INROADS **Real Analysis Exchange**
Vol. 39(2), 2013/14, pp. 403–440

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RIEMANN AND RIEMANN-TYPE INTEGRATION IN BANACH SPACES

Abstract

Riemann, Riemann-Dunford, Riemann-Pettis and Darboux integrable functions with values in a Banach space and Riemann-Gelfand integrable functions with values in the dual of a Banach space are studied in the light of the work of Graves, Alexiewicz and Orlicz, and Gordon. Various properties of these types of integrals and the interrelation between them are established. The Fundamental Theorem of Integral Calculus for these types of integrals is also studied.

1 Introduction.

Riemann integration of Banach space valued functions was first studied by L. M. Graves [27]. In his survey article [26], R. A. Gordon compiled many results of Graves and others, for example, Alexiewicz and Orlicz [2], and Gordon also established the relation between Riemann integral, Darboux integral, and weak forms of these integrals, most of which are works of Alexiewicz and Orlicz, Rejouani, Nemirovski, Ochan, Rejouani and da Rocha. The aim of this paper is to extend the study of the various properties of these integrals.

In this paper we study Riemann and Riemann-type integrable functions such as Riemann-Dunford, Riemann-Pettis integrable functions with values in a Banach space. A relationship between Riemann, Riemann-Lebesgue,

Mathematical Reviews subject classification: Primary: 26A42, 28B05; Secondary: 46G10
Key words: Riemann integral, Darboux integral, Riemann-Dunford integral, Riemann-Pettis integral, Riemann-Gelfand integral, Bochner integrable equivalent.
Received by the editors June 14, 2013
Communicated by: Luisa Di Piazza

Birkhoff, Strongly-Pettis, and Pettis integrable functions is established in Lemma 2. We have studied Bochner integrable equivalent [31, p. 54, Definition 3.1] of Riemann-Dunford and Riemann-Pettis integrable functions. It is shown that a Riemann-Dunford integrable function with relatively weakly compact range is Riemann-Pettis integrable and has a Bochner integrable equivalent. We have shown that a function of bounded variation is Darboux integrable. It is known that if $f = 0$ almost everywhere on $[a, b]$ with respect to Lebesgue measure then f may not be even Riemann integrable. However we show that f is Darboux integrable if Lebesgue measure is replaced by Jordan content. We study Riemann-Pettis and Darboux integrability of a function when its range is a limited set and establish Fundamental Theorem of Integral Calculus for Riemann-Dunford, Riemann-Pettis, Riemann and Darboux integrable functions. At the end of the paper Riemann-Gelfand integrable functions with values in the dual of a Banach space will have been defined and studied.

2 Notations and Preliminaries.

Throughout the paper, X stands, if not stated otherwise, for a real Banach space with dual X^*. The closed unit ball of X and X^* will be denoted by B_X and B_{X^*} respectively. Also, $[a, b]$ stands for a closed bounded interval of $\mathbb{R}$, Σ stands for the sigma algebra of the Lebesgue measurable subsets of $[a, b]$ and λ the Lebesgue measure on Σ so that $([a, b], \Sigma, \lambda)$ becomes a finite measure space. The Borel sigma algebra of $[a, b]$ will be denoted by $\mathfrak{B}$ so that $\mathfrak{B} \subset \Sigma$. A partition of the interval $[a, b]$ is a finite set of points $\{t_i : 0 \leq i \leq n\}$ in $[a, b]$ that satisfy $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. A tagged partition of $[a, b]$ is a partition $\{t_i : 0 \leq i \leq n\}$ of $[a, b]$ together with a set of points $\{s_i : 1 \leq i \leq n\}$ that satisfy $s_i \in [t_{i-1}, t_i]$ for each i. Let $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$. The points $\{t_i : 0 \leq i \leq n\}$ are called the points of the partition, the intervals $\{[t_{i-1}, t_i] : 1 \leq i \leq n\}$ are called the intervals of the partition, the points $\{s_i : 1 \leq i \leq n\}$ are called the tags of the partition, and the norm $|\mathcal{P}|$ of the partition is defined by $|\mathcal{P}| = \max\{t_i - t_{i-1} : 1 \leq i \leq n\}$. If $f \in X^{[a,b]}$, then $f(\mathcal{P})$ will denote the Riemann sum $\sum_{i=1}^{n} f(s_i)(t_i - t_{i-1})$. Finally, the (tagged)partition $\mathcal{P}_1$ is a refinement of the (tagged)partition $\mathcal{P}_2$ if the points of $\mathcal{P}_2$ form a subset of the points of $\mathcal{P}_1$. In this case we say that $\mathcal{P}_1$ refines $\mathcal{P}_2$.

Let us recall that if f is defined on $[a, b]$ and $\mathcal{P} = \{t_i : 0 \leq i \leq N\}$ is a partition of $[a, b]$, then

$$\omega(f, \mathcal{P}) = \sum_{i=1}^{N} \omega(f, [t_{i-1}, t_i])(t_i - t_{i-1})$$

where $\omega(f, [t_{i-1}, t_i]) = \sup\{\|f(v) - f(u)\| : u, v \in [t_{i-1}, t_i]\}$ is the oscillation of the function f on the interval $[t_{i-1}, t_i]$.

Let (Ω, Σ_1, μ) be a finite measure space. A function $f \in X^{\Omega}$ is called scalarly measurable if x^*f is measurable, for each $x^* \in X^*$. A function $f \in X^{\Omega}$ is called scalarly integrable or Dunford integrable if $x^*f \in L^1(\mu)$, for each $x^* \in X^*$. In this case for each subset $E \in \Sigma_1$, there exists an element $x_E^{**} \in X^{**}$ such that $x_E^{**}(x^*) = \int_E x^* f d\mu$ for each $x^* \in X^*$. The element x_E^{**} is called the Dunford integral of f over E and is denoted by Dunford-$\int_E f d\mu$. A function $f \in X^{\Omega}$ is called Pettis integrable if f is scalarly integrable and for each $E \in \Sigma_1$ there exists an element $x_E \in X$ such that

$$x^*(x_E) = \int_E x^* f d\mu$$

for all $x^* \in X^*$. The element x_E is called the Pettis integral of f over E and is denoted by Pettis-$\int_E f d\mu$. In this case we obtain a countably additive, $\mu-$continuous vector measure $m_f \in X^{\Sigma_1}$ defined by

$$m_f(E) = \text{Pettis-}\int_E f d\mu$$

for $E \in \Sigma_1$ and it is called the induced vector measure of f. It is well known that the range of the induced vector measure of a Pettis integrable function is relatively weakly compact but not necessarily relatively compact in X. The function f is said to be Strongly-Pettis integrable if f is Pettis integrable and the range of the induced vector measure m_f of f is relatively compact in X.

