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STRONG DERIVATIVES AND INTEGRALS

Abstract

The strong derivative is not, without some caution, a useful tool in the study of McShane's (i.e., Lebesgue's) integral. Even so, the underlying structure of that process of derivation is closely connected to the formulation of the Riemann sums definition that McShane gave for his integral. This article discusses some of the features and traps for the study of those connections.

In this elementary paper we wish to clarify some properties of the strong derivative and the relation that derivative has to integration. There has been some confusion in papers published here ([7], [8]) as to the exact situation. Mostly this stems from an analogy that is easy to push rather too far. The differentiation basis that expresses the ordinary derivative also expresses the Henstock-Kurzweil integral. That fact is the source of the ease with which one can argue back and forth between properties of the integral and properties of the derivative.

The differentiation basis that expresses the strong derivative also expresses the Lebesgue integral in an identical manner, as shown originally by McShane [12]. That might lead one to think that there is an intimate relationship between the Lebesgue integral and the strong derivative and, moreover, that similar arguments can be used in similar situations. The difficulty is that there is no Vitali covering theorem for the strong differentiation basis. What these, admittedly vague, statements mean is clarified in our article.

Henstock's general theory of integration, which he developed in a number of places and most recently in the textbook [4], uses his variational ideas to generalize Vitali arguments to situations where the classical Vitali theorem might be awkward to employ. There is some difficulty in applying the theory

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since, in each specific case, one needs to characterize a notion of “set of inner variation zero.” Henstock warns about this in a comment at the end of his chapter:

> “The theory of Chapter 4 has not yet been clarified. The author surmises that eventually a theory not based on Vitali’s or Sierpinski’s covering theorem will emerge, to prove that a set of inner variation zero is of variation zero in much wider and more general circumstances than are used at present.”

In this article we will illustrate how any attempt to apply his general theory to the strong differentiation basis (or “division space” as Henstock prefers to call it) encounters this road block whereby the Vitali theorem is not available. It can lead, as we shall see, to formally correct statements that have no content.

The next section contains some historical remarks appropriate to the discussion. The reader who wishes immediately to see the technical details may skip ahead to Section 2.

1 Henstock’s program

Henstock’s earliest research interest was in summability theory that he learned from his supervisor Paul Dienes at Birkbeck College. Dienes, however, recommended to his young student that he make his career in integration theory. At the time there was sufficient interest in nonabsolute integration that that would be a reasonable suggestion. Henstock was influenced technically, it seems, most significantly from the works of Saks and Ward, although he clearly had read extensively in the then current literature devoted to nonabsolute integration. His development of the integral that now bears his name arises quite directly from his studies of Ward’s methods. The lemma, often called Henstock’s lemma and used now extensively in the study of the Henstock-Kurzweil integral and its generalizations, is a variant of one due to Saks (as Henstock himself was quick to point out): most authors call it the Henstock-Saks lemma.

By the early 1960s Henstock was working on a grand scheme of unifying various methods of integration including, as he hoped, a host of special integrals of which many of the current generation might never have heard (e.g., the symmetric totalisation of first and second order of Denjoy, the trigonometric T-integral of Marcinkiewicz and Zygmund, the Cesàro-Perron scale of integrals due to Burkill, the P^n integrals of James, and the Abel-Perron integral of S. J. Taylor).

His papers from that time are dense and opaque, basing his methods on a series of arcane axioms that express abstractly the properties that he desired and could exploit. I believe he had virtually no readers. One of his students suggested to him that the theory would be easier and more popular if he could describe a structure, much the way a measure space is the suitable structure for the development of an abstract Lebesgue integral. This led him to the publication [3] that is the first relatively readable account of his very general ideas, but a work that still has its idiosyncrasies. In this he defines what he calls a division space, carrying all of the apparatus that allows him to define integrals, measures and derivatives and investigate their relations.

Among his many contributions (some well-known, some obscured by his exposition) I would rate his use of the variation and inner variations among my favorites. These ideas allow Vitali arguments to be extended to many settings where they would not have naturally been used. They also allow a complete rewriting of the chapters in Saks [18] in which the clumsier concepts of ACG, VBG, ACG_*, and VBG_* are developed. Theorem 20 below, for example, well illustrates this. Prior to Henstock's variational ideas one might have presumed Saks' textbook to be the final word on the properties of VBG_* functions.

As Henstock acknowledges in the paragraph from his book quoted in our introduction, the methods, as so far developed, may in some instances give only formally correct but meaningless assertions. The prime example is the following empty statement (that has already appeared in articles in this Exchange):

> An indefinite Lebesgue integral is strongly differentiable everywhere except on a set of McShane inner variation zero.

This does indeed follow from Henstock's abstract theory.

My own sentiments about his program is that the effort to fabricate an abstract theory that incorporates all the features of a wide variety of special integrals may not be fruitful. It is likely better to use these methods on an *ad hoc* basis as the need arises. Then, for example, the investigation of McShane's integral either in the real-valued case (as in [7], [8], [10], [19], and [26]) or in the vector-valued version (as in [5], [14], [15], [20], and [21]) should not lead to a study of the inner variation and the strong (i.e., unstraddled) derivative but to other more appropriate tools. One sees also this viewpoint explained lucidly in Pfeffer [6], first in his description of the methods of analyzing both the Henstock-Kurzweil and McShane integrals. More tellingly his account of the various ways of generalizing the integral to higher dimensions *depending on the goals of the theory* shows best how these ideas can be used. Although many of Henstock's methods come into play, there is no effort to derive them from his abstract theory. Indeed it is the special features of the application at hand, rather than the general features, that have the greatest interest.

