SURVEY

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THE DENJOY-YOUNG-SAKS THEOREM IN HIGHER DIMENSIONS: A SURVEY

Abstract

The classical theorem of Denjoy, Young and Saks gives a relation between Dini derivatives of a real variable function that holds almost everywhere. We present what is known in the one and two variable case with an emphasis on the latter. Relations that hold a.e. in both the measure and category sense are considered. Classical and approximate derivatives are both discussed.

1 Introduction.

One of the most well-known and probably the first theorem concerning almost everywhere differentiability of real functions is the famous theorem of Henri Léon Lebesgue (1875-1941), [41] stating that every monotone function is differentiable almost everywhere. Further results were subsequently obtained for BV and Lipschitz functions and these investigations culminated in the Denjoy-Young-Saks Theorem, the subject of this survey. The Denjoy-Young-Saks Theorem is about arbitrary real valued functions of one real variable and describes all possible behaviors of derivate numbers a.e. It is a natural question to ask what can be said about differentiability a.e. of real valued functions of several variable; in this survey we mainly restrict our attention to what is known about the simpler, but still intriguing two variable case.

In recent years research has been intense around similar questions in Banach spaces (see e.g. [42], [87]), but this paper is concerned with results that

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are known about the relations that hold a.e. between Dini derivatives of one and two variable real valued functions. Relations that hold a.e. in both the measure and category sense are considered and both the classical and approximate derivatives are discussed. Several classical results, and some new counterexamples are also presented.

Most of the results presented for one variable functions can also be found in [12] or [76]. For further discussion of both the one and several variable case the reader is referred to Saks' *Theory of the integral,*[1] [66] and [13]. Similar and related aspects are also discussed in [43]. Many additional results and a description of the historical context of the one variable theorem can be found in [15]. The current paper is a synthesis of the author's master's thesis [45] at Eötvös Loránd University.

2 Measure.

2.1 Classical derivatives.

2.1.1 One variable functions

The Denjoy-Young-Saks theorem expresses the connection between the Dini derivatives of an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ almost everywhere. The *Dini derivatives* (sometimes also called derivate numbers or derivates) of a function $f : \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$D^+ f(x) := \limsup_{t \to x+} \frac{f(t) - f(x)}{t - x}$$

$$D^- f(x) := \limsup_{t \to x-} \frac{f(t) - f(x)}{t - x}$$

$$D_+ f(x) := \liminf_{t \to x+} \frac{f(t) - f(x)}{t - x}$$

$$D_- f(x) := \liminf_{t \to x-} \frac{f(t) - f(x)}{t - x}$$

The derivates $D^+ f$ and $D_- f$ are called *opposite derivatives* of each other as are $D_+ f$ and $D^- f$. Moreover, the pairs of derivatives $D^+ f$ and $D_+ f$ and their left-sided counterparts $D^- f$ and $D_- f$ are called *one-sided derivatives.* Finally, $D^+ f$ and $D^- f$ are called *upper derivatives* while $D_+ f$ and $D_- f$ are called *lower derivatives* of f. We denote n-dimensional Lebesgue measure by λ_n.

[1]Many results that were original at the time are stated in the first edition of the book. However, unless stated, we always cite from the second (English) edition of the book.

Theorem 1 (Denjoy-Young-Saks [22], [82], [62]). *For an arbitrary function $f : E \to \mathbb{R}$ defined on an arbitrary set $E \subseteq \mathbb{R}$, at λ_1-a.e. point, one of the following three cases holds:*

1. *f is differentiable.*
2. *Either*
   (a) *$D^+f = D_-f$ is finite, $-\infty = D_+f$ and $D^-f = +\infty$, or*
   (b) *$D^-f = D_+f$ is finite, $-\infty = D_-f$ and $D^+f = +\infty$.*
3. *$D_+f = D_-f = -\infty$ and $D^+f = D^-f = -\infty$.*

With the notation established earlier, we can restate this theorem as follows.

Theorem 2 (Denjoy-Young-Saks [22], [82], [62]). *For an arbitrary function $f : E \to \mathbb{R}$ defined on an arbitrary set $E \subseteq \mathbb{R}$, at λ_1-a.e. point, one of the following three cases holds:*

1. *f is differentiable.*
2. *one pair of f's opposite derivatives are finite and equal, the two other opposite derivatives are infinite with the appropriate sign.*
3. *all four derivate numbers of f are $\pm\infty$ with the appropriate sign.*

A function f with one of the previous properties at x is said to have the Denjoy property *at x.*

The theorem in its previous form was proved by Arnaud Denjoy (1884-1974) in [22], for continuous functions. In [82] Grace Chisholm Young (1868-1944) weakened the condition of continuity to measurability. Then in [62], Stanisław Saks (1897-1942) proved the theorem in its final form for arbitrary real functions. Now there are several proofs. For a proof based on Vitali's Covering Theorem see Eugene Harold Hanson's proof in [27]. For a proof based on the Contingents Theorem see Saks' proof [66] Chapter IX.4, and for a direct proof see Riesz–Szőkefalvi-Nagy [56], pages 18-19.

2.1.2 Two variable functions

Two issues immediately arise when attempting to generalize the Denjoy-Young-Saks theorem to higher dimensions.

A. How does one generalize the Dini derivatives?

B. What should replace the λ_1-a.e.?

There are at least two natural ways to answer Question A. First, we define the *directional* or *linear Dini derivatives* of an arbitrary function $f : E \to \mathbb{R}$ defined on $E \subseteq \mathbb{R}^2$ at a point $x \in E$ in a direction $0 \leq \vartheta < 2\pi$ to be

$$\partial^\vartheta f(x) := \limsup_{E\cap l \ni y \to x} \frac{f(y) - f(x)}{|y - x|}$$
$$\partial_\vartheta f(x) := \liminf_{E\cap l \ni y \to x} \frac{f(y) - f(x)}{|y - x|}$$

where l denotes the half-line $l(x, \vartheta)$ extending from the point x in direction ϑ measured counterclockwise from the x-axis. For instance, this means that if $\partial^0 f = \partial_0 f = -\partial^\pi f = -\partial_\pi f$ and is finite, then the partial derivative $\partial_x f$ exists and is equal to $\partial_0 f$. If the function f restricted to $l(x, \vartheta)$ has the Denjoy property as a one variable function, we say that f has the *directional* (or linear) *Denjoy property* at the point x in the direction ϑ. Another approach to answer Question A is through directed derivatives, which are defined in Section 2.1.7.

To answer Question B there are at least two natural candidates. One can either consider whether the Denjoy property holds at λ_2-a.e. point in every direction or λ_3-a.e. in $\mathbb{R}^2 \times [0, 2\pi)$.

Historically, Hans Rademacher (1892-1969) was the first one to examine differentiability a.e. of several variable functions in [52]. In particular, he proved differentiability a.e. of Lipschitz continuous functions. Vyacheslav Vassilievich Stepanoff (1889-1950), [75] gave the following condition for Lebesgue measurable functions.

Theorem 3 (Stepanoff [75])**.** *Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function defined on a Lebesgue measurable set $E \subseteq \mathbb{R}^2$. Then f is differentiable λ_2-a.e. on E if and only if at λ_2-a.e. point $x \in E$*

$$L_f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty.$$

In [17], John Charles Burkill (1900-1993) and Ughtred Shuttleworth Haslam-Jones (1903-1962) uncovered two imprecise statements in Stepanoff's proof. First, in order to avoid questions of non-measurability, Stepanoff supposes that the Lebesgue measurable function $f : E \to \mathbb{R}$ is defined on an F_σ-set $M \subseteq E$ of measure equal to that of E. However, there are simple examples that show that by defining f on an additional null-set, the differentiability properties can change on a set of positive measure. Thus, Stepanoff's proof is only valid for Lebesgue measurable functions defined on F_σ-sets.

The second shortcoming of the proof is that Stepanoff obtains a function f defined on a set P, such that

$$\frac{f(x+h,y)-f(x,y)}{h} \to \frac{\partial f}{\partial x}$$

uniformly as $h \to 0$, for $x \in P$. However, $f(x+h,y)$ may not be defined as $x+h$ need not be in P and even if it is, it is not clear that f is continuous at such points. Moreover, the continuity of $\partial f/\partial x$ is needed in Stepanoff's proof and hinges on this argument.