A function $f \in (X^*)^{\Omega}$ is said to be weak* scalarly measurable if for each $x \in X$ the scalar function xf is measurable.

A function $f \in (X^*)^{\Omega}$ is said to be weak* scalarly integrable or Gelfand integrable , if $xf \in L^1(\mu)$ for each $x \in X$. In this case for each $E \in \Sigma_1$ there is an element $x_E^* \in X^*$ such that

$$x_E^*(x) = \int_E x f d\mu$$

for all $x \in X$. The element x_E^* is called the Gelfand integral of f over E and is denoted by $G - \int_E f d\mu$ for all $E \in \Sigma_1$.

For standard results and properties of Bochner, Dunford, Pettis and Gelfand integrable functions we refer to [16]. Throughout the paper, measurability of a function means that the function is measurable with respect to the Lebesgue measure if not stated otherwise.

We say that a function $g \in X^{[a,b]}$ is a Bochner-integrable equivalent of a Pettis integrable function $f \in X^{[a,b]}$ if g is Bochner-integrable and for any measurable set $A \subset [a,b]$, Pettis-$\int_A f d\lambda =$ Bochner-$\int_A g d\lambda$. It can be easily verified that g is a Bochner-integrable equivalent of f if and only if for each $x^* \in X^*$, $x^* f = x^* g$ almost everywhere (the exceptional set may depend on x^*).

A function $f \in X^{[a,b]}$ is said to be differentiable at $t \in [a,b]$ if there exists an element $x \in X$ such that

$$\lim_{\delta \to 0} \left\| \frac{f(t+\delta) - f(t)}{\delta} - x \right\| = 0.$$

In this case x is called the derivative of f at t and is denoted by $f'(t)$. If f is differentiable at each point $t \in [a,b]$, then f is called differentiable on $[a,b]$. Thus the derivative of a differentiable function $f \in X^{[a,b]}$ is again belongs to $X^{[a,b]}$ which is denoted by f'.

Let $f \in X^{[a,b]}$ and $t \in [a,b]$. A vector $z \in X$ is the approximate derivative of f at t if there exists a measurable set $E \subset [a,b]$ that has t as point of density such that

$$\lim_{s \to t, s \in E} \frac{f(s) - f(t)}{s - t} = z.$$

In this case z is called approximate derivative of f at t and is denoted by $f'_{ap}(t) = z$.

A function $f \in X^{[a,b]}$ is said to be scalarly differentiable at $t \in [a,b]$, if for each $x^* \in X^*$, $x^* f$ is differentiable there as a scalar function.

A function $f \in X^{[a,b]}$ is said to be weakly differentiable at a point $t \in [a,b]$ if f is scalarly differentiable at $t \in [a,b]$ and if there exists an $x \in X$ such that for each $x^* \in X^*$, $(x^* f)'(t) = x^*(x)$. In this case x is called the weak derivative of f at t and is denoted by $f'_w(t)$. Thus, for each $x^* \in X^*$, $(x^* f)'(t) = x^*(f'_w(t))$. If f is weakly differentiable at each point $t \in [a,b]$, then f is called weakly differentiable on $[a,b]$. The weak derivative f'_w of a differentiable function $f \in X^{[a,b]}$ again belongs to $X^{[a,b]}$. It can be easily verified that if a function $f \in X^{[a,b]}$ is differentiable at a point $t \in [a,b]$, then it is weakly differentiable there and $f'(t) = f'_w(t)$.

A function $f \in (X^*)^{[a,b]}$ is said to be weak* scalarly differentiable at $t \in [a,b]$, if for each $x \in X$, xf is differentiable there as a scalar function; f is said to be weak* differentiable at $t \in [a,b]$, if f is weak* scalarly differentiable at t

and if there exists an $x^* \in X^*$ such that for each $x \in X$, $(xf)'(t) = x^*(x)$. In this case, x^* is called the weak* derivative of f at t and is denoted by $f'_{w^*}(t)$.

A function $f \in X^{[a,b]}$ is said to satisfy Lipschitz condition if there exists a real number $M > 0$ such that $\|f(x) - f(y)\| \leq M|x - y|$ for all $x, y \in [a, b]$. It is easy to verify that a function $f \in X^{[a,b]}$ satisfying Lipschitz condition is absolutely continuous on $[a, b]$.

3 Main Results.

Definition 1. A function $f \in X^{[a,b]}$ is said to be Riemann integrable on $[a, b]$ if there exists a vector z in X with the following property : for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(\mathcal{P}) - z\| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that satisfies $|\mathcal{P}| < \delta$.

In this case, the vector z is called the Riemann integral of f over $[a, b]$ and is denoted by $R-\int_a^b f(t)dt$ or simply by $R-\int_a^b f dt$. If f is Riemann integrable on $[a, b]$, then it is so on any subinterval $[c, d]$ of $[a, b]$ [26, p. 927, Theorem 7].

The collection of all Riemann integrable functions of $X^{[a,b]}$ will be denoted by $R([a, b], X)$.

If $X = \mathbb{R}$, then $R([a, b], X)$ is denoted by $R[a, b]$.

A Riemann integrable function is Pettis integrable and hence scalarly measurable but not necessarily measurable [26, p. 930, Example 12] and hence not necessarily Borel measurable. However, it is well known that every Riemann integrable function with values in a finite dimensional vector space is measurable but not necessarily Borel measurable.