2 The strong derivative

The strong derivative[1] has an ancient lineage going back, at least, to Peano [16] who noted that strong differentiability would be equivalent to the existence of a continuous derivative. Peano viewed that as a positive feature, more useful in many applications than the traditional definition of a derivative.

We state first the usual definition for the bilateral extreme derivates:

Definition 1. Let $F : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. We define

$$\overline{D}F(x_0) = \lim_{t\to 0+} \sup \left\{ \frac{F(y) - F(x_0)}{y - x_0} : y \in (x_0 - t, x_0 + t) \setminus \{x_0\} \right\}$$

and

$$\underline{D}F(x_0) = \lim_{t\to 0+} \inf \left\{ \frac{F(y) - F(x_0)}{y - x_0} : y \in (x_0 - t, x_0 + t) \setminus \{x_0\} \right\}.$$

This is just the ordinary process of differentiation and the existence of the derivative $F'(x_0)$ is simply the finiteness and agreement of the two bilateral extreme derivates. The "strong" (or unstraddled) version of this simply computes these same ratios in a neighborhood of x_0 without requiring the point x_0 to be straddled.

Definition 2. Let $F : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. We define

$$\overline{D}^\sharp F(x_0) = \lim_{t\to 0+} \sup \left\{ \frac{F(y) - F(x)}{y - x} : [x, y] \subset (x_0 - t, x_0 + t),\ x \neq y \right\}$$

and

$$\underline{D}^\sharp F(x_0) = \lim_{t\to 0+} \inf \left\{ \frac{F(y) - F(x)}{y - x} : [x, y] \subset (x_0 - t, x_0 + t),\ x \neq y \right\}.$$

[1]The terminology seems to have taken hold, but is unfortunate. Peano used "strict" derivative. "Strong" has been used for derivatives in higher dimensions taken with regard to the basis of all intervals (rather than all cubes). The same word is often used to distinguish between different modes of differentiation for Banach space-valued functions.

Evidently

$$\underline{D}^\sharp F(x_0) \le \underline{D}F(x_0) \le \overline{D}F(x_0) \le \overline{D}^\sharp F(x_0).$$

The function F is said to be *strongly differentiable* at a point x_0 if it is differentiable there and, moreover, $\underline{D}^\sharp F(x_0) = \overline{D}^\sharp F(x_0) = F'(x)$. This is considerably stronger than merely requiring the existence of the derivative. As many will know (and as reviewed below), in order for a function F to be strongly differentiable at a point x_0 that function would have to be Lipschitz in a neighborhood of the point and the derivative $F'(x)$ would have to be continuous at x_0, i.e., continuous relative to the set of points at which it exists.

3 Properties of the strong extreme derivates

The properties are not difficult or surprising (after a moment's reflection). The only elementary reference in the literature [9] with some level of detail is somewhat inadequate and so we give the details here, as transparently as possible, in the hope that they might be useful for instruction.

Theorem 3. *Let $F : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. If*

$$\overline{D}^\sharp F(x_0) < M < \infty$$

then there is a neighborhood $(x_0 - \delta_0, x_0 + \delta_0)$ such that $\overline{D}^\sharp F(x) < M$ at every point x in that neighborhood. In particular, $F(x) - Mx$ is strictly decreasing, F is VBG$_*$, and F is a.e. differentiable in that neighborhood.

Proof. If $\overline{D}^\sharp F(x_0) < t < M$ then, by definition, there is a $\delta_0 > 0$ so that

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} < t$$

for $x_0 - \delta_0 < x_1 < x_2 < x_0 - \delta_0$. From that it follows that $\overline{D}^\sharp F(x) \le t < M$ for all points x in $(x_0 - \delta_0, x_0 + \delta_0)$. That same inequality proves that $F(x) - tx$ is strictly decreasing there. The Lebesgue differentiation theorem shows that F' exists a.e. in that interval. Standard material on VBG$_*$ functions (see Saks [18, pp. 234–235]) supplies the remaining statement. □

The corollaries are immediate applications of the simple inequality in the theorem.

Corollary 4. *Let $F : \mathbb{R} \to \mathbb{R}$. The function $\overline{D}^{\sharp}F(x)$ is upper semicontinuous and the function $\underline{D}^{\sharp}F(x)$ is lower semicontinuous.*

Proof. To check that $\overline{D}^{\sharp}F(x)$ is upper semicontinuous take any point x_0 at which the strong upper derivate is finite, take $\overline{D}^{\sharp}F(x_0) < t$, and use the theorem to find a neighborhood $(x_0-\delta_0, x_0+\delta_0)$ such that $\overline{D}^{\sharp}F(x) < t$ at every point x in that neighborhood. This is the definition of upper semicontinuity. □

Corollary 5. *Let $F : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Then*

$$-\infty < \underline{D}^{\sharp}F(x_0) \leq \overline{D}^{\sharp}F(x_0) < \infty$$

if and only if there is a neighborhood $(x_0-\delta_0, x_0+\delta_0)$ such that F is Lipschitz in that neighborhood.

The relation between strong differentiability and continuity of the derivative was noted long ago by Peano [16]. In this theorem we use the extreme derivates.

