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EQUILATERAL WEIGHTS ON THE UNIT BALL OF $\mathbb{R}^n$

Abstract

An equilateral set (or regular simplex) in a metric space X is a set A such that the distance between any pair of distinct members of A is a constant. An equilateral set is standard if the distance between distinct members is equal to 1. Motivated by the notion of frame functions, as introduced and characterized by Gleason in [6], we define an equilateral weight on a metric space X to be a function $f : X \longrightarrow \mathbb{R}$ such that $\sum_{i \in I} f(x_i) = W$ for every maximal standard equilateral set $\{x_i : i \in I\}$ in X, where $W \in \mathbb{R}$ is the weight of f. In this paper, we characterize the equilateral weights associated with the unit ball B^n of $\mathbb{R}^n$ as follows: For $n \geq 2$, every equilateral weight on B^n is constant.

1 Introduction

Equilateral sets have been extensively studied in the literature for a number of metric spaces [2]. An equilateral set (or regular simplex) in a metric space X is a set A such that the distance between any pair of distinct members of A is ρ, where $\rho \neq 0$ is a constant. The equilateral dimension of X is defined to be $\sup\{|A| : A \text{ is an equilateral set in } X\}$.

Suppose that $\{x_1, \ldots, x_k\}$ is an equilateral set in $\mathbb{R}^n$ (equipped with the ℓ_2-norm). Then the vectors $v_i := x_{i+1} - x_1$ for $i = 1, \ldots, k-1$ are linearly independent. Indeed, let A be the $(k-1) \times (k-1)$ matrix (a_{ij}) defined by

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$a_{ij} := \langle v_i, v_j \rangle$. Then $a_{ij} = \frac{\rho^2}{2}(1+\delta_{ij})$, where $\rho \neq 0$ is a constant and δ_{ij} is the Kronecker delta. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$ and let B be the $n \times (k-1)$ matrix (b_{ij}) defined by $b_{ij} := \langle v_j, e_i \rangle$. Since $A = B^*B$ and A is clearly non-singular, we deduce that B is non-singular; i.e. the vectors $v_i := x_{i+1} - x_1$ for $i = 1, \ldots, k-1$ are linearly independent and therefore $k \leq n+1$. The equilateral dimension of $\mathbb{R}^n$ (equipped with the ℓ_2-norm) is $n+1$. To see this, observe that the set $\{x_1 - c, \ldots, x_k - c\}$, where $c := \frac{1}{k}\sum_{i=1}^k x_i$, has linear dimension $k-1$ and so if $k < n+1$, then there exists a unit vector $u \in \mathbb{R}^n$ such that $u \perp x_i - c$ for each $i = 1, \ldots, k$, and therefore the set $\{x_1, \ldots, x_k\}$ can be enlarged to a bigger equilateral set in $\mathbb{R}^n$. Let us only mention here that the situation is far more complicated for the other ℓ_p-norms [11, 9, 1] (and others).

An equilateral set in $\mathbb{R}^n$ is *standard* if the distance between distinct points is equal to 1. If $\{x_1, \ldots, x_k\}$ is a standard equilateral set in $\mathbb{R}^n$, its centre $\frac{1}{k}\sum_{i=1}^k x_i$ will be denoted by $c(x_1, \ldots, x_k)$. Let us denote by β_k the *radius* of $\{x_1, \ldots, x_k\}$. A simple calculation yields

$$\begin{aligned}\beta_k = \Big\| x_i - c(x_1, \ldots, x_k) \Big\| &= \frac{1}{k} \Big\| \sum_{\substack{1 \leq j \leq k \\ j \neq i}} (x_j - x_i) \Big\| \\ &= \frac{1}{k}\sqrt{k - 1 + \frac{(k-1)(k-2)}{2}} = \sqrt{\frac{k-1}{2k}}.\end{aligned}$$

If x_{k+1} is another point in $\mathbb{R}^n$ such that $\{x_1, \ldots, x_k, x_{k+1}\}$ is again a standard equilateral set, then $x_{k+1} - c(x_1, \ldots, x_k)$ is orthogonal to $x_i - c(x_1, \ldots, x_k)$ for every $i = 1, \ldots, k$, and thus

$$\Big\| x_{k+1} - c(x_1, \ldots, x_k) \Big\| = \sqrt{1 - \beta_k^2} = \sqrt{\frac{k+1}{2k}}.$$

We will call $\alpha_{k+1} := \sqrt{\frac{k+1}{2k}}$ the *perpendicular height* of $\{x_1, \ldots, x_k, x_{k+1}\}$.