It is interesting to note that the Burkill–Haslam-Jones article also contains an incorrect statement, which was discovered by Clarence Raymond Adams (1898-1965) and James Andrew Clarkson (1906-1970) in a 1939 correction, [2] to their own 1936 article, [1] on functions of bounded variation. The statement, Lemma 2 of [17] states that if $f : E \to \mathbb{R}$ is a two variable Lebesgue measurable function, defined on a Lebesgue measurable set $E \subseteq \mathbb{R}^2$, then the partial derivatives, taken where they exist, are also Lebesgue measurable. This is false. Miloš Neubauer (1898-1959) in [49] had already published the following counterexample, originally given by Hans Hahn (1879-1934).[2]

Example 1 (H. Hahn via M. Neubauer [49]). *Let $N \subseteq \mathbb{R}$ be a non-measurable subset of the line and define $f(x,y) := \chi_{\mathbb{Q}}(x)\chi_N(y)$, where χ_M denotes the characteristic function of the set M. The function f is zero outside of a null-set, so it is λ_2-measurable. Using the previous notation, $\partial^0 f$ and $\partial_0 f$ denote the respective upper and lower directional derivatives of f in the positive x-direction, and we have*

$$\begin{array}{llll} \partial^0 f|_{\mathbb{R}\times N^c} \equiv 0 & \partial_0 f|_{\mathbb{R}\times N^c} \equiv 0 & \partial^\pi f|_{\mathbb{R}\times N^c} \equiv 0 & \partial_\pi f|_{\mathbb{R}\times N^c} \equiv 0 \\ \partial^0 f|_{\mathbb{Q}\times N} \equiv 0 & \partial_0 f|_{\mathbb{Q}\times N} \equiv -\infty & \partial^\pi f|_{\mathbb{Q}\times N} \equiv 0 & \partial_\pi f|_{\mathbb{Q}\times N} \equiv -\infty \end{array}$$

since $\mathbb{Q}$ is dense in $\mathbb{R}$. Also, $f|_{\mathbb{Q}\times N} \equiv 1$, and so

$$\partial^0 f|_{\mathbb{Q}^c\times N} \equiv \infty \qquad \partial_0 f|_{\mathbb{Q}^c\times N} \equiv 0 \qquad \partial^\pi f|_{\mathbb{Q}^c\times N} \equiv \infty \qquad \partial_\pi f|_{\mathbb{Q}^c\times N} \equiv 0.$$

Since $\mathbb{Q}^c \times N$ is non-measurable, $\partial^0 f$ is a non-measurable function. The set where the partial derivatives in the direction of the x-axis exist is $\mathbb{R} \times N^c$, which is not Lebesgue measurable.

[2] It is intriguing that Burkill and Haslam-Jones even stated this Lemma 2, since Haslam-Jones already knew of Neubauer's counterexample, see [28], p. 121.

As we shall see later, although the proof is flawed, Stepanoff 's theorem is correct, and can be even extended to arbitrary functions. This follows from either Theorem 11 of Haslam-Jones or Theorem 12 of Saks.

2.1.3 Measurability (one variable functions)

The previous examples show that measurability conditions are delicate points of this investigation. In 1922, for functions of one variable Stefan Banach (1892-1945) showed in [5] that the Dini derivatives of a Lebesgue-measurable function are also Lebesgue-measurable. Wacław Sierpiński (1882-1969), in the same volume of *Fund. Math.* [71], obtained the same result for Borel measurable functions by showing that if f is a real function of class Baire-α, then its Dini derivates are of class $\alpha + 2$ .In fact, Banach also obtained the same result for *bounded* functions of class Baire-α. Stronger relations do not hold; simple examples show there exist Lebesgue measurable functions whose derivate numbers are not Borel measurable as well as arbitrary functions whose derivate numbers are not Lebesgue measurable. In 1925, Herman Auerbach (1901-1942) gave a simple proof for the Lebesgue measurable case in [4].

Remark 1. *We mention some further related results concerning the measurability of derivate numbers: In [26] it is shown that the extreme bilateral derivatives,* $\overline{f} = \max(D^+f, D^-f)$ *and* $\underline{f} = \min(D_+f, D_-f)$ *of arbitrary one variable functions are of Baire class two and in [73] it is shown that this is best possible. Surprisingly in [39] a function of Baire class two is given such that the upper symmetric derivatives are not Borel measurable.*

2.1.4 Measurability (two variable functions)

For functions of two real variables one can speak of the measurability of the $(x, \vartheta) \mapsto \partial^{\vartheta} f(x)$ directional Dini derivatives, or measurability of Dini derivatives in a fixed direction ϑ_0: $x \mapsto \partial^{\vartheta_0} f(x)$; the latter are the sections of the previous function.

First we discuss the measurability of $\partial^{\vartheta_0} f$ in a fixed ϑ_0 direction. If f is continuous, then $\partial^{\vartheta_0} f$ is Borel measurable, however Borel measurability of f does not imply that $\partial^{\vartheta_0} f$ is Borel measurable. If f is Borel measurable, then $\partial^{\vartheta_0} f$ is Lebesgue measurable. For the previous results see Neubauer [49] and also [38], pp. 512-514.[3] If f is merely Lebesgue measurable, then $\partial^{\vartheta_0} f$ is not necessarily Lebesgue measurable, as Example 1 already shows.

[3]Actually, it is shown that the upper [lower] partial Dini derivatives are measurable functions with respect to the sigma-algebra generated by the analytic sets [co-analytic sets].

However, restricted to the smaller set $M := \{x : L_f(x) < \infty\}$, the functions $\partial^{\vartheta_0} f$ are Lebesgue measurable relative to the set where f is defined. Indeed, an argument due to Herbert Federer (1920-2010) [74] p. 268 proves the following.

Theorem 4 (Federer [74])**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lebesgue measurable function. Then the set*

$$M := \{x \in \mathbb{R}^2 : L_f(x) < \infty\}$$

is Lebesgue measurable and the set of points where the function is differentiable is also Lebesgue measurable.

Federer's argument can be extended to prove Stepanoff's theorem for arbitrary functions.

Theorem 5. *Let $f : E \to \mathbb{R}$ be an arbitrary function defined on an arbitrary set $E \subseteq \mathbb{R}^2$. Define*

$$M := \{x \in E : L_f(x) < \infty\}.$$

Then the function is differentiable a.e. on M. Moreover M is a relative F_σ, and $\partial^{\vartheta_0} f$ are relative Lebesgue measurable functions on M for every $\vartheta_0 \in [0, 2\pi)$.

Note that in Example 1, M is empty. For a proof of the theorem, see Appendix A.1.

When looking at questions of measurability, instead of restricting the set of definition, another direction is to impose some restrictions on the class of functions. In this direction, in 1961 James Burton Serrin (1926-2012) extended Lebesgue's theorem on differentiability of monotonic functions to the several variable case.

Theorem 6 (Serrin [67])**.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function and for almost every $c \in \mathbb{R}$ the section $f_c(x) := f(x, c)$ is monotone, then the partial derivative $\partial_x f$ exists λ_2-a.e. in $\mathbb{R}^2$.*

Serrin also remarks that the monotonicity condition of the theorem cannot be replaced by bounded variation, as he gives a Lebesgue measurable function $f : \mathbb{R}^2 \to \mathbb{R}$, with bounded variation sections $f_c(y) = f(c, y)$ for all $c \in \mathbb{R}$, such that $\partial_x f$ does not exist on a set of full λ_2 outer measure. In 1977, Moshe Marcus and Victor Julius Mizel (1931-2005) weakened the condition of monotonicity of the previous theorem in [44] in the following way. For an arbitrary real variable function $f : \mathbb{R} \to \mathbb{R}$ define the *right cluster set* of f at x as

$$C^+ f(x) := \{y \in \mathbb{R} \cup \{\pm\infty\} : \exists x_n \to x^+,\ f(x_n) \to y\},$$

and define the *left cluster set* $C^- f(x)$ similarly. Define $L^\pm f(x) := \sup C^\pm f(x)$ and $L_\pm f(x) := \inf C^\pm f(x)$.

Theorem 7 (Marcus-Mizel [44]). *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function and for almost every $c \in \mathbb{R}$ the section $f_c(x) := f(x, c)$ satisfies*

$$\min(L^+ f_c(x), L^- f_c(x)) \leq f(x, c) \leq \max(L_+ f_c(x), L_- f_c(x)),$$

for all $x \in \mathbb{R}$, then the set of points Ω_x where the partial derivative $\partial_x f$ in the x-axis' direction exists is Lebesgue measurable, and $\partial_x f$ is Lebesgue measurable on Ω_x.