Theorem 6. *Let $F : \mathbb{R} \to \mathbb{R}$ be differentiable at a point $x_0 \in \mathbb{R}$. Then F is strongly differentiable at x_0 if and only if both $\underline{D}F(x)$ and $\overline{D}F(x)$ are continuous at the point x_0.*

Proof. Recall that $\underline{D}^{\sharp}F(x) \leq \overline{D}^{\sharp}F(x)$ and that the larger function is upper semicontinuous and the smaller is lower semicontinuous. If both functions have the same finite value at a point x_0 then both functions are, in fact, continuous at that point. Since

$$\underline{D}^{\sharp}F(x) \leq \underline{D}F(x) \leq \overline{D}F(x) \leq \overline{D}^{\sharp}F(x)$$

the two functions $\underline{D}F(x)$ and $\overline{D}F(x)$ must also be continuous at the point x_0.

Conversely, suppose that $\underline{D}F(x)$ and $\overline{D}F(x)$ are continuous at the point x_0. Let $\epsilon > 0$ and choose a $\delta_0 > 0$ so that, for x in the interval $(x_0-\delta_0, x_0+\delta_0)$,

$$F'(x_0) - \epsilon < \underline{D}F(x) \leq \overline{D}F(x) < F'(x_0) + \epsilon.$$

Usual properties of the extreme derivates supply the inequalities

$$F'(x_0) - \epsilon < \frac{F(x_2) - F(x_1)}{x_2 - x_1} < F'(x_0) + \epsilon$$

for $x_0 - \delta_0 < x_1 < x_2 < x_0 - \delta_0$. This is exactly the requirement that F is strongly differentiable at the point x_0. □

Finally we state a necessary and sufficient condition for strong differentiability that uses only the derivative (which, of course, may fail to exist on a set of measure zero).

Theorem 7. *Let $F : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Then F is strongly differentiable at x_0 if and only if there is a neighborhood $(x_0 - \delta_0, x_0 + \delta_0)$ such that F is Lipschitz in that neighborhood and F' is continuous at the point x_0 (i.e., continuous relative to the set of points at which it exists).*

Proof. By Theorem 6 we know that, if F is strongly differentiable at x_0 then the upper extreme derivate $\overline{D}F(x)$ is continuous at x_0. Since $F'(x) = \overline{D}F(x)$ at every point of differentiability it follows that F' is continuous at the point x_0.

In the converse direction a simple direct proof is useful. Assuming that F' is continuous at the point x_0 and F is Lipschitz in a neighborhood of x_0 choose a smaller neighborhood $(x_0 - \delta_0, x_0 + \delta_0)$ within which, at almost every point x,

$$F'(x_0) - \epsilon < F'(x) < F'(x_0) - \epsilon.$$

Usual properties of a.e. derivatives of Lipschitz (or absolutely continuous) functions supply the inequalities

$$F'(x_0) - \epsilon < \frac{F(x_2) - F(x_1)}{x_2 - x_1} < F'(x_0) + \epsilon$$

for $x_0 - \delta_0 < x_1 < x_2 < x_0 - \delta_0$. This is exactly the requirement that F is strongly differentiable at the point x_0. □

It might be worth repeating the version in [9]. This simply uses the fact that a function F is Lipschitz in an interval if and only if it absolutely continuous and the derivative F' is bounded (i.e., bounded on the set of points at which it exists).

Corollary 8. *Let $F : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Then F is strongly differentiable at x_0 if and only if there is a neighborhood $(x_0 - \delta_0, x_0 + \delta_0)$ such that F is absolutely continuous in that neighborhood and F' is continuous at the point x_0 (i.e., continuous relative to the set of points at which it exists).*

4 Strong differentiability of indefinite integrals

Strong differentiability can be used to give a characterization of indefinite Riemann integrals. This gives a formal answer to a question posed by Erik Talvila [22] in this Exchange. The characterization that we gave earlier in [24] is more informative since strong differentiability is not easy to check without applying Theorem 7. But in that case it is rather obvious since the characterization essentially is just the assertion that the integrand is bounded and a.e. continuous.

Theorem 9. *Let $F : [a, b] \to \mathbb{R}$. Then F is the indefinite integral of a Riemann integrable function if and only if F is Lipschitz on $[a, b]$ and strongly differentiable at almost every point of (a, b).*

Proof. If F is Lipschitz and strongly differentiable at almost every point of (a, b) then, by Theorem 6, $\overline{D}F(x)$ is continuous a.e. and bounded. Consequently it is Riemann integrable and has F for its indefinite integral. Conversely, if F is the indefinite integral of a Riemann integrable function f then, F is strongly differentiable at every point of continuity f, hence it is strongly differentiable almost everywhere. □

A local version is also helpful, even though obvious.

Theorem 10. *Let $F : [a, b] \to \mathbb{R}$ be the indefinite integral of a bounded, Lebesgue integrable function f. Then F is strongly differentiable at a point $x_0 \in (a, b)$ if and only if f is equivalent to a function continuous at the point x_0.*

The Henstock-Kurzweil indefinite integral has an ordinary derivative at almost every point. In contrast, even though the Lebesgue integral permits a similar definition that is closely related to strong differentiation, there is no analogous property (cf. the misleading assertions in [7, Theorem 8] and [8, Theorem 1.2]).

Theorem 11. *There exists a bounded, Lebesgue integrable function whose indefinite integral is not strongly differentiable at any point.*

Proof. We need only an example of a bounded, Lebesgue integrable function f that is not equivalent to a function with a point of continuity. Take a measurable set V with the property that $0 < |V \cap (c, d)| < d - c$ for any interval (c, d). See Rudin [17] for an elementary construction of such a set. The characteristic function of V supplies our example on any interval. □

We might point out another feature of this example of an indefinite integral that is a.e. differentiable but nowhere strongly differentiable. The set of points here at which the derivative exists is full measure, but it is a set of the first category. One can prove that if F is continuous then the set of points at which F has a derivative, but not a strong derivative, is necessarily first category (see

Jurek [11] or, perhaps easier to find, [23, p. 138]). (Related papers, for the interested reader, are [1] and [28].)