We shall now introduce the notion of equilateral weights. The motivation behind this definition is the notion of frame functions. These were introduced and characterized by Gleason [6] in his famous theorem describing the measures on the closed subspaces of a Hilbert space. Gleason's Theorem is of utmost importance in the laying down of the foundations of quantum mechanics [12, 10, 7, 4, 8] (and others). Let $S(0,1)$ denote the unit sphere of a Hilbert space H. A function $f : S(0,1) \to \mathbb{R}$ is called a frame function on H if there is a number $w(f)$, called the weight of f, such that $\sum_{i \in I} f(u_i) = w(f)$ for every orthonormal basis $\{u_i : i \in I\}$ of H. We recall that a bounded operator T on

H is of trace-class if the series $\sum_{i\in I}\langle Tu_i, u_i\rangle$ converges absolutely for any orthonormal basis $\{u_i : i \in I\}$ of H. (It is well-known that if the series converges for an orthonormal basis $\{u_i : i \in I\}$, then it converges for any orthonormal basis and the sum does not depend on the choice of the basis.) Clearly, if T is self-adjoint and of trace-class, then the function $f_T(x) = \langle Tx, x\rangle$ ($x \in S(0,1)$) defines a continuous frame function on H. Gleason's Theorem says that when $\dim H \geq 3$, every bounded frame function arises in this way. The heart of the proof of Gleason's Theorem is the treatment of the case when H is the real three-dimensional Hilbert space $\mathbb{R}^3$. In fact, all the other cases can be reduced to this case. Thus, as a matter of fact, it can be said that the crux of this theorem can be rendered to the following statement: *For every bounded frame function f on $\mathbb{R}^3$ there exists a symmetric matrix T on $\mathbb{R}^3$ such that $f(u) = \langle Tu, u\rangle$ for every unit vector $u \in \mathbb{R}^3$.* The notion of frame functions and the fact that an orthonormal basis of $\mathbb{R}^3$ is simply a maximal equilateral set on the unit sphere of $\mathbb{R}^3$ suggest the following definition:

Definition 1. *Let X be a metric space and let $W \in \mathbb{R}$. An* equilateral weight on X with weight W *is a function $f : X \longrightarrow \mathbb{R}$ such that*

$$\sum_{i\in I} f(x_i) = W$$

whenever $\{x_i : i \in I\}$ is a maximal standard equilateral set in X.

Given a metric space, can one describe the equilateral weights associated with it?

Example 2. *Every equilateral weight on $\mathbb{R}^2$ is constant. First observe that for every pair of points x and y in $\mathbb{R}^2$ there are points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^2$ such that $\|x_1 - x\| = \|x_{i+1} - x_i\| = \|y - x_n\| = 1$ for every $i = 1, \ldots, n-1$. Thus, it suffices to show that $f(x) = f(y)$ for all $x, y \in \mathbb{R}^2$ satisfying $\|x - y\| = 1$. Let $x, y \in \mathbb{R}^2$ such that $\|x - y\| = 1$. Observe that if $\{a, b, c\}$ and $\{d, b, c\}$ are the vertices of two unit equilateral triangles and f is an equilateral weight, then $f(a) = f(d)$. Thus, f takes the constant value $f(x)$ on the circle with centre x and radius $\sqrt{3}$ and the constant value $f(y)$ on the circle with centre y and radius $\sqrt{3}$. Since these circles intersect, it follows that $f(x) = f(y)$. Using a similar argument, but replacing $\sqrt{3}$ with $2\alpha_{n+1}$, one can easily show that every equilateral weight on $\mathbb{R}^n$ is constant. The same cannot be said for $\mathbb{R}$ – it is easy to find non-trivial equilateral weights on $\mathbb{R}$.*

Example 3. *Let S be the sphere in a Hilbert space H with centre 0 and radius $1/\sqrt{2}$. Two vectors u and v in S satisfy $\|u - v\| = 1$ if and only if $\langle u, v\rangle = 0$. Thus, each maximal standard equilateral set in S corresponds to a rescaling of* *some orthonormal basis of H by a factor of $1/\sqrt{2}$. It is clear, therefore, that the equilateral weights on S correspond to the frame-functions on H (composite with a rescaling by a factor of $\sqrt{2}$). Thus, in view of Gleason's Theorem, if $\dim H \geq 3$ and f is a bounded equilateral weight on S, then there exists a self-adjoint, trace-class operator T such that*

$$f(u) = \langle Tu, u\rangle$$

for all $u \in S$. Let us emphasize that such a description does not hold when $\dim H = 2$ and that the assumption of boundedness is not redundant when $\dim H$ is finite. It known that $\mathbb{R}^n$ admits frame functions that are unbounded and that therefore cannot be described by such an equation (see [4, Proposition 3.2.4]).

By contrast, the boundedness assumption is superfluous when the space is infinite dimensional. This surprising result is due to Dorofeev and Sherstnev [3] and allows us to describe the equilateral weights associated with the metric space S of an infinite dimensional Hilbert space directly from Gleason's Theorem.