For definition of Riemann-Lebesgue and Birkhoff integrable functions with values in a Banach space we refer to [30, p. 51, Definition 1.1] and [13, p. 260, Definition 1] respectively. The interrelation between Riemann, Riemann-Lebesgue, Birkhoff, Strongly-Pettis and Pettis integrable functions are revealed in the following Lemma:

Lemma 2. *Let $f \in X^{[a,b]}$. Let us consider the following statements:*

(a) $f \in R([a, b], X)$.

(b) f is Riemann-Lebesgue integrable.

(c) f is Birkhoff integrable.

(d) f is Strongly-Pettis integrable.

(e) f is Pettis integrable.

Then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$.

PROOF. $(a) \Rightarrow (b)$ Follows from [31, p. 53, Theorem 2.4].

$(b) \Rightarrow (c)$ Follows from [30, p. 51] and [13, p. 268, Proposition 2.6].

$(c) \Rightarrow (d)$ It is well known that a Birkhoff integrable function is Pettis integrable. From [13, p. 274, Corollary 3.6] it follows that indefinite integral of a Birkhoff integrable function is relatively norm compact and hence the function is Strongly-Pettis integrable.

$(d) \Rightarrow (e)$ Trivial. □

Definition 3. A function $f \in X^{[a,b]}$ is said to be scalarly Riemann integrable on $[a,b]$ if x^*f is Riemann integrable on $[a,b]$ for each $x^* \in X^*$.

From definition, it follows that a function $f \in X^{[a,b]}$ is scalarly Riemann integrable on $[a,b]$ if and only if for each $x^* \in X^*$, x^*f is bounded and continuous almost everywhere on $[a,b]$.

It is clear that a function is scalarly Riemann integrable on $[a,b]$ if and only if it is so on every subinterval $[c,d]$ of $[a,b]$.

An application of the Uniform Boundedness Principle shows that every scalarly Riemann integrable function is bounded. Also, it is clear that every scalarly Riemann integrable function is Dunford integrable, and hence, it is also said to be Riemann-Dunford integrable. Thus, if $f \in X^{[a,b]}$ is scalarly Riemann integrable on $[a,b]$, then for each $E \in \sum$, there exists an element x_E^{**} in X^{**} such that $x_E^{**}(x^*) = \int_E x^*f d\lambda$ for all $x^* \in X^*$. The element x_E^{**} is called the Riemann-Dunford integral of f over E and is denoted by $RD - \int_E f d\lambda$. Thus $RD - \int_E f d\lambda \in X^{**}$ and $(RD - \int_E f d\lambda)(x^*) = \int_E x^*f d\lambda$ for all $x^* \in X^*$. The induced vector measure of f has clearly bounded average range, and hence it is absolutely continuous and countably additive.

The Riemann-Dunford integral of f over any subinterval $[c,d]$ of $[a,b]$ is denoted by $RD - \int_c^d f(t)dt$ or simply by $RD - \int_c^d f dt$. Now we define $F(t) = RD - \int_a^t f(t)dt$ for all $t \in [a,b]$. Then clearly $F \in (X^{**})^{[a,b]}$ and $x^*F(t) = \int_a^t x^*f(t)dt$ for all $t \in [a,b]$. The function F is said to be the indefinite Riemann-Dunford integral of f over $[a,b]$.

The collection of all scalarly Riemann integrable functions of $X^{[a,b]}$ will be denoted by $RD([a,b],X)$.

Lemma 4. *A function* $f \in RD([a,b],X)$ *if and only if there exists a vector* z'' *in* X^{**} *with the following property: for each* $\epsilon > 0$ *and for each* x^* *in* X^* *there exists* $\delta > 0$ *such that* $|x^*(f(\mathcal{P})) - z''(x^*)| < \epsilon$ *whenever* $\mathcal{P}$ *is a tagged partition of* $[a,b]$ *that satisfies* $|\mathcal{P}| < \delta$.

Theorem 5. *Let $f \in RD([a,b],X)$ and let F be the indefinite Riemann-Dunford integral of f over $[a,b]$. Then the following results hold:*

(a) F satisfies Lipschitz condition and hence is absolutely continuous on $[a,b]$.

(b) If for some $x^* \in X^*$, x^*f is continuous at a point $t \in [a,b]$, then x^*F is differentiable at that point and $(x^*F)'(t) = x^*f(t)$.

(c) For each $x^* \in X^*$, x^*F is differentiable almost everywhere on $[a,b]$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$ (the exceptional set may depend on x^*).

(d) At each point t of weak continuity of f, the function F is weakly differentiable and $F'_w(t) = f(t)$.

(e) At each point t of continuity of f, the function F is differentiable and $F'(t) = f(t)$.

(f) If F is approximately differentiable at $t \in [a,b]$, then F is differentiable thereat.

PROOF. (a) We have, for $t_1, t_2 \in [a,b]$ with $t_1 < t_2$,

$$\begin{aligned}\|F(t_2) - F(t_1)\| &= \left\| RD - \int_{t_1}^{t_2} f dt \right\| \\ &= \sup_{x^* \in B_{X^*}} \left| x^* \left(\int_{t_1}^{t_2} f dt \right) \right| \\ &\leq \sup_{x^* \in B_{X^*}} \int_{t_1}^{t_2} |x^* f| dt \\ &\leq M(t_2 - t_1)\end{aligned}$$

where M is an upper bound of f on $[a,b]$. This shows that F satisfies Lipschitz condition and hence is absolutely continuous.

(b) Let $x^* \in X^*$. Since $F(t) = \int_a^t f(t)dt$, $x^*F(t) = \int_a^t x^*f(t)dt$. So if x^*f is continuous at the point $t \in [a,b]$, then x^*F must be differentiable at that point and $(x^*F)'(t) = x^*f(t)$.

(c) Since $x^*f \in R[a,b]$, x^*f is continuous almost everywhere for each $x^* \in X^*$ and hence by part (b), x^*F is differentiable almost everywhere on $[a,b]$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$.

(d) If t is a point of weak continuity of f, then x^*f is continuous at t for all $x^* \in X^*$. Hence, the result follows from (b).