Theorem 11 and its proof suggest that there is likely an abundance of examples of such functions, i.e., that the typical bounded, Lebesgue integrable function has an indefinite integral that is not strongly differentiable at any point. In fact we can prove the following.

Theorem 12. *The collection of bounded, Lebesgue integrable functions on an interval* $[a, b]$ *whose indefinite integral is strongly differentiable at at least one point is a nowhere dense subset of* $L_\infty([a, b])$.

PROOF. Let $B(f, r)$ for any bounded, Lebesgue integrable function f and any $r > 0$ denote the open ball in $L_\infty([a, b])$ centered at f and with radius r. The proof is based on the following fact about the function χ_V where V is the measurable set from the proof of Theorem 11 with the property that $0 < |V \cap (c, d)| < d - c$ for any interval (c, d).

Let s and t be any positive real numbers and consider any function $f \in B(s\chi_V, t)$ with indefinite integral F. We know that, for every $x \in [a, b]$ excepting a set of measure zero,

$$s + t > f(x) > s - t \qquad (\text{if } x \in V)$$

and

$$t > f(x) > -t \qquad (\text{if } x \notin V).$$

Consequently, for any interval (u, v),

$$(s+t)|V \cap (u, v)| + t|(u, v) \setminus V| \geq \int_u^v f(x)\, dx \geq (s-t)|V \cap (u, v)| - t|(u, v) \setminus V|.$$

Almost every point of $[a, b]$ is a point of density or a point of dispersion of V and a point of differentiability for F. If z is a point of density of V and $F'(z)$ exists, then we see from these inequalities that

$$s + t \geq F'(z) \geq s - t$$

while, if w is a point of dispersion of V and $F'(w)$ exists, then

$$t \geq F'(w) \geq -t.$$

Now we show that the collection described in the statement of the theorem is a nowhere dense subset of $L_\infty([a, b])$. Take any ball $B(g_1, r_1)$. Select a simple measurable function g_2 so that $\|g_1 - g_2\|_\infty < r_1/4$. (This just uses the fact that every bounded measurable function is the uniform limit of a sequence of measurable simple functions.) Write

$$g_2(x) = \sum_{i=1}^{n} c_i \chi_{E_i}$$

where $c_i \in \mathbb{R}$ are distinct and where $\{E_i\}$ are disjoint measurable sets. Let c be a positive number that is smaller than the distance between any of the values 0, c_1, c_2, …, c_n. Consider the ball $B(g_2 + s\chi_V, t)$ where s and t are chosen as positive numbers so that $s < r_1/4$, $5s < c$, and $t = s/4$.

The ball $B(g_2 + s\chi_V, t)$ is entirely contained in $B(g_1, r_1)$ and, as we now show, consists entirely of functions whose indefinite integrals are not strongly differentiable at any point. This will complete the proof.

To see this suppose that $f_1 \in B(g_2+s\chi_V, t)$. Then $f = f_1 - g_2 \in B(s\chi_V, t)$. Let G_2 be an indefinite integral of g_2, F_1 an indefinite integral of f_1 and F an indefinite integral of f. We know that $G_2'(x)$ exists almost everywhere in $[a, b]$ and assumes one of the values 0, c_1, c_2, …, c_n. We know that F is differentiable at almost every point of $[a, b]$; in fact, if z is a point of density of V and w is a point of dispersion of V, then

$$c/4 > 5s/4 = s + t \geq F'(z) \geq s - t = 3s/4$$

and

$$s/4 = t \geq F'(w) \geq -t = -s/4 > -c/4.$$

So

$$c/2 \geq |F'(z) - F'(w)| \geq s/2.$$

But $F = F_1 - G_2$ and so $F_1'(x) = F'(x) + G_2'(x)$ almost everywhere. Since every interval contains points w and z as above the derivative F_1' is discontinuous at every point at which it exists. Consequently F_1 is nowhere strongly differentiable because of Theorem 10. □

5 Vitali covering theorem

The study of differentiation and integration on the real line leads quite naturally to the following concepts. Let β be any collection of interval-point pairs $([u, v], w)$. (Sometimes, but not always, we might require $w \in [u, v]$.) Any such finite collection

$$\pi = \{([u_1, v_1], w_1), ([u_2, v_2], w_2), ([u_3, v_3], w_3), \ldots, ([u_n, v_n], w_n)\}$$

is said to be a packing if $[u_i, v_i]$ and $[u_j, v_j]$ do not overlap if $i \neq j$. The *total length* of such a packing π is denoted

$$\ell(\pi) = \sum_{i=1}^{n} (v_i - u_i).$$

For any collection β write

$$V(\ell, \beta) = \sup\{\ell(\pi) : \pi \subset \beta,\ \pi \text{ a packing}\}.$$

Definition 13. A collection β of interval-points pairs $([u, v], w)$ is said to be a *full cover* of a set $E \subset \mathbb{R}$ if, for each $w \in E$, there is a $\delta > 0$ so that all pairs $([u, v], w)$ for which $w \in [u, v]$ and $v - u < \delta$ must belong to β.

Definition 14. Dually, β is said to be a *fine cover* of E if for each $w \in E$ and every $\epsilon > 0$ there must exist at least one pair $([u, v], w)$ in β for which $w \in [u, v]$ and $v - u < \epsilon$.