Proposition 4. *Let H be an infinite dimensional Hilbert space, and let S be the sphere in H with centre 0 and radius $1/\sqrt{2}$. If f is an equilateral weight on S, then there exists a self-adjoint, trace-class operator T on H such that $f(u) = \langle Tu, u\rangle$ for every vector u in S.*

The aim of the present paper is to describe the equilateral weights associated with another bounded metric space; namely the unit ball of $\mathbb{R}^n$.

2 Standard equilateral sets in the unit ball of $\mathbb{R}^n$

In what follows, we will be interested in standard equilateral sets contained in the (closed) unit ball of $\mathbb{R}^n$, denoted by B^n. It is clear that the equilateral dimension of B^n is equal to that of $\mathbb{R}^n$. We start by exhibiting some properties of standard equilateral sets in B^n.

Proposition 5. *Let $\{x_1, \dots, x_k\}$ $(k \leq n+1)$ be a standard equilateral set in B^n. Then $\|c(x_1, \dots, x_k)\| \leq \alpha_{k+1}$.*

Proof. First observe that

$$2\langle x_i, x_j\rangle = \|x_i\|^2 + \|x_j\|^2 - \|x_i - x_j\|^2 \leq 1,$$

and therefore,

$$\begin{aligned}\|c(x_1,\ldots,x_k)\|^2 &= k^{-2}\Big\langle \sum_{i=1}^k x_i, \sum_{i=1}^k x_i \Big\rangle \\ &= k^{-2}\Big[\sum_{i=1}^k \|x_i\|^2 + \sum_{\substack{1\le i,j\le k\\ i\ne j}} \langle x_i, x_j\rangle\Big] \\ &\le k^{-2}\Big[k + \frac{k(k-1)}{2}\Big] \\ &= \alpha_{k+1}^2.\end{aligned}$$

□

In the extremal case $k = n+1$, the bound obtained in Proposition 5 can be improved as shown in Proposition 7 below. This improvement is needed to prove Proposition 8. We first prove a lemma.

Lemma 6. *Let $\{x_1, x_2, \ldots, x_{n+1}\}$ be a maximal standard equilateral set in $\mathbb{R}^n$ with centre at the origin and let $x \in \mathbb{R}^n$ satisfy $\langle x, x_i\rangle \ge 0$ for $i = 2, 3, \ldots, n+1$. If $\|x\| \ge 1$, then $\langle x, x_2 + x_3 + \cdots + x_{n+1}\rangle \ge 1/2$.*

Proof. Let $v := x_2 + x_3 + \cdots + x_{n+1}$ and let

$$K := \big\{x \in \mathbb{R}^n : \langle x, v\rangle \le 1/2,\ \langle x, x_i\rangle \ge 0 \text{ for each } i = 2, 3, \ldots, n+1\big\}.$$

K is the intersection of half-spaces, and therefore a point of K is an extreme point if and only if it is the intersection of n hyperplanes whose normals form a basis of $\mathbb{R}^n$. Using the fact that $\langle x_i, x_j\rangle$ is independent of i, j (when $i \ne j$) it is easy to see that the extreme points of K are $\{0, x_2 - x_1, x_3 - x_1, \ldots, x_{n+1} - x_1\}$. The norm, being a strictly convex function, i.e.

$$\|\lambda x + (1-\lambda)y\| < \max(\|x\|, \|y\|), \quad x \ne y,\ 0 < \lambda < 1 \qquad (\star)$$

takes a maximum value at an extremal point, and therefore, since $\|x_i - x_1\| = 1$ ($i = 2, 3, \ldots, n+1$), it follows that $\|x\| \le 1$ for every $x \in K$. From the strict inequality of $(\star)$ and from the fact that each of the vectors $x_i - x_1$ ($i = 2, 3, \ldots, n+1$) lies in the hyperplane $\langle x, v\rangle = 1/2$, it follows that if $x \in \mathbb{R}^n$ satisfies $\langle x, x_i\rangle \ge 0$ ($i = 2, 3, \ldots, n+1$) and $\langle x, v\rangle < 1/2$, then $\|x\| < 1$. □

Proposition 7. *Let $\{u_1, \ldots, u_{n+1}\}$ be a standard equilateral set in B^n. Then $\|c(u_1, \ldots, u_{n+1})\| \le \beta_{n+1}$.*

PROOF. Let $\{u_1, u_2, \dots, u_{n+1}\}$ be a maximal standard equilateral set in B^n. Then $\{0, u_2 - u_1, \dots, u_{n+1} - u_1\}$ is again a maximal standard equilateral set in B^n. Let us denote its centre by c. Note that $\|c\| = \beta_{n+1}$. For each $i = 1, 2, \dots, n+1$, let $x_i := u_i - u_1 - c$. Then $\{x_1, x_2, \dots, x_{n+1}\}$ is a maximal standard equilateral set with centre at the origin. Note that

$$c(u_1, u_2, \dots, u_{n+1}) = c(x_1, x_2, \dots, x_{n+1}) + u_1 + c = u_1 + c.$$