(e) Let f be continuous at a point $t_0 \in [a,b]$. Then, x^*f is continuous at the point t_0 for all $x^* \in X^*$. Therefore, by part (b), $(x^*F)'(t_0) = x^*f(t_0)$ for all $x^* \in X^*$. Since f is continuous at t_0, $\{x^*f : x^* \in B_{X^*}\}$ is equi-continuous at t_0. Therefore, for $\epsilon > 0$ there is a $\delta > 0$ such that for all $x^* \in X^*$, $|x^*f(t) - x^*f(t_0)| < \epsilon$ whenever $|t - t_0| < \delta$.

Now

$$\begin{aligned}
\left\| \frac{F(t_0+h) - F(t_0)}{h} - f(t_0) \right\| &= \left\| \frac{1}{h} \int_{t_0}^{t_0+h} (f(t) - f(t_0))dt \right\| \\
&= \frac{1}{|h|} \sup_{x^* \in B_{X^*}} \left| x^* \left(\int_{t_0}^{t_0+h} (f(t) - f(t_0))dt \right) \right| \\
&\leq \frac{1}{|h|} \sup_{x^* \in B_{X^*}} \int_{t_0}^{t_0+h} |x^*f(t) - x^*f(t_0)|dt \\
&< \frac{1}{|h|} \epsilon |h| = \epsilon \quad \text{whenever} \quad |h| < \delta
\end{aligned}$$

Hence, $\left\| \frac{F(t_0+h)-F(t_0)}{h} - f(t_0) \right\| < \epsilon$ whenever $|h| < \delta$. This shows that F is differentiable at t_0 and $F'(t_0) = f(t_0)$. This is true for every point of continuity of f and the result follows.

(f) Follows from (a) and [25, p. 80, Theorem 27]. □

Note: It is well known that approximate differentiability of a function $F \in X^{[a,b]}$ at a point $t \in [a,b]$ need not imply differentiability at that point. However, in the above theorem, we have shown that this is true if F is the indefinite Riemann-Dunford integral of a function $f \in RD([a,b], X)$.

Definition 6. A function $f \in X^{[a,b]}$ is said to be Riemann-Pettis integrable on $[a,b]$ if it is scalarly Riemann integrable and Pettis integrable.

If f is Riemann-Pettis integrable, then Pettis-$\int_a^b f(t)dt$ is defined to be the Riemann-Pettis integral of f over $[a,b]$ and is denoted by $RP{-}\int_a^b f(t)dt$ or simply by $RP{-}\int_a^b fdt$. It is clear that every Riemann integrable function over $[a,b]$ is Riemann-Pettis integrable over $[a,b]$ and $R - \int_a^b f(t)dt = RP - \int_a^b f(t)dt$. Also, a Riemann-Pettis integrable function over $[a,b]$ is Riemann-Dunford integrable over $[a,b]$ and $RD - \int_c^d fdt = RP - \int_c^d fdt$ for all subintervals $[c,d]$ of $[a,b]$.

The collection of all Riemann-Pettis integrable functions of $X^{[a,b]}$ will be denoted by $RP([a,b], X)$.

Lemma 7. *A function $f \in RP([a,b],X)$ if and only if there exists a vector z in X with the following property: for each $\epsilon > 0$ and for each x^* in X^* there exists $\delta > 0$ such that $|x^*(f(\mathcal{P})) - x^*(z)| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that satisfies $|\mathcal{P}| < \delta$.*

From the very definition, we have the following result:

Theorem 8. *If X has Lebesgue PIP, then $RD([a,b],X) = RP([a,b],X)$.*

Since Mazur property implies Lebesgue PIP, the following result follows from the above theorem:

Corollary 9. If X has Mazur property, then $RD([a,b],X) = RP([a,b],X)$.

Theorem 10. *Let $f \in RP([a,b],X)$. If $RP-\int_a^b f$ is a point of sequential continuity of the set consisting of all the Riemann sums of f, then $f \in R([a,b],X)$.*

PROOF. Let us assume that $f \notin R([a,b],X)$. Then there exists a positive real number η such that for each positive real number δ there exists a tagged partition $\mathcal{P}_\delta$ of $[a,b]$ such that $|\mathcal{P}_\delta| < \delta$ and $\|f(\mathcal{P}_\delta) - (RP - \int_a^b f dt)\| > \eta$. So for each positive integer n, we choose a tagged partition $\mathcal{P}_n$ of $[a,b]$ such that $|\mathcal{P}_n| < \frac{1}{n}$ and $\|f(\mathcal{P}_n) - (RP - \int_a^b f dt)\| > \eta$. This implies that the sequence $\{f(\mathcal{P}_n)\}$ of Riemann sums of f can not converge in norm to $RP - \int_a^b f dt$. Let $x^* \in X^*$. Since $f \in RP([a,b],X)$, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| x^* f(P) - \int_a^b x^* f dt \right| < \epsilon \text{ whenever } \ |P| < \delta.$$

Let n_0 be a positive integer $> \frac{1}{\delta}$. Then, for $n > n_0$, we have $\frac{1}{n} < \frac{1}{n_0} < \delta$ which implies that $|\mathcal{P}_n| < \delta$. Hence

$$\left| x^* f(\mathcal{P}_n) - x^* \left(RP - \int_a^b f dt \right) \right| < \epsilon \quad \text{ for all } \ n > n_0.$$

This shows that $f(P_n) \to RP - \int_a^b f dt$ weakly. Since $RP - \int_a^b f$ is a point of sequential continuity of the set of all Riemann sums, it follows that $f(\mathcal{P}_n) \to RP - \int_a^b f dt$ in norm - a contradiction. Hence, we must have $f \in R([a,b],X)$. $\square$

Let us recall the following definition from [24, p. 540, Definition 3.1]:

Let $f \in X^{[a,b]}$, let $B \in \Sigma$ and π be a finite partition of B into Lebesgue measurable sets. Let $S_f(\pi, B)$ denote the closure of the convex set $\sum_{E \in \pi} \overline{co} f(E) \lambda(E)$.

If π be a finite partition of B into Jordan measurable sets, then we use the notation $J_f(\pi, B)$ instead of $S_f(\pi, B)$.

For any $f \in RD([a,b], X)$ and $E \in \Sigma$, we define $J_f(E) = \{x \in X : x^*(x) = \int_E x^* f d\lambda$, for all $x^* \in X^*\}$.

