \pi(E) = \sup \left\{ \sum_{i=1}^{m} |\pi(Q_i)| \right\},$$

where the supremum is taken over all the δ-fine E-anchored partial partition $\mathcal{P} = \{(x_1, Q_1), (x_2, Q_2), \cdots, (x_m, Q_m)\}$ of Q.

The *critical variation* of π on E is defined as

$$V\pi(E) = \inf V^\delta \pi(E),$$

where the infimum is taken over all gauges δ on E.

It is easy to prove that the extended real-valued function $V\pi \colon E \rightsquigarrow V\pi(E)$ is a metric outer measures on Q. Therefore, by the Carathéodory criterion ([8, Theorem 1.5]), $V\pi$ is a Borel measure.

We note that the measure $V\pi$ is said to be *absolutely continuous* with respect to μ (or *μ-AC*) on Q if, for each $E \subset Q$ with $\mu(E) = 0$, we have $V\pi(E) = 0$.

Theorem 6.1. *If f is HK-integrable on a cell Q and if F is its indefinite HK-integral, then the critical variation VF is μ-AC on Q.*

Proof. Let $E \subset Q$ such that $\mu(E) = 0$. We set

$$h(x) = \begin{cases} f(x), & \text{for } x \in Q \setminus E, \\ 0, & \text{for } x \in E. \end{cases}$$

It is clear that F is also the indefinite HK-integral of h. Then, by Lemma 4.2, given $\varepsilon > 0$ we can find a gauge δ on Q such that

$$\sum_{i=1}^{m} |F(Q_i) - h(x_i)\mu(Q_i)| < \varepsilon,$$

for each δ-fine partial partition $\mathcal{P} = \{(x_1, Q_1), (x_2, Q_2), \cdots, (x_m, Q_m)\}$ of Q. In particular, if $\mathcal{P}$ is anchored in E, then we have

$$\sum_{i=1}^{m} |F(Q_i)| < \varepsilon.$$

Hence, by the arbitrariness of ε, it follows that $VF(E) = 0$. Thus, VF is μ-AC on Q.

□

Theorem 6.2. *If π is differentiable μ-almost everywhere on a cell Q and $V\pi$ is μ-AC on Q, then π' is HK-integrable on Q, and π is the indefinite HK-integral of π' on Q.*

Proof. We denote by E the μ-negligible set of all $x \in Q$ at which π is not differentiable, and we define

$$f(x) = \begin{cases} \pi'(x), & \text{for } x \in Q \setminus E, \\ 0, & \text{for } x \in E. \end{cases}$$

It suffices to show that f is HK-integrable on Q and that π is the indefinite HK-integral of f. Since $V\pi$ is μ-AC, given $\varepsilon > 0$ there exists a gauge δ_1 on E such that $\sum_{i=1}^{p} |\pi(A_i)| < \varepsilon/2$ for each δ_1-fine E-anchored partial partition $\{(y_1, A_1), \cdots, (y_p, A_p)\}$ of Q.

Moreover, given $x \in Q \setminus E$ there exists $\delta_2(x) > 0$ such that

$$|\pi(B) - f(x)\mu(B)| < \frac{\varepsilon}{2\mu(Q)}\mu(B),$$

for each subset B of Q such that $B \in \mathcal{F}$, $x \in B$, and $\operatorname{diam}(B) < \delta_2(x)$. Now, we define a gauge δ on Q by setting

$$\delta(x) = \begin{cases} \delta_1(x), & \text{for } x \in E, \\ \delta_2(x), & \text{for } x \in Q \setminus E, \end{cases}$$

and we choose a δ-fine E-anchored partial partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ of Q. Then,

$$\begin{aligned} \sum_{i=1}^{m} |\pi(Q_i) - f(x_i)\mu(Q_i)| &\le \sum_{x_i \in E} |\pi(Q_i)| + \sum_{x_i \notin E} |\pi(Q_i) - f(x_i)\mu(Q_i)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\mu(Q)} \sum_{x_i \notin E} \mu(Q_i) = \varepsilon, \end{aligned}$$

since $f(x_i) = 0$ for $x_i \in E$ and $\sum_{x_i \notin E} \mu(Q_i) = \mu(Q \setminus E) = \mu(Q)$. Therefore f is HK-integrable on Q and π is the indefinite HK-integral of f. □

Definition 6.1. Let Q be a cell. We say that π is $BV^{\triangle}$ on $E \subset Q$ if there exists a gauge δ on E such that $V^{\delta}\pi(E) < \infty$.