Thus

$$\|c(u_1, u_2, \dots, u_{n+1})\|^2 = \|u_1 + c\|^2 = \|u_1\|^2 + \|c\|^2 + 2\langle u_1, c\rangle,$$

and therefore, for the proposition to hold, we require

$$\left\langle \frac{-u_1}{\|u_1\|}, c \right\rangle \geq \frac{\|u_1\|}{2}. \qquad (\star)$$

To this end, we calculate

$$\begin{aligned} 1 \geq \|u_i\|^2 =& \|x_i + c\|^2 + \|u_1\|^2 + 2\langle u_1, x_i + c\rangle \\ =& 1 + \|u_1\|^2 + 2\langle u_1, x_i\rangle + 2\langle u_1, c\rangle, \end{aligned}$$

which implies

$$\left\langle \frac{-u_1}{\|u_1\|}, x_i \right\rangle \geq \frac{\|u_1\|}{2} - \left\langle \frac{-u_1}{\|u_1\|}, c \right\rangle \qquad (\star\star).$$

for each $i = 2, 3, \dots, n+1$. Now, if the right hand side of $(\star\star)$ is ≤ 0, then $(\star)$ is satisfied. On the other-hand, if the right hand side of $(\star\star)$ is greater than 0, then Lemma 6 can be applied to conclude

$$\frac{\|u_1\|}{2} \leq \frac{1}{2} \leq \left\langle \frac{-u_1}{\|u_1\|}, x_2 + x_3 + \dots + x_{n+1} \right\rangle = \left\langle \frac{-u_1}{\|u_1\|}, -x_1 \right\rangle = \left\langle \frac{-u_1}{\|u_1\|}, c \right\rangle,$$

which completes the proof. □

Proposition 8. *Every standard equilateral set in B^n can be enlarged to one having size $n+1$ such that its members all lie in B^n.*

PROOF. Let $\{x_1, \dots, x_k\}$ $(1 \leq k \leq n)$ be a standard equilateral set in B^n. We show that there exists a vector $x_{k+1} \in B^n$ such that $\{x_1, \dots, x_k, x_{k+1}\}$ is a standard equilateral set. The proof will then follow by induction.

Let $N := \operatorname{span}\{x_i - c(x_1, \dots, x_k) : 1 \leq i \leq k\}$ and set

$$a := (I - P_N)c(x_1, \dots, x_k),$$

where P_N is the projection of $\mathbb{R}^n$ into N and I is the identity. The intersection of B^n with the translation $a+N$ is a $(k-1)$-dimensional ball with centre a and radius $\sqrt{1-\|a\|^2}$. The set $\{x_1,\dots,x_k\}$ is a standard equilateral set in $(a+N)\cap B^n$, and thus, in view of Proposition 7, it follows that

$$\|c(x_1,\dots,x_k)-a\| \le \beta_k.$$

Set $u := -\alpha_{k+1}v$, where $v := a/\|a\|$ if $a \neq 0$ and any unit vector in $N^\perp$ if $a=0$. Then $\|a+u\| \le \|u\| = \alpha_{k+1}$ since $\alpha_{k+1} \ge \beta_k = \|c(x_1,\dots,x_k\}\| \ge \|a\|$. Put $x_{k+1} := c(x_1,\dots,x_k)+u$. The set $\{x_1,\dots,x_k,x_{k+1}\}$ is a standard equilateral set in $\mathbb{R}^n$. Moreover,

$$\begin{aligned}\|x_{k+1}\|^2 &= \|c(x_1,\dots,x_k)+u\|^2\\ &= \big\|c(x_1,\dots,x_k)-a\big\|^2 + \|a+u\|^2\\ &\le \beta_k^2+\alpha_{k+1}^2\\ &= 1.\end{aligned}$$

□

3 Equilateral weights on B^n

In this section, we shall prove that the only admissible equilateral weights on the unit ball of $\mathbb{R}^n$ are those that take a constant value.

For any linear subspace M of $\mathbb{R}^n$, $a \in M$ and $r>0$, we denote the closed ball in M with centre a and radius r by $B^M(a,r)$; i.e.

$$B^M(a,r) = \{x \in M : \|x-a\| \le r\}.$$

We will also denote by $S^M(a,r)$ the sphere in M with centre a and radius r; i.e. $S^M(a,r) = \{x \in M : \|x-a\| = r\}$. We will write $B(a,r)$ (resp. $S(a,r)$) instead of $B^{\mathbb{R}^n}(a,r)$ (resp. $S^{\mathbb{R}^n}(a,r)$. We will need the following definition.