The nature of the duality is explained in [25, Exer. 81, 82]. Realizing this helps gain an understanding of the concepts, but it is not essential. If you are familiar with Vitali covers you will recognize the fine covers as intimately related to that idea and derived from it.

The full and fine covers are useful tools in working with the ordinary derivative and the Henstock-Kurzweil integral. One might hope that a strong version of these concepts would play the same role in the study of the strong derivative and the McShane (i.e., Lebesgue) integral. The analogous definitions would be these:

Definition 15. A collection β of interval-points pairs $([u, v], w)$ is said to be a *full McShane-cover* of a set $E \subset \mathbb{R}$ if, for each $w \in E$, there is a $\delta > 0$ so that all pairs $([u, v], w)$ for which $[u, v] \subset (w - \delta, w + \delta)$ must belong to β.

Definition 16. Dually, β is said to be a *fine McShane-cover* of E if for each $w \in E$ and every $\epsilon > 0$ there must exist at least one pair $([u, v], w)$ in β for which $[u, v] \subset (w - \epsilon, w + \epsilon)$.

Each of these four notions of a cover gives rise to a measure on the real line.

Definition 17. For any set $E \subset \mathbb{R}$ define

$$\ell_*(E) = \inf\{V(\ell, \beta) :\ \beta \text{ a fine cover of } E\},$$

$$\ell^*(E) = \inf\{V(\ell, \beta) :\ \beta \text{ a full cover of } E\},$$

$$\ell_\sharp(E) = \inf\{V(\ell,\beta) : \ \beta \text{ a fine McShane-cover of } E\},$$

and,

$$\ell^\sharp(E) = \inf\{V(\ell,\beta) : \ \beta \text{ a full McShane-cover of } E\}.$$

Each of these is a Borel measure on the real line (also known as a "metric outer measure" in the language of Carathéodory). The relations

$$\ell_\sharp(E) \leq \ell_*(E) \leq \ell^*(E) \leq \ell^\sharp(E)$$

are immediate simply because any full cover is a fine cover, any full McShane-cover is also a full cover, and any fine cover is also a fine McShane-cover. In Henstock's terminology the two smaller measures that are defined using fine covers and fine McShane-covers are known as inner variations. The measures ℓ_* and ℓ^* play a key role in studies of the Henstock-Kurzweil integral; the fact that they are equal is just the classical Vitali theorem.

Analogously one might be led to believe that the other two measures play the same role in the study of the McShane (i.e., Lebesgue) integral. The following theorem reveals where the analogy breaks down. The Vitali theorem supplies the useful content (i.e., the identity of ℓ_* and ℓ^*) while the fact that there is no Vitali theorem for fine McShane-covers is expressed by the fact that $\ell_\sharp$ is useless. (Overlooking this fact can lead to meaningless claims [cf. [7, Theorems 8, 9]]).

Theorem 18. *For any set $E \subset \mathbb{R}$*

$$\ell_*(E) = \ell^*(E) = \ell^\sharp(E) = |E| \ \textit{(the Lebesgue outer measure of } E\textit{)}$$

and $\ell_\sharp(E) = 0$.

Proof. The first statement of the theorem not only follows directly from the Vitali covering theorem for Lebesgue measure, it is equivalent to the Vitali theorem. See [25, pp. 148–151] for an account.

The proof that the measure $\ell_\sharp$ is trivial is not difficult. Let $\epsilon > 0$ and let $r_1, r_2, r_3, \dots$ be an enumeration of the rationals. Consider the collection β of interval-point pairs of the form

$$\left([r_i - \epsilon 2^{-i}, r_i + \epsilon 2^{-i}], x\right)$$

for any $x \in \mathbb{R}$ and any $i = 1, 2, 3, \dots$. This is easily verified to be a fine McShane-cover of any subset E of the real line. Evidently

$$\ell_\sharp(E) \leq V(\ell,\beta) \leq \sum_{i=1}^{\infty} 2\epsilon 2^{-i} = 2\epsilon.$$

Consequently $\ell_\sharp(E) = 0$. ☐

6 Variation of a function

The variation of a function $F : \mathbb{R} \to \mathbb{R}$ can be expressed by introducing measures that generalize the measures of the preceding section. The total variation of F over a packing

$$\pi = \{([u_1, v_1], w_1), ([u_2, v_2], w_2), ([u_3, v_3], w_3), \dots, ([u_n, v_n], w_n)\}$$

is denoted

$$V(F, \pi) = \sum_{i=1}^{n} (|F(v_i) - F(u_i)|.$$

For any collection β write

$$V(F, \beta) = \sup\{V(F, \pi)) : \pi \subset \beta,\ \pi \text{ a packing}\}.$$

Then, exactly as we did in Definition 17, we define

Definition 19. For any function $F : \mathbb{R} \to \mathbb{R}$ and any set $E \subset \mathbb{R}$ define

$$F_*(E) = \inf\{V(F, \beta) :\ \beta \text{ a fine cover of } E\},$$

$$F^*(E) = \inf\{V(F, \beta) :\ \beta \text{ a full cover of } E\},$$

$$F_\sharp(E) = \inf\{V(F, \beta) :\ \beta \text{ a fine McShane-cover of } E\},$$

and,

$$F^\sharp(E) = \inf\{V(F, \beta) :\ \beta \text{ a full McShane-cover of } E\}.$$

Again the relations

$$F_\sharp(E) \le F_*(E) \le F^*(E) \le F^\sharp(E)$$

are immediate and each of these is a Borel measure on the real line. The two measures based on full and fine covers express useful properties of functions and can be used to some advantage in the theory of the Henstock-Kurzweil integral. The two measures based on full McShane-covers and fine McShane-covers, however, do not help much.