We say that π is $BVG^{\triangle}$ on Q if there exists a countable sequence of closed sets $\{E_k\}_k$ such that $\bigcup_k E_k = Q$ and π is $BV^{\triangle}$ on E_k, for each $k \in \mathbb{N}$.

Definition 6.2. Let Q be a cell. We say that π is $AC^{\triangle}$ on $E \subset Q$ if for $\varepsilon > 0$ there exists a gauge δ on E and a positive constant η such that the condition $\sum_{i=1}^{m} \mu(Q_i) < \eta$ implies $\sum_{i=1}^{m} |\pi(Q_i)| < \varepsilon$, for each δ-fine E-anchored partial partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ of Q.

We say that π is $ACG^{\triangle}$ on Q if there exists a countable sequence of closed sets $\{E_k\}_k$ such that $\bigcup_k E_k = Q$ and π is $AC^{\triangle}$ on E_k, for each $k \in \mathbb{N}$.

Theorem 6.3. *Let E be a compact subset of a cell Q. If π is $AC^{\triangle}$ on E, then π is $BV^{\triangle}$ on E.*

Proof. Since π is $AC^{\triangle}$ on E, there exists a gauge δ on Q and a positive constant η such that $\sum_{i=1}^{m} |\pi(Q_i)| < 1$ whenever $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ is a δ-fine E-anchored partial partition of Q with $\sum_{i=1}^{m} \mu(Q_i) < \eta$.

Moreover, since μ is non-atomic, for each $x \in Q$ there exists an open neighborhood G of x such that $\mu(G) < \eta$. Then, by the compactness of E, there exist open sets $G_1, G_2, \cdots, G_p$ with $\mu(G_j) < \eta$, for $j = 1, 2, \cdots, p$, and $E \subset \bigcup_{j=1}^{p} G_j$. Given $x \in E$, let $j \in \{1, \cdots, p\}$ such that $x \in G_j$, and define $\delta_1(x) = \min\{\delta(x), \mathrm{d}(x, \partial G_j)\}$.

Let $\{(x_i, Q_i)\}_{i=1}^{m}$ be an arbitrary δ_1-fine E-anchored partial partition, and let $I_j = \{i : Q_i \subset G_j\}$. Therefore, we have

$$\sum_{i=1}^{m} |\pi(Q_i)| \leq \sum_{j=1}^{p} \sum_{i \in I_j} |\pi(Q_i)| \leq p < \infty,$$

since $\mu\left(\bigcup_{i \in I_j} Q_i\right) \leq \mu(G_j) < \eta$. Hence, $V^{\delta_1}\pi(E) < \infty$, and the proof is complete. □

Theorem 6.4. *If f is HK-integrable on a cell Q and F is its indefinite HK-integral, then there exists a sequence $\{E_k\}_k$ of closed sets such that $Q = \bigcup_{k=1}^{\infty} E_k$ and that f is Lebesgue integrable on E_k for each $k \in \mathbb{N}$.*

Proof. By Theorem 5.3, $|f|$ is μ-measurable. For each natural number m, let

$$A_m = \{x \in Q : |f(x)| \leq m\}.$$

Since μ is a Radon measure, we have $A_m = N_m \cup \bigcup_{i=1}^{\infty} A_{m,i}$ where N_m is μ-null and the $A_{m,i}$, $i = 1, 2, \cdots$, are closed sets.
Now, let $N = \bigcup_{m=1}^{\infty} N_m$ and let $\{C_k\}_k$ be a rearrangement of $\{A_{m,i}\}_i$. Moreover, let

$$Q = N \cup \bigcup_{k=1}^{\infty} C_k,$$

and let

$$h(x) = \begin{cases} f(x), & \text{for } x \in \bigcup_{k=1}^{\infty} C_k, \\ 0, & \text{for } x \in N. \end{cases}$$