Definition 9. *Let $a,b \in B^n$, $a \neq b$ and $N := (b-a)^\perp$. For any subspace $M \neq \{0\}$ of $\mathbb{R}^n$, define*

$$\gamma^M(a,b) := \sup\left\{ r>0 \,:\, \frac{a+b}{2} + B^{M\cap N}(0,r) \subseteq B^n \right\}.$$

Note that the set involved in the definition of $\gamma^M(a,b)$ is not empty and bounded above by 1. Instead of $\gamma^{\mathbb{R}^n}(a,b)$, we will simply write $\gamma(a,b)$. It is easy to see that $\gamma^M(a,b)$ is in fact equal to the maximum of the set of its definition. In addition, if M_1 and M_2 are subspaces of $\mathbb{R}^n$ such that $M_1 \subseteq M_2$, then $\gamma^{M_2}(a,b) \le \gamma^{M_1}(a,b)$. The motivation behind this definition lies in the following observation.

Lemma 10. *Let $a, b \in B^n$ such that $\|b-a\| = 2\alpha_{n+1}$ and $\gamma(a,b) \ge \beta_n$. Then $f(a) = f(b)$ for every equilateral weight f on B^n.*

Proof. Let $N := (b-a)^\perp$ and let $\{x_1, \dots, x_n\}$ be a standard equilateral set in

$$\frac{a+b}{2} + S^N(0, \beta_n) \subseteq B^n.$$

Each x_i can be written as $(a+b)/2 + n_i$, where $n_i \in N$ and $\|n_i\| = \beta_n$. Thus,

$$\|x_i - a\|^2 = \left\| \frac{b-a}{2} + n_i \right\|^2 = \alpha_{n+1}^2 + \beta_n^2 = 1.$$

Similarly, $\|x_i - b\| = 1$, i.e. $\{a, x_1, \dots, x_n\}$ and $\{b, x_1, \dots, x_n\}$ are maximal standard equilateral sets in B^n, and therefore,

$$f(a) + \sum_{i=1}^{n} f(x_i) = f(b) + \sum_{i=1}^{n} f(x_i)$$

for every equilateral weight f on B^n. □

Lemma 11. *Let $a, b \in B^n$, $a \neq b$, and let T be a two-dimensional subspace of $\mathbb{R}^n$ containing a and b. Then $\gamma^T(a,b) = \gamma(a,b)$.*

Proof. We show that $\gamma(a,b) \ge \gamma^T(a,b)$. Let u be a unit vector in T such that $\langle u, b-a \rangle = 0$ and $\langle u, b+a \rangle \ge 0$. Set $x_0 := (a+b)/2$. Let $r > 0$ such that $\|x_0 + ru\| \le 1$, and let $x \in (b-a)^\perp$ such that $\|x\| \le r$. Then $P_T x = \lambda u$, where $|\lambda| \le \|x\| \le r$. Hence,

$$\begin{aligned}
\|x_0 + x\|^2 &= \|x_0\|^2 + \|x\|^2 + 2\langle x_0, x \rangle \\
&\le \|x_0\|^2 + \|x\|^2 + 2|\langle P_T x_0, x \rangle| \\
&= \|x_0\|^2 + \|x\|^2 + 2|\lambda|\langle x_0, u \rangle \\
&\le \|x_0\|^2 + r^2 + 2r\langle x_0, u \rangle \\
&= \|x_0 + ru\|^2 \\
&\le 1,
\end{aligned}$$

and therefore, $\gamma(a,b) \ge \gamma^T(a,b)$ as required. □

Lemma 12. *Let f be an equilateral weight on B^n, where $n \geq 2$. There exists $0 \leq \lambda_n < 1$ such that f is constant in $\{x \in B^n : \|x\| \geq \lambda_n\}$.*

Proof. It suffices to show that there exists $0 \leq \lambda_n < 1$ such that f is constant in $\{x \in B^n \cap T : \|x\| \geq \lambda_n\}$ for every two-dimensional subspace T of $\mathbb{R}^n$.

Fix an arbitrary two-dimensional subspace T, and let D denote the closed unit disc $B^n \cap T$. To make calculations easier, we fix a rectangular coordinate system in D with origin o at the centre of D (see Figure 1.). Consider the points $w(0,-1)$, $x(-1,0)$, $y(0,1)$ and $z(1,0)$. Let C_w (resp. C_x, C_y, C_z) be the circular arc with centre w (resp. x, y, z) and radius $2\alpha_{n+1}$. The arcs C_w and C_x meet in D at the point a, the coordinates of which can be easily calculated:

$$a\Big(\frac{-1+\sqrt{8\alpha_{n+1}^2-1}}{2}, \frac{-1+\sqrt{8\alpha_{n+1}^2-1}}{2}\Big).$$

Similarly, let $b, c, d \in D$ such that $C_x \cap C_y = \{b\}$, $C_y \cap C_z = \{c\}$ and $C_z \cap C_w = \{d\}$. Let C_a (resp. C_b, C_c and C_d) denote the circular arc in D having centre a and radius $2\alpha_{n+1}$ (see Figure 1 below).