2.1.5 Directional derivatives, λ_3-a.e. on $\mathbb{R}^2 \times [0, 2\pi)$

In 1937, Augustus John Ward (1911-1984) obtained the following theorem on measurability of the $(x, \vartheta) \mapsto \partial^\vartheta f(x)$ directional Dini derivatives:

Theorem 8 (Ward, [80]). *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Borel measurable function, then the function $(x, \vartheta) \mapsto \partial^\vartheta f(x)$ is also Borel measurable.*

From this, he shows that the two-variable case for Borel measurable functions can be reduced to the one variable Denjoy-Young-Saks theorem:

Theorem 9 (Ward, [80]). *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Borel measurable function, then f has the directional Denjoy property at the point x on the line $l(x, \vartheta)$ for λ_3-a.e. (x, ϑ).*

Remark 2. *Actually more is obtained from his results on approximate derivatives; at λ_2-a.e. point the directional derivatives are not independent of each other: there exists a derivate plane, such that λ_1-a.e. directional derivative belongs to this plane.*

However if f is only Lebesgue measurable, Ward remarks that he cannot prove Lebesgue measurability of the directional Dini derivatives. Luckily, since Roy Osborne Davies in [21] shows that this need not be the case. In fact, he constructs a function using transfinite methods, whose $(x, \vartheta) \mapsto \partial^\vartheta f(x)$ Dini derivatives are not Lebesgue measurable. He remarks that his construction can be modified in such a way that one obtains a Lebesgue measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ with the following property. At every point of a subset $H \subseteq \mathbb{R}^2$ of positive measure, the set of directions in which the lower directional Dini derivatives have values a and c and the upper directional Dini derivatives have values b and d has full outer measure for every $a \leq 0 \leq b$, $c \leq 0 \leq d$.

2.1.6 Directional derivatives, at λ_2-a.e. point, in all directions

The question whether a certain stronger form of Ward's Theorem 9 holds, was answered negatively by Abram Samuel Besicovitch (1891-1970) in a remarkable construction in his paper [8]. By this stronger form we mean the following question:

> Is it true that an $f : E \to \mathbb{R}$ function of the appropriate class (e.g. continuous) defined on an appropriate set E, at λ_2- a.e. point of the set E has the directional Denjoy property in all directions?

In [8], Besicovitch constructed a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and a set $G \subseteq \mathbb{R}^2$ of positive measure such that for each point of G there exists a direction (even $\mathfrak{c}$ many directions), in which the function does not have the directional Denjoy property. In these directions the function has one upper derivative that is $+\infty$ on one side, and all three other derivate numbers are 0. The construction can be sketched as follows:

Besicovitch constructs pairwise disjoint sets $\mathcal{D}_n \subseteq [0,1]^2$, each a union of finitely many disks, with the following properties. At each point x of a set G of positive measure, there exist $\mathfrak{c}$ many lines through x such that each line on one side of x locally intersects only finitely many disks, and on the other side of x it intersects one from each system $\mathcal{D}_n$ for $n \geq N$ for a certain N. Then by taking $\cup_n \mathcal{D}_n$ as the support of the function f, and placing right cones of appropriate height on each disk of $\mathcal{D}_n$ (so that f is continuous), his result follows.

For a function of one variable the Denjoy-Young-Saks Theorem shows it is impossible for two derivate numbers be finite and distinct on a set of positive measure. Besicovitch's example does not answer the seemingly natural question whether anything can be said in this sense for two variable functions, for λ_2-a.e. point of E, in all directions. In [40] we answer this question in the negative by giving a continuous function such that for each point of a set of positive measure there exist $\mathfrak{c}$ many directions in which all derivate numbers are finite and distinct.

2.1.7 Directed derivatives

In order to avoid the previously encountered problems posed by non- measurability, Haslam-Jones in [28] introduced the notion of *directed derivatives*. This notion is slightly less natural, however stronger types of relations hold for these derivatives as Haslam-Jones had already shown by obtaining positive results concerning the directed derivatives for λ_2-a.e. points in all directions.

The definition is the following. For $\vartheta \in [0, 2\pi)$ denote by e_ϑ the unit vector at an angle ϑ with the $\{(x,0) : x \geq 0\}$ half-line. Denote by $S_\alpha(\rho, \eta)$ the open angular sector originating from 0, with radius ρ, direction α and central angle

2η:

$$S_\alpha(\rho,\eta) := \{re_\vartheta : \vartheta \in [\alpha-\eta,\alpha+\eta], r \in [0,\rho]\}.$$

When α, ρ and η are of no importance, we simply write $S\angle$. Putting

$$B^\alpha(\rho,\eta,x) := \sup_{r\in S_\alpha(\rho,\eta)} \frac{f(x+r)-f(x)}{|r|},$$

the *directed upper derivative* is defined as

$$D^\alpha f(x) := \lim_{\eta\to 0}\lim_{\rho\to 0} B^\alpha(\rho,\eta,x).$$

The *directed lower derivative* is defined similarly and is denoted $D_\alpha f$. Haslam-Jones also gives the following definitions:

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to have an *upper differential* at the point x if there exists a $d^+f(x) \in \mathbb{R}^2$ such that for every α and η:

$$\limsup_{\rho\to 0,\, r\in S_\alpha(\rho,\eta)} \frac{f(x+r)-f(x)-\langle d^+f(x), r\rangle}{|r|} = 0.$$

The *lower differential* is defined analogously. The following definition of Saks is equivalent:

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to have an *upper differential* at the point x_0 if there exists a $d^+f(x_0) \in \mathbb{R}^2$ such that

$$\limsup_{r\to 0} \frac{f(x_0+r)-f(x_0)-\langle d^+f(x_0), r\rangle}{|r|} = 0,$$

and the contingent of the graph of f at the point $(x_0, f(x_0)) \in \mathbb{R}^3$ contains the plane

$$\left\{(x,z) \in \mathbb{R}^2 \times \mathbb{R} : \langle d^+f(x_0), x-x_0\rangle = z - f(x_0)\right\}.$$

For definition of contingents see the following Subsection 2.1.8. We define analogously the *lower differential*. If both exist at x, then they are necessarily equal, and the function is totally differentiable at the point x.

Concerning the measurability of directed derivatives, Haslam-Jones in [28] obtained that two variable Lebesgue measurable functions have Lebesgue measurable directed $D^\vartheta f$ Dini derivatives in every fixed direction ϑ. This does not hold for non-measurable functions: if M is a set, such that both M and its complement M^c are of full outer measure, then the directed Dini derivatives of the characteristic function of M are not measurable for any fixed direction.

It was already proven by G. C. Young in [81] that if f is an arbitrary function of one variable , then except for a countable set of points

$$D^+f(x) \geq D_-f(x) \qquad \text{and} \qquad D^-f(x) \geq D_+f(x).$$

Ward in [78] by examining the structure of plane sets and using directed derivatives, obtained the following two dimensional analogue of Young's theorem.

Theorem 10 (Ward, [78])**.** *For an arbitrary function $f : E \to \mathbb{R}$, defined on an arbitrary set $E \subseteq \mathbb{R}^2$, then at every point except a countable set of points there exists a direction α, such that*

$$D^\alpha f + D^{\alpha+\pi} \geq 0.$$

However, the set of points at which there exists an α such that $D^\alpha f + D^{\alpha+\pi} f < 0$ need not be countable as the following example shows. Let $f(x, y) = -|x|$. Then at each point of the y-axis and in the direction, $\alpha = (1, 0)$ (the direction of the positive x-axis) $D^\alpha f + D^{\alpha+\pi} f < 0$. However, the fact that this set is always of measure 0 follows from the following theorems.

Haslam-Jones [28] obtained what can be considered as the analogue of the Denjoy-Young-Saks theorem for directed derivatives of measurable functions. Subsequently Saks, in [66] removed some measurability conditions and finally, Ward extended the result to arbitrary functions in [79].

Theorem 11 (Haslam-Jones–Ward, [28], [79])**.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary function, then at λ_2-a.e. point x one of the following relations is satisfied.*

- *The function, f is totally differentiable at x,*
- *There exists an upper [lower] derivate plane at x, and $D_\alpha f(x) = -\infty$ [$D^\alpha f(x) = +\infty$] in all directions α at the point x,*
- *$D^\alpha f(x) = \infty$ and $D_\alpha f(x) = -\infty$ in all directions α at x.*

To understand Sak's version we need some additional notation. For an arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}$ define

$$H := \left\{ x \in \mathbb{R}^2 : \exists S\angle : \limsup_{S \ni h \to 0} \frac{f(x+h) - f(x)}{|h|} < \infty \right\}$$

and

$$K := \left\{ x \in \mathbb{R}^2 : \exists S\angle : \liminf_{S \ni h \to 0} \frac{f(x+h) - f(x)}{|h|} > -\infty \right\}.$$

Saks' theorem published in the second edition of [66] p. 311, slightly extends Theorem 11 above.