We remark that h is still HK-integrable on Q and that F is its indefinite HK-integral. Therefore, by Lemma 4.2, there exists a gauge δ on Q such that

$$\sum_{i=1}^{m} |F(Q_i) - h(x_i)\mu(Q_i)| < 1, \tag{3}$$

for each δ-fine partial partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ of Q. Then, in particular,

$$\sum_{i=1}^{m} |F(Q_i)| < 1, \tag{4}$$

for each δ-fine N-anchored partial partition $\mathcal{P} = \{(\xi_i, Q_i)\}_{i=1}^{m}$ of Q.
For each natural number k, let

$$W_k = \left\{ x \in N : \delta(x) \geq \frac{1}{k} \right\}.$$

It is clear that $N = \bigcup_{i=1}^{\infty} W_k$. Hence, $N \subset \bigcup_k \overline{W}_k$. Then, $Q = \bigcup_k \overline{W}_k \cup \bigcup_k C_k$.

The function h is Lebesgue integrable on C_k, for $k = 1, 2, \cdots$, since it is measurable and bounded. Then to complete the proof, it is enough to show that h is Lebesgue integrable on $\overline{W}_k$, for $k = 1, 2, \cdots$. To this aim, for each $q \in \mathbb{N}$, we remark that the function $h_q(x) = \min\{|h(x)|, q\}$ is measurable and bounded; therefore, $h_{q,k} := h_q \, \chi_{\overline{W}_k}$ is Lebesgue integrable on Q. Hence, by Theorem 4.1, $h_{q,k}$ is HK-integrable on Q. Let $F_{q,k}$ be the indefinite HK-integral of $h_{q,k}$ with respect to μ (or the indefinite HK-integral of h_q with respect to μ_k, with $\mu_k(E) = \mu(E \cap \overline{W}_k)$); then by Lemma 4.2 there exists a gauge δ_1 on Q such that $\delta_1(x) < \inf\{\delta(x), 1/k\}$, for each $x \in Q$, and

$$\sum_{i} |F_{q,k}(Q_i) - h_q(x_i)\mu_k(Q_i)| < 1,$$

for each δ_1-fine partial partition $\{(x_i, Q_i)\}_i$ of Q. Let $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ be a fixed δ_1-fine partition of Q, and let $I = \{i : W_k \cap Q_i^{\circ} \neq \emptyset\}$. Then,

- If $i \notin I$, we have $(Q_i \cap \overline{W}_k) \subseteq \partial Q_i$; so, by condition (c),

$$0 \leq \sum_{i \notin I} F_{q,k}(Q_i) = \sum_{i \notin I} \int_{Q_i \cap \overline{W}_k} h_q \, d\mu \leq \sum_{i \notin I} \int_{\partial Q_i} h_q \, d\mu = 0;$$

- If $i \in I$, there exists $\xi \in Q_i \cap W_k$; so $\{(\xi_i, Q_i)\}_i$ is a δ_1-fine W_k-anchored partial partition.

Thus, by (3) and (4) we have

$$\begin{aligned}\sum_{i\in I} |h_q(\xi_i)\,\mu_k(Q_i)| &\le \sum_{i\in I} |h(\xi_i)\,\mu(Q_i)| \\ &\le \sum_{i\in I} |h(\xi_i)\,\mu(Q_i) - F(Q_i)| + \sum_{i\in I} |F(Q_i)| \\ &\le 1 + 1 = 2.\end{aligned}$$

Hence,

$$\begin{aligned}F_{q,k}(Q) &= \sum_{i=1}^{m} |F_{q,k}(Q_i)| = \sum_{i\in I} |F_{q,k}(Q_i)| \\ &\le \sum_{i\in I} |F_{q,k}(Q_i) - h_q(\xi_i)\mu_k(Q_i)| + \sum_{i\in I} |h_q(\xi_i)\,\mu_k(Q_i)| \\ &\le 1 + 2 = 3.\end{aligned}$$