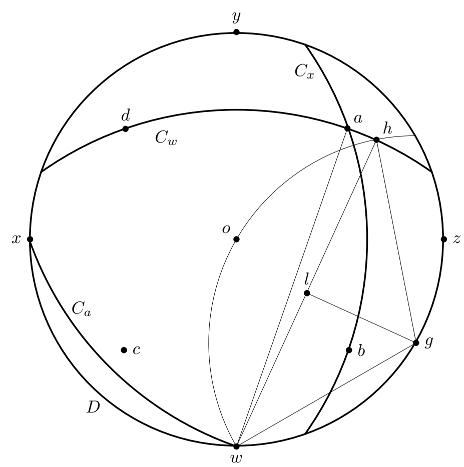

Figure 1

First we show that $\gamma^T(a,w) \geq \beta_n$. Let g be the point $\big(\frac{\sqrt{3}}{2}, -\frac{1}{2}\big)$. Since $2\alpha_{n+1} \leq \sqrt{3}$, it easy to see that the circular arc in D having centre g and radius 1 intersects C_w, say at h. Observe that if l is the midpoint of the line segment wh, then $|lg| = \beta_n$. So to show that $\gamma^T(w,a) \geq \beta_n$, it suffices to show that the angle $\widehat{owa}$ is less than or equal to the angle $\widehat{owh}$. To this end, it is enough to show that $\sin\widehat{owa} \leq \sin\widehat{owh}$. Since $\widehat{doa} = \frac{\pi}{2}$, we have

$$\begin{aligned}\sin\widehat{owa} &= \sin(\pi/4 - \widehat{oaw})\\ &= \frac{1}{\sqrt{2}}\big(\cos\widehat{oaw} - \sin\widehat{oaw}\big).\end{aligned}$$

Applying the sine rule for triangle oaw we deduce that

$$\sin\widehat{oaw} = \frac{\sin 3\pi/4}{2\alpha_{n+1}} = \frac{1}{2}\sqrt{\frac{n}{n+1}} \quad \text{and} \quad \cos\widehat{oaw} = \frac{1}{2}\sqrt{\frac{3n+4}{n+1}}.$$

Thus,

$$\sin\widehat{owa} = \frac{1}{2\sqrt{2}}\left(\sqrt{3+\frac{1}{n+1}} - \sqrt{1-\frac{1}{n+1}}\right).$$

On the other-hand,

$$\begin{aligned}\sin\widehat{owh} &= \sin(\pi/3 - \widehat{lwg})\\ &= \frac{1}{2}(\sqrt{3}\cos\widehat{lwg} - \sin\widehat{lwg})\\ &= \frac{1}{2}(\sqrt{3}\alpha_{n+1} - \beta_n)\\ &= \frac{1}{2\sqrt{2}}\left(\sqrt{3+\frac{3}{n}} - \sqrt{1-\frac{1}{n}}\right).\end{aligned}$$

Thus, $\sin\widehat{owa} \leq \sin\widehat{owh}$ and therefore $\gamma^T(w,a) \geq \beta_n$.

It is clear (see Figure 1.) that $\gamma^T(u,a) \geq \gamma^T(w,a)$ for every $u \in C_a$. Thus, in view of Lemma 10 and Lemma 11, it follows that f is constant on C_a. By symmetry, it follows that f is constant on the circuit $C_a \cup C_b \cup C_c \cup C_d$. If $\{w', x', y', z'\}$ is another quadruple of points on the circumference of D such that $w'y'$ and $x'z'$ are perpendicular, then we can repeat the same argument as above to deduce that f is constant on the corresponding circuit joining the points w', x', y' and z'. Moreover, since any two such circuits intersect, it follows that f is constant in the annulus $\{u \in D : |ou| \geq 2\alpha_{n+1} - |oa|\}$. Let $\lambda_n := 2\alpha_{n+1} - |oa|$. From the coordinates of a one can calculate

$$\lambda_n = \frac{1}{\sqrt{2}}\left(1 + \sqrt{4+\frac{4}{n}} - \sqrt{3+\frac{4}{n}}\right).$$

□

For each $\rho \in [\beta_n, 1]$, define $\eta_n(\rho) := \alpha_{n+1} - \sqrt{\rho^2 - \beta_n^2}$. Observe that the value $\eta_n(\rho)$ decreases strictly from α_{n+1} (when $\rho = \beta_n$) to 0 (when $\rho = 1$), and $\eta_n(\rho) = \rho$ if and only if $\rho = \beta_{n+1}$. Thus, $\eta_n(\rho) \geq \rho$ for every $\rho \in [\beta_n, \beta_{n+1}]$, and $\eta_n(\rho) < \rho$ when $\rho \in (\beta_{n+1}, 1]$. The geometric meaning of $\eta_n(\rho)$ becomes apparent from the following lemma.