Theorem 12 (Saks, [66]). *For an arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}$ the following conditions hold.*

- *f is totally differentiable at λ_2-a.e. point of $H \cap K$,*
- *At λ_2-a.e. point of H [K] there exists an upper [lower] differential,*
- *The set*
$$M := \left\{ x \in \mathbb{R}^2 : \exists S\angle : \lim_{S \ni h \to 0} \frac{|f(x+h) - f(x)|}{|h|} = \infty \right\}$$
has measure zero.

Note that by definition, at every point $x \in H^c \cap K^c$ the directed derivatives of Haslam-Jones, $D^\alpha f(x)$ and $D_\alpha f(x)$ are infinite, with the appropriate sign in every direction α. Consequently, this case corresponds to the third case of Haslam-Jones' theorem. The proof of Saks' theorem relies on the Contingents Theorem in higher dimensions.

Remark 3. *A recent result of Luděk Zajíček [87] generalizes the theorem of Haslam-Jones and Ward to separable Banach spaces under certain conditions, with the exceptional sets being Γ-null sets.*

2.1.8 Contingent of sets

The first definition of the contingent of a set is apparently due to Georges Bouligand (1889-1979) in [11]. A further examination of the contingent can be found in the thesis of Jean Mirguet [46] and also in [47]. Besicovitch in [7] obtains part of the Contingents Theorem; subsequently, Andrei Nikolayevich Kolmogorov (1903-1987) and Ivan Yakovlevich Verčenko in [35], [36] obtain the more complete form of the theorem for plane sets. Both Saks [64] and Fréderic Roger, a student of Maurice René Fréchet (1878-1973) and Denjoy rediscover the theorem, with Roger generalizing it to higher dimensions. This was then presented by Émile Borel (1871-1956) at the French Academy of Sciences in [57], [59], [58] and [61]. Roger called Saks' attention to his results, who stated and proved Roger's theorem in the supplement of [64]. In a subsequent paper [60], Roger remarks that this supplement states only his "simplest results", and that the article "stays silent" on his less simple, and more important results.[4] In response, Saks gives a simple proof of Roger's theorem in [65].

[4] *"Aussi n'est-il pas étonnant qu'elles conduisent, comme le fait remarquer l'Auteur dans un supplément á son article, aux plus simples de mes résultats de l'espace. Cependant la tournure plus géométrique des méthodes que j'ai employées permet peut-être une extensions plus facile, notamment aux résultats d'oú je vais tirer un critère d'analyticité et sur lesquels la remarque de M. Saks reste muette."*[60]

We state the definitions and the Contingents Theorem for the case $\mathbb{R}^3$.

Define the *contingent* of a set $E \subseteq \mathbb{R}^3$ at a point x as the set $C(x)$ of all half-lines issuing from x, with the property that for each $l \in C(x)$ there exist l_n half-lines issuing from x, converging to l such that there exist $x_n \in l_n \cap E$ such that $x_n \to x$. Here, by convergence of half-lines issuing from the same point we mean the convergence of the directions of these half-lines. We say that the contingent of the set E at the point x is a half-space if $\cup C(x)$ is congruent to the half-space $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$. We say that the contingent of the set E at x is the whole space if $\cup C(x)$ is $\mathbb{R}^3$, and we say that the contingent of the set E at x is a plane, if $\cup C(x)$ is an affine plane. In the following, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.

Theorem 13 (Contingents Theorem)**.** *Any set $E \subseteq \mathbb{R}^3$ can be decomposed into two sets P and Q such that*

- *at each point of P the contingent of E is the whole space,*
- *at $\mathcal{H}^2$-a.e. point of Q the contingent is either a plane or a half-space*
- *Q is $\mathcal{H}^2$ σ-finite.*

A proof of the theorem can be found in [66], p. 307.

Remark 4. *Ward's theorem states that if $f : \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable, then f has the Denjoy property in λ_1-a.e. direction at λ_2-a.e. point. However, it is not evident whether the domain can be partitioned to sets of positive measure, such that in λ_1-a.e. direction at λ_2-a.e. point of these sets the Denjoy-behaviour of f is the same in that direction. Gholam-Hossein Mossaheb (1910-1979) answered this question in the negative, [48], by constructing a continuous function, such that at every point of a set of positive measure there are sets of directions of positive measure, on which the Denjoy-behaviour of the function is different.*

For one variable functions, Jerome Raymond Ravetz in [53] examined the Hausdorff dimension of the exceptional set where the Denjoy relations do not hold. He constructed a continuous function $f : \mathbb{R} \to \mathbb{R}$, such that there exists a set $H \subseteq \mathbb{R}$ of Hausdorff dimension one, on which three of the Dini derivatives are 0, and one is $+\infty$.

2.2 Approximate derivatives.

2.2.1 Definitions (One variable functions)

In the same year, Denjoy [23] and Aleksandr Yakovlevich Khintchine (1894-1959) [31] introduce the notion of approximate derivative, which Khintchine's referred to as an "asymptotic" derivative. These derivatives are defined by ignoring a set of outer density zero around each point.

A point $x \in E$ is called a *point of dispersion* of the set $E \subseteq \mathbb{R}$, if E has outer density 0 at the point x. The set of all dispersion points of E is denoted $\mathrm{dsp}(E)$ and $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$. The *approximate limits* of an arbitrary function $f : E \to \mathbb{R}$ defined on a set $E \subseteq \mathbb{R}$ are defines as follows.

$$\begin{aligned}
A^+ \lim_{y\to x} f(y) &:= \inf \left\{K \in \mathbb{R}^* : x \in \mathrm{dsp}(\{y \in E : y > x \text{ and } f(y) > K\})\right\}, \\
A_+ \lim_{y\to x} f(y) &:= \sup \left\{K \in \mathbb{R}^* : x \in \mathrm{dsp}(\{y \in E : y > x \text{ and } f(y) < K\})\right\}, \\
A^- \lim_{y\to x} f(y) &:= \inf \left\{K \in \mathbb{R}^* : x \in \mathrm{dsp}(\{y \in E : y < x \text{ and } f(y) > K\})\right\}, \\
A_- \lim_{y\to x} f(y) &:= \sup \left\{K \in \mathbb{R}^* : x \in \mathrm{dsp}(\{y \in E : y < x \text{ and } f(y) < K\})\right\}.
\end{aligned}$$

Defining derivatives using approximate limits one obtains the *approximate Dini derivatives* of f.

$$\begin{aligned}
AD^+ f(x) &:= A^+ \lim_{y\to x} \frac{f(y) - f(x)}{y - x}, \\
AD_+ f(x) &:= A_+ \lim_{y\to x} \frac{f(y) - f(x)}{y - x}, \\
AD^- f(x) &:= A^- \lim_{y\to x} \frac{f(y) - f(x)}{y - x}, \\
AD_- f(x) &:= A_- \lim_{y\to x} \frac{f(y) - f(x)}{y - x}.
\end{aligned}$$

2.2.2 Definitions (Two variable functions)

If $E \subseteq \mathbb{R}^2$ and $f : E \to \mathbb{R}$, we can define *directional approximate Dini derivatives* using approximate limits in the same way that the classical directional Dini derivatives are defined using normal limits. Denote

$$E^+(x, \vartheta, K) := \{y \in E \cap l(x, \vartheta) : f(y) > K\},$$

and

$$E^-(x, \vartheta, K) := \{y \in E \cap l(x, \vartheta) : f(y) < K\},$$

where $l(x, \vartheta)$ denotes the half-line extending from the point x in the direction ϑ. We first define the *directional* (or linear) *approximate limits* of such a function in the direction $0 \le \vartheta < 2\pi$. If E is a linear set, lin-dsp denotes the linear dispersion points of E.

$$A^{\vartheta} \lim_{y \to x} f(y) := \sup \left\{ K \in \mathbb{R}^* : x \in \text{lin-dsp}(E^+(x, \vartheta, K)) \right\},$$
$$A_{\vartheta} \lim_{y \to x} f(y) := \inf \left\{ K \in \mathbb{R}^* : x \in \text{lin-dsp}(E^-(x, \vartheta, K)) \right\}$$

Using directional approximate limits, the definition of *directional approximate Dini derivatives* follows:

$$A\partial^{\vartheta} f(x) := A^{\vartheta} \lim_{y \to x} \frac{f(y) - f(x)}{y - x},$$

$$A\partial_{\vartheta} f(x) := A_{\vartheta} \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Approximate directed derivatives $AD^{\vartheta} f(x)$ and $AD_{\vartheta} f(x)$ can also be defined similarly by first definining approximate directed limits and then the approximate directed derivatives using angular sectors. We omit the details.