Lemma 11. *Let $f \in RD([a,b], X)$. Then, for any subinterval $[c,d]$ of $[a,b]$, $\bigcap_{\pi\in\Pi} S_f(\pi, [c,d]) \subset \bigcap_{\pi\in\Pi'} J_f(\pi, [c,d]) \subset J_f([c,d])$, where Π and Π' are the collection of all finite partitions of $[c,d]$ into Lebesgue and Jordan measurable subsets respectively of $[c,d]$.*

PROOF. Let $[c,d]$ be any subinterval of $[a,b]$. The first inclusion trivially follows from the fact that each Jordan measurable subset of any subinterval $[c,d]$ of $[a,b]$ is a Lebesgue measurable subset of $[c,d]$.

If $\bigcap_{\pi\in\Pi'} J_f(\pi, [c,d])$ is a null set, then the second inclusion is trivial. So we assume that $\bigcap_{\pi\in\Pi'} J_f(\pi, [c,d])$ is not a null set.

Let z be an arbitrary element of $\bigcap_{\pi\in\Pi'} J_f(\pi, [c,d])$ so that $z \in J_f(\pi, [c,d])$ for all $\pi \in \Pi'$. Let $x^* \in X^*$ be fixed and $\epsilon > 0$. Since $f \in RD([a,b], X)$, we must have $f \in RD([c,d], X)$. So $x^* f \in R[c,d]$ and hence it is the uniform limit of a sequence of functions of the form $\sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where A_i is a Jordan measurable subset of $[c,d]$ for $i = 1, 2, ..., n$ [42, p. 628, Theorem 2](vide [5, p. 395, Theorem]). So we have a finite partition $\pi \in \Pi'$ of $[c,d]$ such that if E is in π and if u, v are points in E, then $|x^* f(u) - x^* f(v)| < \epsilon$. Now proceeding as in the proof of [24, p. 541, Theorem 3.2], we must have $x^*(z) = \int_c^d x^* f(t) dt$. This implies that $z \in \{x \in X : x^*(x) = \int_c^d x^* f(t) dt\}$. This is true for each $x^* \in X^*$. Hence, $z \in J_f([c,d])$ and the result follows. □

Theorem 12. *Let $f \in RD([a,b], X)$. Then the following statements are equivalent:*

(a) $f \in RP([a,b], X)$.

(b) $J_f(E) \neq \phi$ for all $E \in \Sigma$.

(c) $J_f(E) \neq \phi$ for all $E \in \mathfrak{B}$.

(d) $J_f(E) \neq \phi$ for all Jordan measurable subsets E of $[a,b]$.

(e) $J_f([c,d]) \neq \phi$ for any subinterval $[c,d]$ of $[a,b]$.

(f) *For any subinterval* $[c,d]$ *of* $[a,b]$*, the set* $\bigcap_{\pi\in\Pi} S_f(\pi,[c,d])$ *is non-empty where* Π *is the collection of all finite partitions of* $[c,d]$ *into Lebesgue measurable subsets of* $[c,d]$*.*

(g) *For any subinterval,* $[c,d]$ *of* $[a,b]$*, the set* $\bigcap_{\pi\in\Pi'} J_f(\pi,[c,d])$ *is non-empty where* Π' *is the collection of all finite partitions of* $[c,d]$ *into Jordan measurable subsets of* $[c,d]$*.*

(h) $J_f(E)$ *is a singleton set for all* $E\in\Sigma$*.*

(i) $J_f(E)$ *is a singleton set for all* $E\in\mathfrak{B}$*.*

(j) $J_f(E)$ *is a singleton set for all Jordan measurable subsets* E *of* $[a,b]$*.*

(k) $J_f([c,d])$ *is a singleton set for any subinterval* $[c,d]$ *of* $[a,b]$*.*

(l) *For any subinterval* $[c,d]$ *of* $[a,b]$*,* $\bigcap_{\pi\in\Pi} S_f(\pi,[c,d])$ *is a singleton set where* Π *is the collection of all finite partitions of* $[c,d]$ *into Lebesgue measurable subsets of* $[c,d]$*.*

(m) *For any subinterval* $[c,d]$ *of* $[a,b]$*,* $\bigcap_{\pi\in\Pi'} J_f(\pi,[c,d])$ *is a singleton set where* Π' *is the collection of all finite partitions of* $[c,d]$ *into Jordan measurable subsets of* $[c,d]$*.*

(n) *For any subinterval* $[c,d]$ *of* $[a,b]$*,* $f\in RP([c,d],X)$*.*

Proof. $(a)\Rightarrow(b)$ Let $f\in RP([a,b],X)$. Then for each $E\in\Sigma$ there exists an element $x_E\in X$ such that $x^*(x_E)=\int_E x^* f\,d\lambda$ for each $x^*\in X^*$. This shows that $x_E\in J_f(E)$, and hence, $J_f(E)\neq\phi$ for any $E\in\Sigma$. Therefore, (b) follows.

$(b)\Rightarrow(c)$ and $(h)\Rightarrow(i)$ Follow from the fact that $\mathfrak{B}\subset\Sigma$.

$(c)\Rightarrow(e)$ and $(i)\Rightarrow(k)$ Follow from the fact that any subinterval $[c,d]$ of $[a,b]$ is contained in $\mathfrak{B}$.

$(e)\Rightarrow(n)$ From hypothesis, we have $J_f([s,t])\neq\phi$ for any subinterval $[s,t]$ of $[c,d]$. Hence, the result follows from [39, p. 133, Theorem A.4.15] and the fact that the induced vector measure of f is countably additive.

$(n)\Rightarrow(a)$ Trivial, as $[a,b]$ itself is a subinterval of $[a,b]$.

$(b)\Rightarrow(d)$ and $(h)\Rightarrow(j)$ Follow from the fact that every Jordan measurable subset of $[a,b]$ is contained in Σ

$(d)\Rightarrow(e)$ and $(j)\Rightarrow(k)$ Follow from the fact that any subinterval $[c,d]$ of $[a,b]$ is Jordan measurable.