Thus, $0 \le \int_Q h_q\ d\mu_k = F_{q,k}(Q) \le 3$; i.e., h_q is Lebesgue integrable on Q. In conclusion, since $h_q \to |h|$, by the Monotone Convergence Theorem, we have

$$(L)\int_Q |h|\ d\mu_k = \lim_{k\to\infty} (L)\int_Q h_q\ d\mu_k \le 3;$$

i.e., h is Lebesgue integrable on $\overline{W}_k$. □

Theorem 6.5. *Let f be HK-integrable on a cell Q and let F be its indefinite HK-integral. If f is Lebesgue integrable on a closed subset A of Q, then F is $AC^{\triangle}$ on A.*

Proof. By Lemma 4.2, for each $\varepsilon > 0$ there exists a gauge δ_1 on Q such that

$$\sum_{i=1}^{m} |F(Q_i) - f(x_i)\mu(Q_i)| < \frac{\varepsilon}{3}, \tag{5}$$

for each δ_1-fine partial partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ of Q. Moreover, since f is Lebesgue integrable on A, the function $f\chi_A$ is HK-integrable on Q. We set $f_A := f\chi_A$, and we denote by $F_A(Q)$ the indefinite HK-integral of f_A on Q. Therefore, by Lemma 4.2, there exists a gauge δ_2 on Q such that

$$\sum_{i=1}^{m} |F_A(Q_i) - f_A(\xi_i)\mu(Q_i)| = \sum_{i=1}^{m} |F_A(Q_i) - f(\xi_i)\mu(Q_i)| < \frac{\varepsilon}{3}, \tag{6}$$

for each δ_2-fine A-anchored partial partition $\{(\xi_i, Q_i)\}_{i=1}^m$ of Q. Now, since f is Lebesgue integrable on A, the function F_A is μ-AC on A. Consequently, we can find a positive η such that the condition $\mu\left(\bigcup_{i=1}^m Q_i\right) = \sum_{i=1}^m \mu(Q_i) < \eta$ implies

$$\sum_{i=1}^{m} |F_A(Q_i)| \le \sum_{i=1}^{m} \int_{Q_i \cap A} |f|\, d\mu \le \int_{\bigcup Q_i \cap A} |f|\, d\mu < \frac{\varepsilon}{3}. \tag{7}$$

Therefore, by (5), (6) and (7), we infer

$$\sum_{i=1}^{m} |F(Q_i)| \le \sum_{i=1}^{m} |F(Q_i) - f(\xi_i)\mu(Q_i)| + \sum_{i=1}^{m} |f(\xi_i)\mu(Q_i) - F_A(Q_i)| + \sum_{i=1}^{m} |F_A(Q_i)| < \varepsilon,$$

for each δ-fine A-anchored partial partition $\{(\xi_i, Q_i)\}_{i=1}^m$, where

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}.$$

Hence, F is $AC^{\triangle}$ on A. □

Theorem 6.6. *If f is HK-integrable on a cell Q and if F is its indefinite HK-integral, then F is $ACG^{\triangle}$ on Q.*

Proof. By Theorem 6.4, there exists a sequence $\{E_k\}_k$ of closed sets such that $Q = \bigcup_{k=1}^{\infty} E_k$ and f is Lebesgue integrable on E_k for each $k \in \mathbb{N}$. Moreover, by Theorem 6.5, F is $AC^{\triangle}$ on E_k for each k. Therefore, F is $ACG^{\triangle}$ on Q. □

Theorem 6.7. *If π is $ACG^{\triangle}$ on a cell Q, then $V\pi$ is μ-AC on Q.*

Proof. By hypothesis, there exists a sequence of closed sets $\{E_k\}_k$ such that $\bigcup_k E_k = Q$ and that π is $AC^{\triangle}$ on E_k for each $k \in \mathbb{N}$. Therefore, for $\varepsilon > 0$ there exists a gauge δ on E_k and a positive η such that the condition $\sum_{i=1}^m \mu(Q_i) < \eta$ implies $\sum_{i=1}^m |\pi(Q_i)| < \varepsilon$ for each δ-fine E_k-anchored partial partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$ of Q. Let $E \subset Q$ be μ-null. Since $E \cap E_k$ is μ-null, for each $k \in \mathbb{N}$, there exists an open set G_k such that $E \cap E_k \subset G_k$ and $\mu(G_k) < \eta$.