2.2.3 Measurability (one and two variable functions)

The distinction of the one and several variable case for questions of measurability of approximate derivatives is not necessary here, as the following result shows.

Theorem 14 (Khintchine-Saks, [32], [66]). *Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function defined on a Lebesgue measurable set $E \subseteq \mathbb{R}^2$. For any fixed direction ϑ, the approximate partial Dini derivatives $x \mapsto A\partial^{\vartheta} f(x)$ and $x \mapsto A\partial_{\vartheta} f(x)$ are also Lebesgue measurable.*

The result for one variable functions was obtained by Khintchine [32]. Besicovitch [9] obtained the theorem for continuous functions of one variable, which result has been extended to Lebesgue measurable functions by Burkill and Haslam-Jones [16] independently of Khintchine's result. The first mention of the multivariable case stating measurability of all sections can be found in Saks [66], p. 299, Theorem 11.2. The following theorem of Ward [80] examines the measurability of $(x, \vartheta) \mapsto A\partial_{\vartheta} f(x)$.

Theorem 15 (Ward, [80]). *Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function defined on a Lebesgue measurable set $E \subseteq \mathbb{R}^2$. Then f has Lebesgue measurable approximate directional Dini derivatives, $(x, \vartheta) \mapsto A\partial^{\vartheta} f(x)$ and $(x, \vartheta) \mapsto A\partial_{\vartheta} f(x)$.*

Remark 5. *Further results on measurability of approximate ordinary derivatives (as opposed to approximate Dini derivates) of one variable functions can be found in [25] and [37]. In [25] it is shown that if f is an approximately differentiable function on an interval, then ADf is of Baire class one. In [37] it is shown that for any $f : \mathbb{R} \to \mathbb{R}$, if $Q \subseteq \mathbb{R}$ denotes the points where f is approximately differentiable and all points of Q are points of outer density, then ADf is of Baire class two with respect to Q.*

2.2.4 One variable functions

Denjoy in [23], pp. 208–209 and Khintchine in [32] and [33] p. 212 independently discover the following theorem. Besicovitch also discovers this same theorem, but only for continuous functions [9].

Theorem 16 (Denjoy-Khintchine [23], [32])**.** *Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function defined on a set $E \subseteq \mathbb{R}$; then one of the following two Denjoy-properties for approximate derivatives holds λ_1-a.e.:*

- *f is approximately differentiable.*
- *all four approximate derivate numbers of f are $\pm\infty$ with the appropriate signs.*

Saks remarks in *Theory of the Integral* [66] that the Denjoy-Khintchine Theorem can be extended to arbitrary functions by a "slight modification" of the proof; see page 297, discussion following Theorem (10.1) in Chapter IX. However, Shu-Er Chow in [19] constructs a simple example, showing that the Denjoy-Khintchine Theorem stated above does not generalize to arbitrary functions.

Example 2. *Take $(0,1) = I \cup J$ where I and J are disjoint and non-measurable sets of outer measure 1. Then by taking $f(x) := \chi_I(x)$ to be the characteristic function of I, we obtain that at each point x of I,*

$$AD^+f(x) = AD_-f(x) = 0 \text{ and } AD_+f(x) = -AD^-f = -\infty.$$

But on J:

$$AD^+f(x) = -AD_-f(x) = +\infty \text{ and } AD_+f(x) = AD^-f(x) = 0.$$

This contradicts the conclusion of an arbitrary functions version of the Denjoy-Khintchine Theorem.

However, upon closer inspection Saks states that the "slight modification" should be made in such a way that the definition of approximate differentiability should be modified for arbitrary functions. According to the new definition, the point x is said to be an point of *approximate differentiability in the modified sense*, if there exists a set, of which x is a point of outer density, and on which the function is differentiable. If interpreted in this light, Chow's counterexample fails since f is approximately differentiable in the modified sense at all points by taking I as this set for points $x \in I$, and by taking J for points $x \in J$.

We now return to the discussion of the original definition. Burkill and Haslam-Jones in a 1931 paper, [16] knowing of Besicovitch's result, first extend that result from continuous functions to Lebesgue measurable functions. Then, in their subsequent 1933 paper, [18], p. 238, Theorem 10, they obtain a result for non-measurable functions. Ward in the same year obtains a partial result in [77], and completes the result of Burkill and Haslam-Jones in [79] p. 344, Theorem II. All of these extensions are obtained using the notion of λ-*approximate derivates*; that is instead of sets of density 0, sets of density λ are ignored. The following theorem is not stated in terms of λ-approximate derivates, but it follows directly from their result.

Theorem 17 (Burkill–Haslam-Jones–Ward [18], [79])**.** *If $f : E \to \mathbb{R}$ is an arbitrary function defined on an arbitrary set $E \subseteq \mathbb{R}$, then λ_1-a.e. one of the following Denjoy relations holds for approximate derivates:*

- *f is approximately differentiable.*
- *One pair of f's opposite approximate derivatives*

  $$(AD^+f, AD_-f) \text{ or } (AD_+f, AD^-f)$$

  are finite and equal, the other pair opposite derivatives

  $$(AD_+f, AD^-f) \text{ or } (AD^+f, AD_-f)$$

  are infinite with the appropriate signs.
- *All four approximate derivate numbers of f are $\pm\infty$ with the appropriate signs.*

Ralph Lent Jeffery (1889-1975) in [29], (cf. also [30] pp. 198-199) also obtained the previous result for arbitrary functions, however his results are based on the notion of metric separability[5] and the paper has been criticized for the unusual definition of the approximate derivatives, cf. [63]. Chow in his paper [19] also obtained the same result, using the metrical upper and lower boundaries $u(x), l(x)$ originally introduced by Henry Blumberg (1886-1950) in [10].

Remark 6. *Alberti-Csörnyei-Laczkovich-Preiss in [3] examined the valid relations that hold for approximate derivates at $\mathcal{H}^1$-a.e. point of the graph of an arbitrary real function.*

2.2.5 Approximate directional derivatives, λ_3-a.e. on $\mathbb{R}^2 \times [0, 2\pi)$

Stepanoff in [75] obtains the following theorem (cf. also Saks [66] p. 300):

Theorem 18 (Stepanoff, [75])**.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function, then it is approximately differentiable a.e. if and only if f possesses approximate partial derivatives a.e..*

Here by approximate partial derivatives we mean the approximate version of the usual partial derivatives, in the directions of the x and y axes.

Saks in [66], p. 312 also obtained further conditions on approximate differentiability of two variable Lebesgue measurable functions. Ward in [80] extended the Saks result:

Theorem 19 (Saks-Ward, [66], [80])**.** *If $f : E \to \mathbb{R}$ is a Lebesgue measurable function defined on a Lebesgue measurable set $E \subseteq \mathbb{R}^2$, then for λ_2-a.e. x in E, either*

- *In λ_1-a.e. ϑ direction $A\partial^\vartheta f(x) = \infty$, and there exists no approximate derivative plane, or*
- *There exists an approximate derivative plane $(a(x)\cos\vartheta + b(x)\sin\vartheta)$ and in λ_1-a.e. ϑ direction:*

$$A\partial^\vartheta f(x) = A\partial_\vartheta f(x) = a\cos\vartheta + b\sin\vartheta.$$

Andrew Michael Bruckner and Melvin Rosenfeld in [14] obtain that if a Lebesgue measurable function on the plane has approximate partial derivatives almost everywhere, then it has approximate directional derivatives at

[5] Two sets of finite outer measure are said to be *metrically separable* if for every $\varepsilon > 0$ there exist neighborhoods of the sets, such that the intersection of these neighborhoods has measure less than ε. Or, the same: if the sum of their outer measures is the outer measure of their union. A function $f : E \to \mathbb{R}$ is said to be metrically separable if for every c the sets $E(f < c)$ and $E(f \geq c)$ are metrically separable.

λ_2-a.e. point, in λ_1-a.e. direction. Note that this result also follows from the previous two theorems.

Since the approximate Denjoy-relations hold for arbitrary one variable functions a.e., the question arises whether anything could be said about the approximate directional behaviour of an arbitrary two variable function a.e.. By a transfinite construction we obtain that the answer is negative.

Theorem 20. *There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$, such that at each point x of a set of positive outer measure M, and for each ϑ of a set of positive outer measure $\Theta \subseteq [0, \pi]$, f does not have the linear approximate Denjoy property at x in the direction ϑ.*

For the proof, see Appendix A.2.