$(a) \Rightarrow (h)$ Let $E \in \Sigma$. Then, $J_f(E) \neq \phi$ by (b). If possible, let $x_1, x_2 \in J_f(E)$. Then,

$$x^*(x_1) = \int_E x^* f = x^*(x_2), \text{for all } \; x^* \in X^*$$

$$\text{i.e. } x^*(x_1) = x^*(x_2), \text{for all } \; x^* \in X^*.$$

Therefore $x_1 = x_2$ and hence (h) follows.

$(k) \Rightarrow (e)$ Trivial.

$(a) \Rightarrow (f)$ Follows from [24, p. 541, Theorem 3.2].

$(f) \Rightarrow (g) \Rightarrow (e)$ Follow from Lemma 11.

$(f) \Rightarrow (l)$ Let $[c,d]$ be any subinterval of $[a,b]$ and let the set $\bigcap_{\pi \in \Pi} S_f(\pi, [c,d])$ be non-empty. Since (f) and (k) are equivalent, the set $J_f([c,d])$ is a singleton set. Thus $\bigcap_{\pi \in \Pi} S_f(\pi, [c,d])$ is a non-empty subset of the singleton set $J_f([c,d])$, by Lemma 11. Hence $\bigcap_{\pi \in \Pi} S_f(\pi, [c,d])$ must be a singleton set.

$(l) \Rightarrow (f)$ Trivial.

$(g) \Rightarrow (m)$ Similar to the proof of $(f) \Rightarrow (l)$.

$(m) \Rightarrow (g)$ Trivial. □

Note. In view of the above theorem $((a) \Leftrightarrow (n))$ and [26, p. 944 and p. 927, Theorem 7(d)], it follows that $R([a,b],X) \subset RP([a,b],X)$ and for any $f \in R([a,b],X)$, $R - \int_c^d f(t)dt = RP - \int_c^d f(t)dt$ for any subinterval $[c,d]$ of $[a,b]$.

Corollary 13. Let $f \in RD([a,b],X)$. Then, the following statements are equivalent:

(a) $f \in RP([a,b],X)$.

(b) For $t \in [a,b]$, $\bigcap_{\pi \in \Pi} S_f(\pi, [a,t]) \neq \phi$, where Π is the collection of all finite partitions of $[a,t]$ into Lebesgue measurable subsets of $[a,t]$.

(c) For $t \in [a,b]$, $\bigcap_{\pi \in \Pi'} J_f(\pi, [a,t]) \neq \phi$, where Π' is the collection of all finite partitions of $[a,t]$ into Jordan measurable subsets of $[a,t]$.

(d) The indefinite Riemann-Dunford integral F of f over $[a,b]$ takes its values in X.

Proof. $(a) \Rightarrow (b)$ Follows from the above theorem.

$(b) \Rightarrow (c)$ Follows from Lemma 11.

$(c) \Rightarrow (d)$ For $t \in [a,b]$, let $\bigcap_{\pi \in \Pi'} J_f(\pi, [a,t]) \neq \phi$. Then, there is an element $z \in X$ such that $z \in \bigcap_{\pi \in \Pi'} J_f(\pi, [a,t])$. Hence, by Lemma 11, $z \in J_f([a,t])$. So $x^*(z) = \int_a^t x^* f(t)dt = x^* F(t)$ for all $x^* \in X^*$ which implies that $F(t) = z \in X$ and the result follows.

$(d) \Rightarrow (a)$ For any subinterval $[c,d]$ of $[a,b]$, $F(c), F(d) \in X$. Let $x = F(d) - F(c)$ so that $x \in X$ and for any $x^* \in X^*$,

$$\begin{aligned} x^*(x) &= x^*(F(d) - F(c)) \\ &= x^*(F(d)) - x^*(F(c)) \\ &= x^*(\int_a^d f(t)dt) - x^*(\int_a^c f(t)dt) \\ &= \int_a^d x^* f(t)dt - \int_a^c x^* f(t)dt \\ &= \int_c^d x^* f(t)dt, \end{aligned}$$

which implies that $x \in J_f([c,d])$. Thus, for each subinterval $[c,d]$ of $[a,b]$, $J_f([c,d]) \neq \phi$, and the result follows from the above theorem ($(e) \Rightarrow (a)$). $\square$

Corollary 14. If $f \in RP([a,b], X)$, then for any subinterval $[c,d]$ of $[a,b]$, $\bigcap_{\pi \in \Pi} S_f(\pi, [c,d]) = \bigcap_{\pi \in \Pi'} J_f(\pi, [c,d]) = J_f([c,d])$.

Proof. If $f \in RP([a,b], X)$, then by Theorem 12, for any subinterval $[c,d]$ of $[a,b]$, the set $\bigcap_{\pi \in \Pi} S_f(\pi, [c,d])$ is non-empty and the set $J_f([c,d])$ is singleton. Thus, from Lemma 11, the set $\bigcap_{\pi \in \Pi} S_f(\pi, [c,d])$ is a non-empty subset of the singleton set $J_f([c,d])$. Hence by the same lemma, we have
$J_f([c,d]) = \bigcap_{\pi \in \Pi} S_f(\pi, [c,d])$ and the result follows. $\square$

Following [24, p. 543, Definition 4.2], let us define for any bounded function $f \in X^{[a,b]}$ and for any Borel measurable set E of $[a,b]$,

$$H_f(E) = \bigcap_{A \subset E} \overline{co}[f(E \setminus A)\lambda(E) + 3M\lambda(A)B_X],$$

where the intersections are taken over all Borel measurable proper subsets of E and M is the supremum of $\|f(t)\|$ in $[a,b]$ which may be assumed to be positive; otherwise $f(t)=0$ for all $t\in[a,b]$ which is a trivial case.

Theorem 15. *Let $f\in RD([a,b],X)$ be such that x^*f is Borel measurable for each $x^*\in X^*$. Then*

(a) for each Borel measurable subset E of $[a,b]$, $H_f(E)$ is compact.

(b) $f\in RP([a,b],X)$ if and only if $H_f(E)\neq\phi$ for every Borel measurable subset E of $[a,b]$ of positive measure.