Lemma 13. *(a) Let $1 \geq \rho \geq \beta_n$, and let $x \in B^n$ such that $\|x\| = \eta_n(\rho)$. Then there exists a standard equilateral set $\{x_1, x_2, \dots, x_n\}$ such that $\|x_i\| = \rho$ and $\|x_i - x\| = 1$ for every $i = 1, 2, \dots, n$.*

(b) Conversely, if $\{x_1, x_2, \dots, x_{n+1}\}$ is a maximal standard equilateral set in B^n and $\|x_i\| = \rho$ for every $i = 1, 2, \dots, n$, then $\rho \geq \beta_n$, and if $\operatorname{conv}(x_1, \dots, x_{n+1})$ contains 0, then $\|x_{n+1}\| = \eta_n(\rho)$.

PROOF. (a) First note that if $\rho = 1$, then $0 = \eta_n(\rho) = \|x\|$, and therefore, the statement is true in this case. Suppose that $\beta_n \leq \rho < 1$. Let $\{u_1, u_2, \dots, u_n\}$ be a maximal standard equilateral set in $x^\perp$ with centre 0. Then $\|u_i\| = \beta_n$. It is easy to check that the vectors

$$x_i := u_i - \sqrt{\rho^2 - \beta_n^2}\,\frac{x}{\|x\|} \qquad (i = 1, 2, \dots, n)$$

satisfy the required conditions.

(b) The locus of points in $\mathbb{R}^n$ equidistant from each of the x_i's ($i = 1, \dots, n$) is the line passing through 0 and parallel to $x_{n+1} - c(x_1, \dots, x_n)$. The point on this line with shortest distance from any (and therefore from each) of the x_i's ($i = 1, \dots, n$) is that with position vector $c(x_1, \dots, x_n)$. Thus,

$$\beta_n = \big\|c(x_1, \dots, x_n) - x_i\big\| \leq \|x_i\| = \rho \qquad (i = 1, 2, \dots, n).$$

If $0 \in \operatorname{conv}(x_1, \dots, x_{n+1})$, then $0 = \lambda x_{n+1} + (1 - \lambda) c(x_1, \dots, x_n)$ for some $\lambda \in [0, 1]$. Thus,

$$\begin{aligned}\alpha_{n+1} = \big\|x_{n+1} - c(x_1, \dots, x_n)\big\| &= \|x_{n+1}\| + \big\|c(x_1, \dots, x_n)\big\| \\ &= \|x_{n+1}\| + \sqrt{\rho^2 - \beta_n^2}.\end{aligned}$$

□

Lemma 14. *Let f be an equilateral weight on B^n taking the constant value δ in $\{x \in B^n : \|x\| \geq \rho_0\}$, where $\rho_0 \in [\beta_n, 1]$. Then f takes the constant value $W - n\delta$ in $B(0, \eta_n(\rho_0))$, where W is the weight of f. If $\rho_0 \leq \beta_{n+1}$, then f takes the constant value $\frac{W}{n+1}$ in B^n.*

Proof. Let $x \in B(0, \eta_n(\rho_0))$. The inequality $0 \leq \|x\| \leq \eta_n(\rho_0)$ implies that there exists $1 \geq \rho \geq \rho_0$ such that $\eta_n(\rho) = \|x\|$. Thus, by Lemma 13, there are vectors $\{x_1, x_2, \ldots, x_n\}$ such that $\|x_i\| = \rho$ for $1 \leq i \leq n$ and such that $\{x, x_1, x_2, \ldots, x_n\}$ is a maximal standard equilateral set in B^n. So, $f(x) + n\delta = W$.

If $\rho_0 \leq \beta_{n+1}$, then $\eta_n(\rho_0) \geq \rho_0$, i.e.

$$\{x \in B^n : \|x\| \geq \rho_0\} \cap B(0, \eta_n(\rho_0)) \neq \emptyset,$$

and thus, $W - n\delta = \delta$. □

We are now ready to prove the result announced in the abstract.

Theorem 15. *Let $n \geq 2$. Every equilateral weight on B^n is constant.*

Proof. Set $\mu_n(\rho) := 1 - \eta_n(\rho)$ and $\nu_n(\rho) := \rho - \mu_n(\rho)$ when $\rho \in [\beta_n, 1]$. Observe that μ_n is strictly increasing with range $[1 - \alpha_{n+1}, 1]$. It is easy to check that ν_n is strictly decreasing and that $\nu_n(1) = 0$. Thus, $\mu_n(\rho) < \rho$ for all $\rho \in [\beta_n, 1)$.