2.2.6 Approximate directional derivatives, at λ_2-a.e. point, in all directions

We may ask a question analogous to the one in Paragraph 2.1.6, this time only for approximate derivatives. The answer is again negative:

Theorem 21. *There exists a continuous function f, such that at each point of a set of positive measure in $\mathfrak{c}$ many directions ϑ, f has finite and different one-sided approximate derivatives:*

$$\pm\infty \neq A\partial^{\vartheta} f = A\partial_{\vartheta} f \neq -A\partial^{\vartheta+\pi} f = -A\partial_{\vartheta+\pi} f \neq \pm\infty.$$

For the proof, see Appendix A.3.

2.2.7 Approximate directed derivatives

The approximate version of the directed derivatives AD^{ϑ} introduced by Haslam-Jones have been examined by Ward in [79], who obtained the following theorem:

Theorem 22 (Ward [79])**.** *If $f : E \to \mathbb{R}$ is an arbitrary real function defined on an arbitrary set $E \subseteq \mathbb{R}^2$, then for λ_2-a.e. point $x \in E$ we have either*

- $AD^{\vartheta} f(x) = \infty$ *for all* $\vartheta \in [0, 2\pi]$*, or*
- *there exists an upper approximate derivate plane at x.*

Note that this theorem is of the utmost generality; i.e. the strongest type of statement holds for arbitrary functions.

Remark 7. *Fedor Isaakovich Shmidov using notions of approximate contingents also examined the problem of approximate differentiability for two variable functions in several papers, cf. [68], [69], [70], [72].*

3 Category.

3.1 Classical derivatives.

3.1.1 One variable real functions

The relation between the Dini derivatives from the category point of view was already examined by William Henry Young (1863-1942) in [83] where he proved that for continuous functions $D^+f = D^-f$ and $D_+f = D_-f$, except on a set of first category (cf. also Christoph J. Neugebauer (1927-2012) [50]).

Ludek Zajíček [85] and simultaneously Belna-Cargo-Evans-Humke [6] found the following theorem, which may be regarded as the category version of the Denjoy-Young-Saks theorem (cf. also [76] p. 176):

Theorem 23 (Zajíček, [85])**.** *For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ one of the following relations must hold, except on a first category set:*

- *The upper derivatives are equal ($D^+f = D^-f$) and the lower derivatives are equal ($D_+f = D_-f$);*
- *The opposite derivatives are infinite with the appropriate sign $D_-f = -\infty$, $D^+f = \infty$ [$D_+f = -\infty$, $D^-f = \infty$] and $D_+f \leq D^-f$ [$D_-f \leq D^+f$].*

If a function satisfies one of the previous relations at the point x, we say that it has the Zajíček property *at the point x.*

In fact it is shown that the exceptional set is σ-porous.

3.1.2 Directional derivatives at a generic point, in a generic direction

If a two variable function f restricted to the line $l(x, \vartheta)$ satisfies the Zajíček property at the point x, we say that ϑ is a *Zajíček direction* of f at the point x. Adapting Ward's proof of Theorem 8, [80] almost verbatim, replacing the measure theoretic notions by their category equivalents we obtain the following theorem.

Theorem 24. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function (Baire measurable) then for a generic (x, ϑ) where ϑ is a* Zajíček-direction *at x, one of the following must hold on a residual set $Q \subseteq \mathbb{R}^2 \times [0, 2\pi)$.*

- *The upper directional derivatives are equal and the lower directional derivatives are equal.*
- *$\partial_{\vartheta+\pi} f = -\infty$, $\partial^{\vartheta} f = \infty$ and $\partial_{\vartheta} f \le \partial^{\vartheta+\pi} f$.*

For several variables, Ravetz in two articles [54], [55] examines from the category point of view the properties of directed derivatives introduced by Haslam-Jones. He calls a point a *tangential singularity* when there exists a direction μ, in which one of the following holds:

- $\vartheta \mapsto D^{\vartheta} f(x)$ or $\vartheta \mapsto D_{\vartheta} f(x)$ is not continuous in $\vartheta = \mu$.
- $D^{\mu} f(x) > \partial^{\mu} f(x)$ or $D_{\mu} f(x) < \partial_{\mu} f(x)$.
- $D^{\mu} f(x) = \infty$ or $D_{\mu} f(x) = -\infty$.

Theorem 25 (Ravetz [54])**.** *For a continuous function $f : E \to \mathbb{R}$ on an arbitrary set $E \subseteq \mathbb{R}^2$, there exists a set $H \subseteq E$, residual in E and which is the disjoint union of two relative open sets U and V, such that*

- *neither point of V is a tangential singularity (at these points the directed derivatives $\vartheta \mapsto D^{\vartheta} f$ and $\vartheta \mapsto D_{\vartheta} f$ are continuous, finite and equal to the directional (linear) derivatives), and*
- *at every point x of U there is a direction μ such that in every direction $\vartheta \in [\mu, \mu + \pi]$:*
$$D^{\vartheta} f(x) = -D_{\vartheta+\pi} f(x) = \infty.$$

From this theorem he obtains an analogue of W. H. Young's theorem stated at the beginning of this section:

Theorem 26 (Ravetz [54])**.** *If $f : E \to \mathbb{R}$ is a continuous function defined on an arbitrary set $E \subseteq \mathbb{R}^2$ and ϑ_0 is a fixed direction, then the set of points x where $\partial^{\vartheta_0} f(x) \ne \partial_{\vartheta_0+\pi} f(x)$ is a set of first category.*

3.1.3 Directional derivatives at a generic point, in all directions

A recent result of Zajíček [86] shows that there exists a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$, such that for all $\vartheta_0 \in [0, 2\pi)$, the sections of f on a.e. line of direction ϑ_0 are C^{∞}, but at each point of a residual set $U \subseteq \mathbb{R}^2$, there exists a direction in which f is nondifferentiable.

3.2 Approximate derivatives.

For approximate derivatives of one variable functions, Zajíček [84] obtained a result similar to the category version of the Denjoy-Young-Saks Theorem.. This was later strengthened in a joint paper with David Preiss in [51] to the following.

Theorem 27 (Preiss-Zajíček [51]). *For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ for all x except a set of first category at least one of the following must hold:*

- *Both upper and lower approximate derivatives are equal*

$$AD^+ f(x) = AD^- f(x) \text{ and } AD_+ f(x) = AD_- f(x).$$

- *Two opposite derivatives are infinite with the appropriate sign. That is, at least one of the relations*

$$AD^+ f(x) = -AD_- f(x) = \infty \text{ or } AD^- f(x) = -AD_+ f(x) = \infty$$

  holds.

In their paper it is also shown that for any given four extended real numbers satisfying any of the theorem's relations, there exists a function such that the relation is satisfied on a residual subset of the line. In this sense, the theorem is the strongest possible.

Remark 8. *It should be mentioned that Michael Jon Evans and Lee Larson in [24] examined what is in some sense the category analogue of approximate derivatives, i.e. instead of ignoring a set of outer density zero around each point, a set of first category is ignored. These are also known as qualitative derivatives, and are originally due to Solomon Marcus. Evans and Larson obtain a Denjoy type theorem for qualitative derivatives of arbitrary functions that hold on a residual set, similar to the theorem of Zajíček.*

A Proof of theorems.

A.1 Proof of Theorem 5.

For the sake of completeness, we repeat the theorem here:

Theorem. *Let $f : E \to \mathbb{R}$ be an arbitrary function defined on an arbitrary set $E \subseteq \mathbb{R}^2$. Define*

$$M := \{x \in E : L_f(x) < \infty\}.$$

Then the function is differentiable a.e. on M. Moreover M is relative F_σ, and $\partial^{\vartheta_0} f$ are relative Lebesgue measurable functions on M for every $\vartheta_0 \in [0, 2\pi)$.

PROOF. For each n, take the points x in which f restricted to the $1/n$ neighbourhood of x is Lipschitz with constant n:

$$M_n := \left\{ x \in E : |f(x+h) - f(x)| \le n|h|, \quad \forall |h| < \frac{1}{n},\, x+h \in E \right\}.$$

Since $L_f(x) = K < \infty$ means that for a small enough neighborhood U_x of x, at every $y \in U_x$ $|f(y) - f(x)| \le K|y - x|$ holds, it follows that $M = \cup_n M_n$.