PROOF. (a) Let $g(t)=\frac{1}{M}f(t)$ for all $t\in[a,b]$. Then $g\in RD([a,b],X)$ and g takes its values in the unit ball B_X of X. Also x^*f is Borel measurable for each $x^*\in X^*$. It is easy to verify that $H_f(E)=MH_g(E)$, for each Borel measurable subset E of $[a,b]$. Since Lebesgue measure on the Borel measurable subsets of $[a,b]$ is perfect, it follows from [24, p. 544, Theorem 4.4] that $H_g(E)$ is compact. Then $H_f(E)$ is clearly compact.

(b) If g is as in (a) above, then clearly $g\in RP([a,b],X)$ if and only if $f\in RP([a,b],X)$, and $H_f(E)\neq\phi$ if and only if $H_g(E)\neq\phi$ for every Borel measurable subset E of $[a,b]$. According to [24, p. 545, Theorem 4.6], $g\in RP([a,b],X)$ if and only if $H_g(E)\neq\phi$ for every Borel measurable set $E\subset[a,b]$ of positive measure. Hence, the result follows immediately. □

Let $f\in RP([a,b],X)$ and let us define the function F by

$$F(t)=RP-\int_a^t f(t)dt \text{ for all } t\in[a,b].$$

Then, from the above theorem, it follows that $F\in X^{[a,b]}$. The function F is said to be the indefinite Riemann-Pettis integral of f over $[a,b]$.

Let us recall that a Banach space X is called a Gelfand space if each absolutely continuous function $f\in X^{[a,b]}$ is differentiable almost everywhere on $[a,b]$.

Theorem 16. *Let X be a Gelfand space and let $f\in RP([a,b],X)$. If F be the indefinite Riemann-Pettis integral of f over $[a,b]$, then F is differentiable almost everywhere on $[a,b]$ and at each point of weak continuity of f, $F'(t)=f(t)$.*

PROOF. From Theorem 5, F is absolutely continuous. Hence, from the definition of the Gelfand space it follows that F is differentiable almost everywhere on $[a,b]$. The next part follows from Theorem 5(d). □

Theorem 17. *Let X be separable and $f \in X^{[a,b]}$. Let us consider the following statements:*

(a) $f \in RD([a,b],X)$.

(b) $f \in RP([a,b],X)$.

(c) f *is bounded and measurable and hence Bochner integrable.*

Then $(a) \Leftrightarrow (b) \Rightarrow (c)$.

Proof. $(a) \Rightarrow (b)$ Follows from [2, p. 130, Theorem 3(a)].

$(b) \Rightarrow (a)$ Trivial.

$(a) \Rightarrow (c)$ Since $f \in RD([a,b],X)$, f is bounded. Again, since X is separable, f is measurable. Therefore, f is Bochner integrable. □

Theorem 18. *Let B_{X^*} be separable and $f \in X^{[a,b]}$. Let us consider the following statements:*

(a) $f \in RD([a,b],X)$.

(b) $f \in RP([a,b],X)$.

(c) f *is bounded and weakly continuous almost everywhere.*

(d) f *is bounded and measurable and hence Bochner integrable.*

Then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (c)$ Since B_{X^*} is separable, there exists a countable subset $\{x_n^*\}$ which is dense in B_{X^*}. Then, for each n, $x_n^* f \in R[a,b]$ and hence $x_n^* f$ is continuous almost everywhere on $[a,b]$. So there exists a Lebesgue null set $E_n \subset [a,b]$ such that $x_n^* f$ is continuous on $[a,b] \setminus E_n$. Let $E = \cup_{n=1}^{\infty} E_n$. Then E is Lebesgue null and each $x_n^* f$ is continuous on $[a,b] \setminus E$. Let $x^* \in X^*$. Let $\epsilon > 0$. Then, $\frac{x^*}{2\|x^*\|} \in B_{X^*}$. So there exists an $x_k^* \in \{x_n^*\}$ such that

$$\left\| x_k^* - \frac{x^*}{2\|x^*\|} \right\| < \frac{\epsilon}{8M\|x^*\|} \quad \text{where} \quad M = \sup_{t \in [a,b]} \|f(t)\|.$$

Let $t_0 \in [a,b] \setminus E$. Since $x_k^* f$ is continuous on $[a,b] \setminus E$, there exists a $\delta > 0$ such that

$$|x_k^* f(t) - x_k^* f(t_0)| < \frac{\epsilon}{4\|x^*\|} \quad \text{whenever} \quad |t - t_0| < \delta.$$

Now,

$$
\begin{aligned}
&|x^*f(t) - x^*f(t_0)| = 2\|x^*\| \left| \frac{x^*}{2\|x^*\|} f(t) - \frac{x^*}{2\|x^*\|} f(t_0) \right| \\
&= 2\|x^*\| \left| \frac{x^*}{2\|x^*\|} f(t) - x_k^* f(t) + x_k^* f(t) - x_k^* f(t_0) + x_k^* f(t_0) - \frac{x^*}{2\|x^*\|} f(t_0) \right| \\
&\leq 2\|x^*\| \left[\left| \frac{x^*}{2\|x^*\|} f(t) - x_k^* f(t) \right| + |x_k^* f(t) - x_k^* f(t_0)| \right. \\
&\qquad\qquad\qquad\qquad \left. + \left| x_k^* f(t_0) - \frac{x^*}{2\|x^*\|} f(t_0) \right| \right] \\
&\leq 2\|x^*\| \left[\left\| \frac{x^*}{2\|x^*\|} - x_k^* \right\| \|f(t)\| + |x_k^* f(t) - x_k^* f(t_0)| + \left\| x_k^* - \frac{x^*}{2\|x^*\|} \right\| \|f(t_0)\| \right] \\
&\leq 2\|x^*\| \left[\left\| \frac{x^*}{2\|x^*\|} - x_k^* \right\| M + |x_k^* f(t) - x_k^* f(t_0)| + \left\| x_k^* - \frac{x^*}{2\|x^*\|} \right\| M \right] \\
&= 2\|x^*\| \left[\left\| \frac{x^*}{2\|x^*\|} - x_k^* \right\| 2M + |x_k^* f(t) - x_k^* f(t_0)| \right] \\
&< 2\|x^*\| \left[2\frac{\epsilon M}{8M\|x^*\|} + \frac{\epsilon}{4\|x^*\|} \right] \quad \text{whenever} \quad |t - t_0| < \delta \\
&= \epsilon.
\end{aligned}
$$

This shows that $x^*f(t)$ is continuous at t_0 and since $t_0 \in [a,b] \setminus E$ is arbitrary, x^*f is continuous on $[a,b] \setminus E$. Thus, x^*f is continuous on $[a,b] \setminus E$ for each $x^* \in X^*$ and E is Lebesgue null. This shows that f is weakly continuous almost everywhere on $[a,b]$. Hence, (c) follows.