For each $x \in E \cap E_k$, we define $\delta_1(x) = \min\{\delta(x), \mathrm{d}(x, \partial G_k)\}$. So, if $\{(x_i, Q_i)\}_{i=1}^m$ is a δ_1-fine $E \cap E_k$-anchored partial partition of Q, we have $Q_i \subset G_k$, for each i. Therefore, $\sum_{i=1}^m \mu(Q_i) \le \mu(G_k) < \eta$, which implies $\sum_{i=1}^m |\pi(Q_i)| < \varepsilon$. Then, $V^{\delta_1}\pi(E \cap E_k) \le \varepsilon$ and $V\pi(E \cap E_k) \le \varepsilon$. By the arbitrariness of ε, it follows that $V\pi(E\cap E_k)=0$. Hence, since $V\pi$ is an outer measure and $E=\bigcup_{k=1}^{\infty}(E\cap E_k)$, we have

$$V\pi(E)\le\sum_{k=1}^{\infty}V\pi(E\cap E_k)=0.$$

Thus, $V\pi$ is μ-AC on Q. □

We note that a signed measure λ, defined on the σ-algebra of all μ-measurable subsets of Q, is said to be *absolutely continuous* with respect to μ, and we write $\lambda\ll\mu$ if the condition $\mu(E)=0$ implies $|\lambda|(E)=0$ for each μ-measurable $E\subset A$. Here, $|\lambda|(E)$ denotes the variation of λ on E.

Lemma 6.8. *Let A be a closed subset of Q and let λ be a signed measure on Q such that $\lambda\ll\mu$. Then, λ is $AC^{\triangle}$ on A.*

The proof follows easily by [22, Theorem 6.11].

Lemma 6.9. *If π is an additive function of cells that is $AC^{\triangle}$ on a closed subset A of a cell Q, then*

$$E=\left\{x\in A:\lim_{Q\to x}\frac{|\pi(Q)|}{\mu(Q)}\neq 0\right\}\quad \textit{is } \mu\textit{-null.}$$

Proof. Let

$$E_n=\left\{x\in E:\text{there exists }\{Q_k^x\}_k\to x,\ \text{with }\frac{|\pi(Q_k^x)|}{\mu(Q_k^x)}>\frac{1}{n}\text{ for each }k\in\mathbb{N}\right\}.$$

It is trivial to remark that $E=\bigcup_n E_n$; therefore, to end the proof it is enough to show that $\mu(E_n)=0$, for each $n\in N$. Proceeding towards a contradiction, we can suppose that there exists a natural $\bar{n}\in N$ such that $\mu(E_{\bar{n}})\neq 0$. Thus, there exists a compact set $K\subset E_{\bar{n}}$ for which $\mu(K)>0$. Less than substracting from K a μ-null relatively open subset, we can assume that $\mu(K\cap U)>0$ for each open set $U\subset X$ with $K\cap U\neq\emptyset$.

Since K is compact there exists a countable dense subset C of K. Let $H\supset C$ be a μ-null G_δ set. Therefore, $K\cap H$ is a μ-null G_δ subset of K that is dense on K. We show that $V_\pi(K\cap H)>0$, contradicting Theorem 4.7.

Set $D=K\cap H$, and let δ be a gauge on D. We define $D_m=\{x\in D:\delta(x)>1/m\}$, for $m\in\mathbb{N}$. Then, by $D=\bigcup_m D_m$ and by the Baire Category theorem, there exists an open set U such that $D\cap U\neq\emptyset$ and there exists a natural $\bar{m}$ such that $D_{\bar{m}}$ is dense on $D\cap U$, and hence on $K\cap U$.