Let f be an equilateral weight on B^n. In view of Lemma 12, we can define

$$\theta := \inf\{\rho : f \text{ is constant in } B^n \setminus B(0, \rho)\}$$

and note that $\theta \leq \lambda_n$. In view of Lemma 14, the proof would be complete if we could show that $\theta < \beta_{n+1}$. So we suppose that $\theta \geq \beta_{n+1}$ and seek a contradiction. Let ϵ be a positive real number satisfying

$$\epsilon < \min\{\nu_n(\lambda_n), \beta_{n+1} - \beta_n\}.$$

Then $\theta - \epsilon > \beta_n > 1 - \alpha_{n+1}$, and thus, $\mu_n^{-1}(\theta - \epsilon)$ is defined. In addition, it follows that $\mu_n^{-1}(\theta - \epsilon) > \theta$, for if $\mu_n^{-1}(\theta - \epsilon) \leq \theta$, then (since μ_n is strictly increasing) we would have $\theta - \epsilon \leq \mu_n(\theta)$ and this would lead to $\epsilon \geq \nu_n(\theta) \geq \nu_n(\lambda_n)$, which contradicts our choice of ϵ.

Fix $\rho_0 := \mu_n^{-1}(\theta - \epsilon)$. Then, since $\mu_n^{-1}(\theta - \epsilon) > \theta$, f takes a constant value, say δ, in the annulus $\{x \in B^n : \|x\| \geq \rho_0\}$, and therefore, by virtue of Lemma 14, f takes the constant value $W - n\delta$ in $B(0, \eta_n(\rho_0))$, where W is the weight of f. We show that f then must take the constant value δ in the annulus $\{x \in B^n : \|x\| \geq \mu(\rho_0)\}$. This would contradict the definition of θ and thus conclude the proof.

To this end, fix an arbitrary vector $u \in B^n$ such that

$$1 - \eta_n(\rho_0) = \mu_n(\rho_0) \leq \|u\| \leq \rho_0, \qquad (\star)$$

and let $v = -\frac{1-\|u\|}{\|u\|}u$. Then $v \in B^n$ and $1 = \|u - v\| = \|u\| + \|v\|$. From the inequalities

$$1 - \eta_n(\rho_0) + \|v\| \leq \|u\| + \|v\| = 1 \leq \rho_0 + \|v\|$$

we obtain $1 - \rho_0 \leq \|v\| \leq \eta_n(\rho_0)$, and therefore, in virtue of Lemma 14, we obtain $f(v) = W - n\delta$. We can now apply Proposition 8 to obtain an enlargement $\{x_1, \ldots, x_{n-1}, u, v\}$ of $\{u, v\}$ to a maximal standard equilateral set in B^n. Let $w := (u + v)/2$. For each $i = 1, 2, \ldots, n-1$, we have

$$\|x_i\|^2 = \|x_i - w\|^2 + \|w\|^2 = \frac{3}{4} + \left|\|u\| - \frac{1}{2}\right|^2.$$

If $\eta_n(\rho_0) > \frac{1}{2}$, then $\rho_0^2 < 5/4 - \alpha_{n+1}$, and thus,

$$\|x_i\|^2 \geq \frac{3}{4} > \frac{5}{4} - \frac{1}{\sqrt{2}} > \frac{5}{4} - \alpha_{n+1} > \rho_0^2.$$

On the other-hand, if $\eta_n(\rho_0) \leq \frac{1}{2}$, then $(\star)$ implies

$$\frac{1}{2} \leq 1 - \eta_n(\rho_0) \leq \|u\|$$

, and therefore,

$$\begin{aligned}
\|x_i\|^2 &= \frac{3}{4} + \left|\|u\| - \frac{1}{2}\right|^2 \\
&\geq \frac{3}{4} + \left(\frac{1}{2} - \eta_n(\rho_0)\right)^2 \\
&= 1 - \eta_n(\rho_0) + \eta_n(\rho_0)^2 \\
&= (1 - 2\alpha_{n+1})\left(\sqrt{\rho_0^2 - \beta_n^2} - \alpha_{n+1}\right) + \rho_0^2 \\
&\geq \rho_0^2.
\end{aligned}$$

So in both cases we conclude that $f(x_i) = \delta$ for each $i = 1, 2, \ldots, n-1$, and therefore,

$$\begin{aligned}
f(u) =& W - f(v) - \sum_{i=1}^{n-1} f(x_i) \\
=& W - (W - n\delta) - (n-1)\delta = \delta,
\end{aligned}$$

as required. This completes the proof. □

Remark 16. (i) *It follows immediately from the theorem proved here that an equilateral weight on a connected subset of* $\mathbb{R}^n$ *that is the union of unit balls is constant.*

(ii) *Our method of the proof should work also to show that an equilateral weight on an* n*-dimensional (closed) ball with radius greater than* α_{n+1} *is constant. What is not completely clear to us is the case when the radius lies in the interval* $(\beta_{n+1}, \alpha_{n+1}]$.

(iii) *Although we have defined equilateral weights as real-valued functions, it is apparent from the proof that the same conclusion can be drawn if one considers group-valued equilateral weights on the unit ball of* $\mathbb{R}^n$.

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