The sets M_n are relative closed. It is enough to show that if $x_k \in M_n$, and $x_k \to x$, then $x \in M_n$. Take a y, such that $|y - x| < 1/n$. We want to show that

$$|f(y) - f(x)| \le n|y - x|.$$

By taking an x_k close enough to x, one has $|x_k - y| < 1/n$ for all $k > N$ for a certain N. Since $x_k \in M_n$, it follows that $|f(x_k) - f(y)| \le n|x_k - y|$. Using the triangle inequality:

$$|f(y) - f(x)| \le |f(y) - f(x_k)| + |f(x_k) - f(x)| \le n|x_k - y| + n|x_k - x|,$$

and since $|x_k - x| \to 0$ and $|x_k - y| \to |x - y|$, it follows that $x \in M_n$.

The function f restricted to M_n is continuous for each n, since it is locally Lipschitz on M_n.

In the following we show that the function is differentiable a.e. on M. Divide M_n into $M_{n,i}$, each of diameter $1/n$. It is enough to prove that f is differentiable a.e. on $M_{n,i}$ for each i and n. The function $f|_{M_{n,i}}$ is (globally) Lipschitz on $M_{n,i}$ with constant n. By known results (see Mojżesz David Kirszbraun (1904-1942) [34]) $f|_{M_{n,i}}$ can be extended to $\mathbb{R}^2$ with the same Lipschitz constant; denote this function $\tilde{f}$.

By Rademacher's theorem $\tilde{f}$ is a.e. differentiable. Therefore it is enough to show that f is differentiable at those density points of $M_{n,i}$, where $\tilde{f}$ is differentiable. Let $a \in M_{n,i}$ be such a point and let $d = (\tilde{f})'(a)$. Then

$$\lim_{x \to a} \frac{\tilde{f}(x) - \tilde{f}(a) - \langle d, x - a \rangle}{|x - a|} = 0. \tag{1}$$

Our aim is to show that

$$\lim_{x \to a,\, x \in E} \frac{f(x) - f(a) - \langle d, x - a \rangle}{|x - a|} = 0. \tag{2}$$

Let $\varepsilon > 0$ be given. Since a is a density point of $M_{n,i}$, there is a $0 < \delta_1 < 1/(2n)$ such that

$$\lambda_2(M_{n,i} \cap B(a,h)) > (1 - \varepsilon^2/4)\lambda_2(B(a,h))$$

for every $0 < h < \delta_1$. It is easy to check that if $x \in B(a, \delta_1/2),\ x \neq a$ and $|x - a| = r$, then $B(x, \varepsilon r) \cap M_{n,i} \neq \emptyset$.

By (1), there is a $\delta_2 > 0$ such that

$$|\tilde{f}(x) - \tilde{f}(a) - \langle d, x - a\rangle| \leq \varepsilon|x - a|$$

for every $x \in B(a, \delta_2)$. Let $\delta = \min(\delta_1, \delta_2)/2$.

Let $x \in B(a, \delta) \cap E,\ x \neq a$ be arbitrary. Choose an element $y \in B(x, \varepsilon r) \cap M_{n,i}$. Then $y \in M_n$ and $|y - x| < 1/n$, and thus

$$|f(y) - f(x)| \leq n \cdot |y - x| \leq n\varepsilon|x - a|.$$

Therefore,

$$\begin{aligned}
|f(x) - f(a) - \langle d, x - a\rangle| &= |f(x) - \tilde{f}(a) - \langle d, x - a\rangle| \leq \\
&\leq |f(y) - \tilde{f}(a) - \langle d, y - a\rangle| + |f(x) - f(y)| + |\langle d, y - x\rangle| \leq \\
&\leq |\tilde{f}(y) - \tilde{f}(a) - \langle d, y - a\rangle| + n\varepsilon|x - a| + |d| \cdot |y - x| \leq \\
&\leq \varepsilon|x - a| + n\varepsilon|x - a| + |d|\varepsilon \cdot |x - a| = \\
&= (1 + n + |d|)\varepsilon \cdot |x - a|.
\end{aligned}$$

This proves (2).

To prove measurability of the partial derivatives of f, it is enough to show that the partial derivatives of $\tilde{f}$ are measurable. For proving the measurability of the $\partial^0 \tilde{f}$ functions, define $g_k : \mathbb{R}^2 \to \mathbb{R}$ as

$$g_k(x, y) := \frac{\tilde{f}(x + 1/k, y) - \tilde{f}(x, y)}{1/k}$$

Since the $\tilde{f}$ functions are Lipschitz continuous and defined everywhere, it follows that the $g_k|_{M_n}$ functions are also Lipschitz continuous. Since the g_k functions are Lipschitz, they are differentiable a.e., thus $\partial^0 \tilde{f} = \lim_{k\to\infty} g_k$ a.e., thus the $\partial^0 \tilde{f}$ functions are measurable.

□

A.2 Proof of Theorem 20.

For convenience, we repeat the theorem.

Theorem. *There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$, such that at each point x of a set of positive outer measure M, for each ϑ of a set of positive outer measure $\Theta \subseteq [0, \pi]$, f does not have the linear approximate Denjoy property at x in the direction ϑ.*

PROOF. Let $\kappa = \text{non}\,\mathcal{N}$, the least cardinal of any set of positive (Lebesgue) outer measure. We construct sets $M \subseteq \mathbb{R}^2$ and $\Theta \subseteq [0, \pi)$ both of positive outer measure and of cardinality κ with the following additional property. Let Ψ denote the set of directions in the difference set of M:

$$\Psi = \{\vartheta \in [0, \pi) : \overline{ab} = \vartheta,\ a, b \in M\},$$

where $\overline{ab}$ denotes the direction of the line determined by the points a, b. The set Θ is constructed so that $\Theta \cap \Psi = \emptyset$.

First, suppose $\kappa < c$. Then let $M \subseteq \mathbb{R}^2$ be a set of positive outer measure of cardinality κ. Then Ψ also has cardinality $\kappa \cdot \kappa = \kappa$. Then either Ψ has positive outer measure, or Ψ has measure zero. If Ψ has measure zero, then let $H \subseteq [0, \pi]$ be a set of positive outer measure of cardinality κ. Setting $\Theta = H \backslash \Psi$ we obtain a set of positive outer measure in the complement of Ψ of cardinality κ.

If Ψ has positive outer measure, then $(\Psi + x) \cap \Psi = \emptyset$ for some $x \in [0, \pi]$ (mod π). Indeed, $(\Psi + x) \cap \Psi \neq \emptyset$ iff $x = \psi_1 - \psi_2$ for some $\psi_1, \psi_2 \in \Psi$, but the difference set $\Psi - \Psi$ has cardinality κ and there are $\mathfrak{c}$ many $x \in [0, \pi]$. Take such an x, and let $\Theta = \Psi + x \pmod{\pi}$.

Now suppose $\kappa = c$. We construct the sets M and Θ by transfinite induction. Let $\mathcal{F}_2$ denote the closed sets of $\mathbb{R}^2$ of positive measure and let $\mathcal{F}$ denote the closed sets of $[0, \pi]$ of positive measure. Take a well-ordering

$$\mathcal{F}_2 \times \mathcal{F} = (F_\alpha, C_\alpha)_{\alpha < c}.$$

Suppose $p_\beta \in F_\beta$ and $\vartheta_\beta \in C_\beta$ have been selected for $\beta < \alpha$ such that the directions determined by the p_β are distinct from ϑ_γ ($\gamma < \alpha$). Take two orthogonal lines l, $l^\perp$ of directions distinct from ϑ_β ($\beta < \alpha$). Since F_α is a closed set of positive measure, by Fubini's theorem, the orthogonal sections F_α^x have positive measure on points x of a set $H \subseteq l$ of positive outer measure. On the other hand, each such section contains less than $\mathfrak{c}$ many points of the lines $l(p_\beta, \vartheta_\gamma)$, $(\beta, \gamma < \alpha)$. So for such an x we can take a point

$$p_\alpha \in F_\alpha^x \setminus \left(\bigcup_{\beta,\gamma<\alpha} l(p_\beta, \vartheta_\gamma) \cap F_\alpha^x \right).$$

Therefore the directions $\overline{p_\beta p_\alpha}$ $(\beta < \alpha)$ are distinct from $(\vartheta_\gamma)_{\gamma<\alpha}$.

Take a direction ϑ_α in C_α, distinct from $(\vartheta_\gamma)_{\gamma<\alpha}$ and also distinct from the directions $\overline{p_\beta p_\gamma}$ $(\beta, \gamma < \alpha)$. This can be done, since if α is an infinite ordinal, these determine $\alpha + \alpha^2 = \alpha < c$ forbidden directions and C_α has cardinality $\mathfrak{c}$. If α is a finite ordinal, then there are less than $\omega < c$ forbidden directions. Set $M = (p_\alpha)_{\alpha<c}$ and $\Theta = (\vartheta_\alpha)_{\alpha<c}$. These sets have cardinality $\mathfrak{c}$, have positive outer measure and with the same definition of Ψ, $\Psi \cap \Theta = \emptyset$.