Let $\mathcal{B}$ be the system of all cells Q such that $|\pi(Q)|>\mu(Q)/\bar{m}$, and $\operatorname{diam}(Q)<1/\bar{m}$. Therefore, $\mathcal{B}$ is a fine cover of $K\cap U$. Moreover, since

$\mu(K \cap U) > 0$ and since $\mathcal{F}$ is a μ-Vitali family, by the previous remark on the choice of K, there exists a non-overlapping system of cells $\{Q_i \in \mathcal{B}\}_i$ that covers $K \cap U$ up to a μ-null set. Then,

$$\sum_{i=1}^{\infty} |\pi(Q_i)| > \frac{1}{\bar{m}} \sum_{i=1}^{\infty} \mu(Q_i) > \frac{1}{\bar{m}} \mu(K \cap U) = M.$$

So, there exists an integer $p \geq 1$ such that $\sum_{i=1}^{p} |\pi(Q_i)| > M$, and, since μ does not charge the boundaries of cells (condition (c)), the interior of each Q_i meets $K \cap U$. Thus, by the density of $D_{\bar{m}}$ on $K \cap U$, we have $D_{\bar{m}} \cap Q_i \neq \emptyset$, and we can select $x_i \in D_{\bar{m}} \cap B_i$ for each natural i. So, $\{(x_1, B_1), (x_2, B_2), \ldots (x_p, B_p)\}$ is a δ-fine $D_{\bar{m}}$-anchored partial partitions of $K \cap U$, and consequently, $V_{\pi}^{\delta}(D_{\bar{m}}) \geq M$. Then, by the arbitrariness of δ, we have $V_{\pi}(D_{\bar{m}}) \geq M$, the required contradiction. □

Theorem 6.10. *Let π be an additive cell function. If π is $AC^{\triangle}$ on a closed subset A of a cell Q, then π is differentiable μ-almost everywhere on A.*

Proof. Given an arbitrary subset Y of Q, we define the functions

$$V_{+}^{\delta}\pi(Y) = \sup\left\{\sum_{i=1}^{m} (\pi(Q_i))^{+}\right\} \quad \text{and} \quad V_{-}^{\delta}\pi(Y) = \sup\left\{\sum_{i=1}^{m} (\pi(Q_i))^{-}\right\},$$

where $(\pi(Q_i))^{+} = \max\{\pi(Q_i), 0\}$ and $(\pi(Q_i))^{-} = \max\{-\pi(Q_i), 0\}$ are the positive and the negative parts of π, respectively, and the supremum is taken over all δ-fine Y-anchored partial partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ of Q.

As for the definition of $V\pi$, we can define $V_{+}\pi$ and $V_{-}\pi$ by

$$V_{+}\pi(Y) = \inf V_{+}^{\delta}\pi(Y) \quad \text{and} \quad V_{-}\pi(Y) = \inf V_{-}^{\delta}\pi(Y),$$

where the infimum is taken over all gauges δ on E. It is easy to prove that $V_{+}\pi$ and $V_{-}\pi$ are finite measures.

For each measurable set E of Q, we define $\nu^{+}(E) = V_{+}\pi(E \cap A)$ and $\nu^{-}(E) = V_{-}\pi(E \cap A)$. Since π is $AC^{\triangle}$ on A, given $\varepsilon > 0$ there exists a gauge δ on A and $\eta > 0$ such that the condition $\sum_{i=1}^{m} \mu(Q_i) < \eta$ implies $\sum_{i=1}^{m} |\pi(Q_i)| < \varepsilon$ for each δ-fine A-anchored partial partition $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^{m}$ of Q.

Let $E \subset Q$ be μ-null. Therefore, $E \cap A$ is μ-null, and thus, there exists an open set G such that $E \cap A \subset G$ and $\mu(G) < \eta$. By the argument used in the proof of Theorem 6.7, we have $\sum_{i=1}^{m} \mu(Q_i) \leq \mu(G) < \eta$, which implies $\sum_{i=1}^{m} (\pi(Q_i))^{+} \leq \sum_{i=1}^{m} |\pi(Q_i)| < \varepsilon$, for each δ_1-fine $(E \cap A)$-anchored partial partition $\{(x_i, Q_i)\}_{i=1}^{m}$ of Q. Therefore, $V_{+}^{\delta_1}\pi(E \cap A) \leq \varepsilon$

and $\nu^+(E) = V_+\pi(E \cap A) \leq \varepsilon$. Thus, $\nu^+ \ll \mu$. Similarly, we can prove that $\nu^- \ll \mu$.