Just as a sample of the rich theory available for the former measures let us cite, without proof, one interesting property which captures in a single statement many familiar theorems from classical analysis (including the Lebesgue differentiation theorem, de la Vallée Poussin's theorem, and the Vitali covering theorem). It is a vain hope that anything quite so nice holds for the variations based on the McShane-covers. See [25, Chapter 6] for the development of this theory. Theorems 6.8, 6.21, 6.25, 6.30, and 6.32 in that text, in particular, provide the proofs of the following statement.

Theorem 20. *Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a continuous function and let E be a closed set of real numbers. Then the following are equivalent:*

1. *F is VBG$*$ on E.*
2. *F^* is σ-finite on E.*
3. *F^* is σ-finite on every $\mathcal{G}_\delta$ measure-zero subset of E.*
4. *$F_* = F^*$ on every subset of E.*
5. *If D denotes the set of points of differentiability of F and D_∞ the set of points x at which $F'(x) = \pm\infty$, then*
$$|E \setminus D| = F^*\left(E \setminus (D \cup D_\infty)\right) = 0.$$

Moreover, should these conditions be valid, then

$$F^*(E) = F^*(E \cap D_\infty) + \int_{E\cap D} |F'(x)|\, dx.$$

In contrast, note how little information is conveyed by the measures $F_\sharp$ and $F^\sharp$. (The measure $F_\sharp$, while useless, nonetheless plays a key role in a bogus theorem in [8, Theorem 1.4] for example.)

Theorem 21. *For an arbitrary function $F : \mathbb{R} \to \mathbb{R}$ the measure $F_\sharp$ is zero on every set.*

PROOF. The proof is not difficult and is nearly identical with the corresponding part of Theorem 18. We use an old result of W. H. Young [27]. Let us say that F is feebly continuous at a point x if there is at least one sequence of points $\{x_n\}$ convergent to x with the property that

$$\lim_{n\to\infty} F(x_n) = F(x).$$

Young proved that the set of points at which an arbitrary function fails to be feebly continuous is at most countable. (See, e.g., [23, Chapter 2].)

Let $\epsilon > 0$ and let $x_1, x_2, x_3, \dots$ be a sequence of points chosen to be everywhere dense and such that F is weakly continuous at each x_i. For each i, select points $s_i \le x_i \le t_i$ with $|F(t_i) - F(s_i)| < \epsilon 2^{-i}$ and $0 < (t_i - s_i) < 2^{-i}$.

Consider the collection β of interval-point pairs of the form

$$([s_i, t_i], x)$$

for any $x \in \mathbb{R}$ and any $i = 1, 2, 3, \ldots$. This is easily verified to be a fine McShane-cover of any subset E of the real line. Evidently

$$F_{\sharp}(E) \leq V(F, \beta) \leq \sum_{i=1}^{\infty} 2\epsilon 2^{-i} = 2\epsilon.$$

Consequently $F_{\sharp}(E) = 0$. □

For the larger measure based on full McShane-covers we can indicate, just by a local property, that the measure will have limited use.

Theorem 22. *Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a continuous function and let $x_0 \in \mathbb{R}$. Then*

1. $F_*(\{x_0\} = F^*(\{x_0\} = 0$.
2. *Either $F^{\sharp}(\{x_0\}) = 0$ or else $F^{\sharp}(\{x_0\}) = \infty$. The former holds if and only if F has bounded variation in some neighborhood of the point x_0.*

PROOF. The statement that $F_*(\{x_0\} = F^*(\{x_0\} = 0$ is easily proved, merely from the assumption that F is continuous at x_0. If $F^{\sharp}(\{x_0\}) < \infty$ there is a full McShane-cover β of the set $\{x_0\}$ so that $V(F, \beta)$ is finite. There must be a $\delta_0 > 0$ so that all pairs $([u, v], x_0)$ for which $[u, v] \subset [x_0 - \delta_0, x_0 + \delta_0]$ belong to β. Thus the variation of F on that interval cannot exceed $V(F, \beta)$. It follows that F has bounded variation on $[x_0 - \delta_0, x_0 + \delta_0]$.

Since F is continuous it also has a total variation function T that is also continuous on $[x_0 - \delta_0, x_0 + \delta_0]$. It is easy to check that

$$F^{\sharp}(\{x_0\}) \leq T(x_0 + \delta_0) - T(x_0 - \delta_0).$$

From this we see that, if $F^{\sharp}(\{x_0\}) < \infty$ then $F^{\sharp}(\{x_0\}) = 0$. □

7 Equivalence of McShane's and Lebesgue's integrals

One sees, so far, that the strong derivative, fine McShane-covers, and the corresponding inner variations do not play the analogous role in the study of McShane's (i.e., Lebesgue's) integral that the ordinary derivative and fine covers do in the study of the Henstock-Kurzweil integral. Even so, one can exploit full McShane-covers and the strong derivative with some caution.

To illustrate let us prove, using these methods, the following assertion (by now well-known) that characterizes Lebesgue integrability as a McShane integral (at least for bounded functions). Note especially the correct use of full McShane-covers and the strong derivative in the details of the proof.