Set

$$\mathcal{L} = \{l(p, \vartheta) : p \in M, \vartheta \in \Theta\}.$$

By the construction of Θ, each line $l \in \mathcal{L}$ contains exactly one $p \in M$. Now for each such line $l(p, \vartheta)$ we will construct a set $E(p, \vartheta) \subseteq l(p, \vartheta)$, with the property that p is a point of linear density 1 of $E(p, \vartheta)$ in the direction ϑ and such that the sets $E(p, \vartheta)$ $(p \in M, \vartheta \in \Theta)$ are disjoint.

Take a well-ordering

$$\mathcal{L} = (l_\alpha)_{\alpha<\kappa}.$$

Suppose E_β has been defined for all $\beta < \alpha$. Set

$$E_\alpha = l_\alpha \setminus \bigcup_{\beta<\alpha} (l_\alpha \cap l_\beta).$$

If $l_\alpha = l(p, \vartheta)$, set $E(p, \vartheta) = E_\alpha$. Since $\alpha < \kappa$, the set $\{l_\alpha \cap l_\beta : \beta < \alpha\}$ has measure zero, so p is a point of density of $E(p, \vartheta)$. The sets E_α $(\alpha < \kappa)$ are disjoint. Indeed, suppose $\beta < \alpha$, then $E_\beta \subseteq l_\beta$, $E_\alpha \subseteq l_\alpha$, but if $l_\alpha \cap l_\beta = \{q\}$, then $q \notin E_\alpha$ by the definition of E_α.

The linear approximate Dini derivatives can be arbitrarily set for each $p \in M$ on each line $l(p, \vartheta)$ simultaneously, since the sets $E(p, \vartheta)$ are disjoint $(p \in M, \vartheta \in \Theta)$ and $l(p, \vartheta) \setminus E(p, \vartheta)$ has linear measure zero. Finally, define f arbitrarily on $\mathbb{R}^2 \setminus \bigcup \mathcal{L}$.

□

A.3 Proof of Theorem 21.

For convenience, we repeat the theorem:

Theorem. *There exists a continuous function f, such that at each point of a set of positive measure in $\mathfrak{c}$ many directions f has finite and different one-sided approximate derivatives:*

$$\pm\infty \neq A\partial^{\vartheta} f = A\partial_{\vartheta} f \neq A\partial^{\vartheta+\pi} f = A\partial_{\vartheta+\pi} f \neq \pm\infty.$$

Let $M \subseteq \mathbb{R}^2$ be a set. A line $l \subseteq \mathbb{R}^2$ is said to be a *line of accessibility* of M at $x \in M$, if $l \cap M = \{x\}$. We use the following result of Davies [20], p. 231:

Theorem 28 (Davies [20])**.** *For every $\varepsilon > 0$ there exists a set $M \subseteq [0,1]^2$ of measure greater than $1-\varepsilon$, such that to each point p of M there is associated a set of lines of accessibility $L(p)$ with the following properties:*

- *Each angle contains $\mathfrak{c}$ many lines of accessibility $l(p)$ from $L(p)$;*
- *To each p in M there is associated a subset $F(p) \subseteq \bigcup L(p)$, such that for each $l(p)$ in $L(p)$, if we define $E(p) = l(p) \cap F(p)$, then the point p is a linear density point of $E(p)$;*
- *There exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that if $p, p' \in M$ and $d(p,p') > \delta$, then $d(F(p), F(p')) > g(\delta)$, where $d(\cdot,\cdot)$ denotes the Euclidean distance of point sets.*

Remark 9. *The property of g can also be stated in the following way: if $d(q,q') < \varepsilon$ for some $q \in F(p)$, $q' \in F(p')$, then $d(p,p') < g^{-1}(\varepsilon) = \delta$. Notice that g can be supposed to be a nondecreasing function, and that $\lim_{t\to 0+} g(t) = 0$. In particular the sets $F(p)$ $(p \in M)$ are disjoint.*

Davies originally used this theorem for proving the following result: for any given continuous function f defined on a unit square, for any $\varepsilon > 0$, one can give a function g equal to f on a set H of measure greater than $1-\varepsilon$, with the property that through each point of H there pass $\mathfrak{c}$ many lines, each on which g is approximately constant.

However our aims are different, and based on this result we prove Theorem 21.

PROOF OF THEOREM 21. Take the set M of the previous theorem. Define f to be 0 on M. Since the sets $F(p)$ are disjoint, we can independently define f on the points q of $F(p)$ as the distance from p: $f(q) := d(p,q)$. Fix $p \in M$ and

$l \in L(p)$. Since p is a point of linear density of $E(p) = F(p) \cap l$ (meaning a point of linear dispersion of $l \backslash F(p)$), f is approximately differentiable on both sides of l, with linear approximate derivatives both $+1$, so f does not possess the linear approximate Denjoy property on l.

We show that f is uniformly continuous. Fix an $\varepsilon > 0$. First, we show that there exists a δ, such that if $d(q, q') < \delta$ ($q \in F(p)$, $q' \in F(p')$, $p, p' \in M$), then $|f(q) - f(q')| < \varepsilon$. Suppose $d(p, q) > d(p', q')$, then we have

$$\begin{aligned} |f(q) - f(q')| &= |d(p, q) - d(p', q')| \leq |d(p, p') + d(p', q') + d(q, q') - d(p', q')| \\ &= |d(p, p') + d(q, q')| < g^{-1}(\delta) + \delta, \end{aligned}$$

and by taking the limit as $\delta \to 0$, we obtain a δ, such that $g^{-1}(\delta) + \delta < \varepsilon$ holds.

Next, let $q' \in F(p')$ ($p' \in M$) and assume that $d(p, q') < \delta$ holds for some $p \in M$. Then since p is a point of accumulation of $F(p)$, a point $q \in F(p)$ close enough to p (say $d(p, q) = \eta$) can be found such that $d(q, q') < \delta$ also holds, implying $|f(q) - f(q')| < \varepsilon$. We obtain that

$$|f(p) - f(q')| \leq |f(q') - f(q)| + |f(p) - f(q)| < \varepsilon + \eta,$$

for arbitrarily small η. Now it only remains to extend this uniformly continuous function to the whole domain. □

Remark 10. *Note that when considering approximate derivatives, since p is a point of linear dispersion of $l \backslash F(p)$, we can ignore the behaviour of the function on the lines $l' \in L(p')$ for $p' \neq p$. However the point p can still be a point of accumulation of such $l' \cap l$ intersections, thus changing the value of the classical Dini derivatives. It is not clear how a transfinite construction could be executed in this case. A similar construction is achieved in [40] (in preparation).*

B Summary.

In order to create some order to all these results, the following tables recapitulate the most relevant results and articles known in each case. A plus sign (+) denotes positive results for Denjoy type relations, whereas a minus sign (-) denotes the cases where counterexamples have been given. On some occasions depending on the conditions (Borel/Lebesgue/non-measurable) both a + and a - sign can be found in the same entry (Ward/Davies; Ward/Theorem 20).

1-dim	Classical	Approximate
Measure	Denjoy-Young-Saks	Denjoy-Khintchine, Ward [79]
Category	Zajíček [85], Belna-Cargo-Evans-Humke [6]	Zajíček [84], Preiss-Zajíček [51]

2-dim directed (typical point all dir)	Classical	Approximate
Measure	+: Haslam-Jones [28], Saks	+: Ward [79]
Category	Ravetz [54], [55]	

2-dim linear (typical point+dir)	Classical	Approximate
Measure	+: Ward [80], -: Davies [21]	+: Ward [80], -: Theorem 20
Category	+: Theorem 24	

2-dim linear (typical point all dir)	Classical	Approximate
Measure	-: Besicovitch [8], [40] (in preparation)	-: Theorem 21
Category	-: Zajíček [86]	

The following table illustrates known results about the Lebesgue measurability of derivate numbers of Lebesgue measurable functions:

$f : \mathbb{R}^2 \to \mathbb{R}$ λ_2-meas. [measurability]	Classical	Approximate
$x \mapsto \partial^\vartheta f(x)$ sections, ϑ fix	No [Neubauer]	Yes [Khintchine-Saks]
$(x, \vartheta) \mapsto \partial^\vartheta f(x)$	No [Davies]	Yes [Ward]
$x \mapsto D^\vartheta f(x)$ sections, ϑ fix	Yes [Haslam-Jones]	
$(x, \vartheta) \mapsto D^\vartheta f(x)$		

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