So, by the Radon-Nikodym Theorem ([11, Theorem 19.23]), there exist non-negative Lebesgue integrable functions f^+ and f^- on Q such that

$$\nu^+(E) = (L)\int_E f^+ \, d\mu \quad \text{and} \quad \nu^-(E) = (L)\int_E f^- \, d\mu,$$

for every μ-measurable subset E of Q.

We set $f = f^+ - f^-$, and we remark that f is Lebesgue integrable on Q. Therefore, by Theorem 4.1, f is HK-integrable on Q, and $\nu = \nu_+ - \nu_-$ is the indefinite HK-integral of f. Since f is the Radon-Nikodym derivative of ν with respect to μ, we have

$$\lim_{\mathcal{F} \ni R \to x} \frac{\nu(R)}{\mu(R)} = f(x), \tag{8}$$

μ-almost everywhere on A.

Now, by Lemma 6.8, the signed measure ν is $AC^{\triangle}$ on A. Then also, $\pi - \nu$ is $AC^{\triangle}$ on A. Hence, by Lemma 6.9, we have $\lim_{R \to x}(\pi(R) - \nu(R))/\mu(R) = 0$ μ-almost everywhere on A, and by (8) we have $\lim_{R \to x} \pi(R)/\mu(R) = f(x)$, μ-almost everywhere on A; i.e., $\pi'(x) = f(x)$ μ-almost everywhere on A. □

Theorem 6.11. *Let π be an additive cell function. If π is $ACG^{\triangle}$ on a cell Q, then π is differentiable μ-almost everywhere on Q.*

Proof. Since π is $ACG^{\triangle}$ on Q, then there exists a countable sequence of closed sets $\{E_k\}_k$ such that $\bigcup_k E_k = Q$ and π is $AC^{\triangle}$ on E_k, for each $k \in \mathbb{N}$. So, by Theorem 6.10, π is differentiable μ-almost everywhere on E_k for each $k \in \mathbb{N}$. Thus, it is differentiable μ-almost everywhere on Q. □

Main Theorem 1 (of Type A). *Let Q be a cell. A function $f \colon Q \to \mathbb{R}$ is HK-integrable on Q if and only if there exists an additive cell function F that is $ACG^{\triangle}$ on Q and $F'(x) = f(x)$ μ-almost everywhere on Q.*

Proof. Let $f \colon Q \to \mathbb{R}$ be HK-integrable on Q, and let F be its HK-primitive. By Theorem 6.6, F is $ACG^{\triangle}$ on Q, then by Theorem 6.11 F is differentiable μ-almost everywhere on Q. Moreover, by Theorem 6.7, VF is μ-AC on Q. So, by Theorem 6.2, $F'(x) = f(x)$ μ-almost everywhere on Q.

Vice versa, let F be an additive function of cells that is $ACG^{\triangle}$ on Q and such that $F'(x) = f(x)$ μ-almost everywhere on Q. By Theorem 6.7,

VF is μ-AC on Q, and then, by Theorem 6.2, F is the HK-primitive of F'. Thus, the condition $f(x) = F'(x)$, μ-almost everywhere on Q, implies the HK-integrability of f on Q.

□

Main Theorem 2 (of Type B)**.** *Let Q be a cell. A function $f\colon Q \to \mathbb{R}$ is HK-integrable on Q if and only if there exists an additive cell function F such that VF is μ-AC on Q and $F'(x) = f(x)$ μ-almost everywhere on Q.*

Proof. Let $f\colon Q \to \mathbb{R}$ be HK-integrable on Q, and let F be its HK-primitive. By Theorems 6.6 and 6.7, VF is μ-AC. Moreover by Theorems 6.6 and 6.10, F is differentiable μ-almost everywhere on Q, and, by Theorem 6.2, $F'(x) = f(x)$ μ-almost everywhere on Q.

Vice versa, let F be an additive function of cells such that VF is μ-AC on Q and $F'(x) = f(x)$ μ-almost everywhere on Q. Then, by Theorem 6.2, F is the HK-primitive of F' on Q. Thus, the condition $f(x) = F'(x)$, μ-almost everywhere on Q, implies the HK-integrability of f on Q.

□

Acknowledgment. The authors would like to thank Professor Jan Maly for his constructive comments during the preparation of this paper.

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