Lemma 23. *Suppose that f is a bounded function on an interval $[a,b]$. Then a necessary and sufficient condition for f to be Lebesgue integrable there with the function F as its indefinite integral is the following:*

> *For every $\epsilon > 0$, there is a full McShane-cover β of $[a,b]$ so that, for every packing π contained in β,*
> $$\sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v-u)| < \epsilon. \tag{1}$$

Proof. Throughout we are using McShane-covers that employ only subintervals of $[a,b]$ and we restrict all attention to that interval.

In the first part of the proof we show that the stated condition implies that F is Lipschitz and that $F' = f$ almost everywhere in $[a,b]$. Consequently f is Lebesgue integrable and F is an indefinite integral. This proves the lemma in one direction.

Take M as a positive upper bound of $|f|$. The equation (1) implies that, if π is a partition of an interval $[c,d] \subset [a,b]$ and $\pi \subset \beta$, then

$$\begin{aligned} |F(d) - F(c)| &\le \sum_{([u,v],w)\in\pi} |F(v) - F(u)| \\ &\le \sum_{([u,v],w)\in\pi} \{|F(v) - F(u) - f(w)(v-u)| + M(v-u)\} \\ &< \epsilon + M(d-c). \end{aligned}$$

Since every full McShane-cover of $[a,b]$ would have to contain a partition of any such interval $[c,d]$ (cf. Cousin's lemma) we see how to deduce that F is Lipschitz on $[a,b]$ directly from the condition stated in the lemma.

To analyze the set of points x where $F'(x) = f(x)$ might fail, we introduce, for each integer n, the collections

$$N_n = \left\{ x \in [a,b] : \limsup_{y\to x} \left| \frac{F(y) - F(x)}{y-x} - f(x) \right| > \frac{1}{n} \right\}$$

and

$$\beta_n = \left\{ ([u,v],w) : w \in [u,v],\ \left| \frac{F(v) - F(u)}{v-u} - f(w) \right| > \frac{1}{n} \right\}.$$

Every point x where $F'(x) = f(x)$ fails belongs to one of the sets N_n for some integer n and the collection β_n is a fine cover of N_n.

Using the full McShane-cover β of $[a,b]$ stated in the condition we can estimate $V(\ell, \beta \cap \beta_n)$. Take any packing $\pi \subset \beta \cap \beta_n$ and argue that

$$\sum_{([u,v],w)\in\pi} (v-u) \le \sum_{([u,v],w)\in\pi} n|F(v) - F(u) - f(w)(v-u)| < n\epsilon.$$

But the collection $\beta \cap \beta_n$ is a fine cover[2] of N_n and consequently this shows that

$$\ell_*(N_n) \le V(\ell, \beta \cap \beta_n) \le n\epsilon.$$

Each set N_n has, thus, Lebesgue measure zero and, since the set of all points x at which the identity $F'(x) = f(x)$ fails belongs to one of the sets N_n for some integer n, it follows that $F' = f$ almost everywhere as stated. We conclude that f is Lebesgue integrable on $[a,b]$ and that F is an indefinite integral.

In the oppositive direction we assume that f is a bounded Lebesgue integrable function with an indefinite integral F. Again take M as a positive upper bound of $|f|$, let $\epsilon > 0$ and write $\eta = \epsilon[3(b-a)]^{-1}$. Choose an open set G so that f is continuous relative to $[a,b] \setminus G$ and so that $|G| < \epsilon[6M]^{-1}$. (This just uses Lusin's theorem.) Let $\overline{f}$ be a continuous function on $[a,b]$ that agrees with f on the closed set $[a,b] \setminus G$ and for which $|\overline{f}(x)| \le M$ for all $x \in [a,b]$.

Suppose that $\overline{F}$ is an indefinite integral for the continuous function $\overline{f}$. We know, from the material in Section 4, that $\overline{F}$ is strongly differentiable at every point of $[a,b]$ and that $\overline{f}$ is its derivative. Consequently the collection

$$\beta_1 = \left\{([u,v],w) : w \in [a,b],\ \left|\overline{F}(v) - \overline{F}(u) - \overline{f}(w)[v-u]\right| \le \eta(v-u)\right\}$$

is a full McShane-cover of $[a,b]$. Let β consist of all pairs $([u,v],w) \in \beta_1$ for which $w \in [a,b] \setminus G$ as well as all pairs $([u,v],w)$ for which $w \in G$ and $[u,v] \subset G$. This too is a full McShane-cover of $[a,b]$.

Note, first, that if π is any packing consisting of subintervals of $[a,b]$ then

$$\begin{aligned}\sum_{([u,v],w)\in\pi} \left|F(v) - F(u) - [\overline{F}(u) - \overline{F}(v)]\right| &\le \sum_{([u,v],w)\in\pi} \int_u^v \left|f(t) - \overline{f}(t)\right|\, dt \\ &\le \int_a^b \left|f(t) - \overline{f}(t)\right|\, dt \\ &\le 2M|G| \\ &< \epsilon/3.\end{aligned}$$

[2]Note that it is also a fine McShane-cover of N_n but that would lead nowhere (because of Theorem 18).

Note, secondly, that if π is any packing consisting of subintervals of components of the open set G then

$$\left| \sum_{([u,v],w)\in\pi} \overline{f}(w)(v-u) - \sum_{([u,v],w)\in\pi} f(w)(v-u) \right| \leq 2M|G| < \epsilon/3.$$

Finally, note that if π is any packing contained in β but with all associated points in $[a,b]\setminus G$, then π is also a subset of β_1 and hence

$$\sum_{([u,v],w)\in\pi} \left|\overline{F}(v) - \overline{F}(u) - \overline{f}(w)[v-u]\right| \leq \sum_{([u,v],w)\in\pi} \eta(v-u) \leq \eta(b-a)| < \epsilon/3.$$

From these three inequalities (1) follows. □

There are numerous proofs in the literature that establish the equivalence of the Lebesgue integral and the McShane integral, none of which, as best we can tell, make any use of the strong derivative or allude explicitly to full McShane-covers. Our lemma uses an approximation to strongly differentiable functions to show that the variational version of McShane's integral integrates all bounded Lebesgue integrable functions. The other direction uses full McShane-covers and Vitali's theorem (i.e., the equivalence of ℓ_* and Lebesgue outer measure) to show that all bounded McShane integrable functions must be Lebesgue integrable with the same indefinite
integral.

In Henstock's general theory of integration in division spaces [the division space here is the collection of all full McShane-covers] variational integrals are equivalent to Riemann-sum versions. Moreover, monotone convergence theorems for such integrals are also available as part of the general theory. This can be used to establish that the McShane integral is equivalent to the Lebesgue integral in general.

References

[1] C. L. Belna, M. J. Evans, and P. D. Humke, *Symmetric and strong differentiation.* Amer. Math. Monthly **86(2)** (1979), 121–123.

[2] A. M. Bruckner and J. L. Leonard, *Derivatives.* Amer. Math. Monthly **73(4.2)** (1966), 24–56.

[3] R. Henstock, *Generalized integrals of vector-valued functions.* Proc. London Math. Soc. **3(19)** (1969), 509–536.

[4] R. Henstock, *The general theory of integration*. Oxford Mathematical Monographs. New York, 1991. ISBN: 0-19-853566-X.

[5] J.-C. Feauveau, *A generalized Riemann integral for Banach-valued functions.* Real Anal. Exch. **25(2)** (1999/2000), 919–930.

[6] W. F. Pfeffer, *Ralph Henstock's influence on the integration theory.* Sci. Math. Jpn. **67(1)** (2008), 37–50.

[7] J. V. Benitez, F. P. Jamil, and T. S. Chew, *Uniform differentiability.* Real Anal. Exch. **37(2)** (2011/2012), 451–462.

[8] T. S. Chew, *On Henstock's inner variation and strong derivatives.* Real Anal. Exch. **27(2)** (2001/2002), 725–733.

[9] M. Esser and O. Shisha, *A Modified Differentiation.* Amer. Math. Monthly **71(8)** (1964), 904–906.

[10] R. Gordon, *Some comments on the McShane and Henstock integrals.* Real Anal. Exch. **23(1)** (1997/1998), 329–341.

[11] B. Jurek, *Sur les nombres dérivés de fonctions discontinues.* Ceskâ Spol. Nauk Th'da Math. Pfirodovëdeckâ, Vëstnik **1** (1937), 1–22.

[12] E. J. McShane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals.* Mem. Amer. Math. Soc., **88**, Amer. Math. Soc., Providence, 1969.

[13] S. N. Mukhopadhyay, *On differentiability.* Bull. Calcutta Math. Soc. **59** (1967), 181–183.

[14] K. M. Naralenkov, *Several comments on the Henstock-Kurzweil and McShane integrals of vector-valued functions.* Czechoslovak Math. J. **61(4)** (2011), 1091–1106.

[15] K. M. Naralenkov, *A Henstock-Kurzweil integrable vector-valued function which is not McShane integrable on any portion.* Quaest. Math. **35(1)** (2012), 11–21.

[16] G. Peano, *Sur la définition de la dérivée.* Mathesis **2** (1892), 12–14.

[17] W. Rudin, *Well-distributed measurable sets.* Amer. Math. Monthly **90(1)** (1983), 41–42.

[18] Saks, S., *Theory of the Integral*, 2nd revised ed., Dover, New York, 1964.

[19] K. Singh, P. Surinder, and I. K. Rana, *Some alternative approaches to the McShane integral.* Real Anal. Exch. **35(1)** (2010), 229–233.

[20] V. A. Skvortsov and A. P. Solodov, *A variational integral for Banach-valued functions.* Real Anal. Exch. **24(2)** (1998/1999), 799–805.

[21] C. Swartz, *Uniform integrability and mean convergence for the vector-valued McShane integral.* Real Anal. Exch. **23(1)** (1997/1998), 303–311.

[22] E. Talvila, *Characterizing integrals of Riemann integrable functions.* Real Anal. Exch. **33(2)** (2008), 487.

[23] B. S. Thomson, *Real functions*, Lecture Notes in Mathematics, **1170**. Springer-Verlag, Berlin, 1985. ISBN: 3-540-16058-2

[24] B. S. Thomson, *Characterizations of an indefinite Riemann integral.* Real Anal. Exch. **35(2)** (2010), 487–492.

[25] B. S. Thomson, *Theory of the Integral*, CreateSpace (2013). ISBN-13: 978-1467924399.
http://classicalrealanalysis.info/Theory-of-the-Integral.php

[26] R. Výborný, *Kurzweil-Henstock absolute integrable means McShane integrable.* Real Anal. Exch. **20(1)** (1994/1995), 363–366.

[27] W. H. Young, *On the distinction of right and left at points of discontinuity.* Quart. J. of Pure and Appl. Math. **39** (1908), 67–83.

[28] L. Zajíček, *Strict differentiability via differentiability.* 15th Winter school in abstract analysis (Srní, 1987). Acta Univ. Carolin. Math. Phys. **28(2)** (1987), 157–159.