$(c) \Rightarrow (b)$ Follows from [26, p. 944, Corollary 30].

$(b) \Rightarrow (a)$ Trivial.

$(c) \Rightarrow (d)$ Follows from [10, p. 246, Remark 2.8]. □

From the above theorem, the following result follows which is a generalization of [45, p. 153, Corollary 7].

Corollary 19. Let X have a separable dual and $f \in X^{[a,b]}$. Let us consider the following statements:

(a) $f \in RD([a,b],X)$.

(b) $f \in RP([a,b],X)$.

(c) f is bounded and weakly continuous almost everywhere.

(d) f is bounded and measurable and hence Bochner integrable.

Then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d)$.

Theorem 20. *Let $f \in X^{[a,b]}$ be bounded and weakly continuous almost everywhere on $[a,b]$. Then*

(a) $f \in RP([a,b],X)$ and measurable and hence Bochner integrable.

(b) The indefinite Riemann-Pettis integral F of f is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$.

(c) The indefinite Riemann-Pettis integral F of f is constant on $[a,b]$ if and only if $f = 0$ almost everywhere on $[a,b]$.

PROOF. (a) Follows from [26, p. 944, Corollary 30] and [10, p. 246, Remark 2.8].

(b) Since f is Bochner integrable, the result follows from [28, p. 88, After Corollary 2].

(c) From (b), F is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$. If F is constant on $[a,b]$ then $F' = 0$ on $[a,b]$ and hence $f = 0$ almost everywhere on $[a,b]$. The converse part is obvious. □

Corollary 21. Let $f \in X^{[a,b]}$ be weakly continuous on $[a,b]$. Then

(a) $f \in RP([a,b],X)$ and measurable and hence Bochner integrable.

(b) The indefinite Riemann-Pettis integral F of f is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$. Also F is weakly differentiable on $[a,b]$ and $F'_w = f$ on $[a,b]$.

(c) The indefinite Riemann-Pettis integral F of f vanishes on $[a,b]$ if and only if f vanishes on $[a,b]$.

PROOF. (a) Follows from Theorem 20.

(b) First part follows from Theorem 20 and the second part follows from Theorem 5(d).

(c) If F vanishes on $[a,b]$, it is differentiable on $[a,b]$ and $F' = 0$ everywhere on $[a,b]$. So F is weakly differentiable on $[a,b]$ and $F'_w = 0$ on $[a,b]$. Hence by (b), f vanishes on $[a,b]$. The converse part is obvious. □

Corollary 22. If $f \in X^{[a,b]}$ is a function such that x^*f is convex for each $x^* \in X^*$, then $f \in RP([a,b],X)$ and is measurable and hence Bochner integrable.

PROOF. From hypothesis, it follows that for each $x^* \in X^*$, x^*f is bounded on $[a,b]$ [44, p. 18] and is continuous on (a,b) [44, p. 18, Theorem 2.5]. This implies that f is bounded and weakly continuous almost everywhere on $[a,b]$. Hence, the result follows from Theorem 20. □

Theorem 23. *Let $f \in X^{[a,b]}$ be bounded and $x^*f = 0$ almost everywhere on $[a,b]$ with respect to Jordan content for all $x^* \in X^*$ such that the exceptional set depends upon $x^* \in X^*$. Then, $f \in RP([a,b],X)$ and $RP - \int_c^d f dt = 0$ for all subintervals $[c,d]$ of $[a,b]$.*

PROOF. Since $x^*f = 0$ almost everywhere on $[a,b]$ with respect to Jordan content for all $x^* \in X^*$, $x^*f \in D[a,b]$ for all $x^* \in X^*$. Therefore, $f \in RD([a,b],X)$ and $\int_c^d x^*f dt = 0$ for all subsets $[c,d]$ of $[a,b]$ and for all $x^* \in X^*$. So $x^*(RD - \int_c^d f dt) = 0$ for all $x^* \in X^*$. Therefore $RD - \int_c^d f dt = 0 \in X$ which implies that $RD - \int_c^d f dt \in J_f([c,d])$. So $J_f([c,d]) \neq \phi$ and the result follows from Theorem 12($(a) \Leftrightarrow (e)$) and the fact that $RD - \int_c^d f dt = RP - \int_c^d f dt$ for all subintervals $[c,d]$ of $[a,b]$. □

Theorem 24. *Let $f \in RD([a,b],X)$. Then, the following statements are equivalent:*

(a) $f \in RP([a,b],X)$.

(b) There exists a function $F \in X^{[a,b]}$ which satisfies Lipschitz condition such that for each $x^* \in X^*$, x^*F is derivable and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$.

(c) There exists an absolutely continuous function $F \in X^{[a,b]}$ such that for each $x^* \in X^*$, x^*F is derivable and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$.

PROOF. $(a) \Rightarrow (b)$ Let $f \in RP([a,b],X)$ and let F be the indefinite integral of f over $[a,b]$. Then the result follows from Theorem 5(a) and (c).

$(b) \Rightarrow (c)$ Follows from the fact that if a function satisfies Lipschitz condition, then it must be absolutely continuous.

$(c) \Rightarrow (a)$ Follows from [35, p. 746, Theorem 5.1]. □

We know that a Riemann integrable function with values in a Banach space may not be Bochner integrable. The following example shows that a Riemann integrable function with values in a Banach space may not even have a Bochner-integrable equivalent.

Example 25. Let us consider the function $f : [0,1] \to L_\infty[0,1]$ defined by $f(t) = \chi_{[t,1]}$, for $t \in [0,1]$. Then, f is Riemann integrable but does not have a Bochner-integrable equivalent [31, p. 55–57, Example].

Theorem 26. *If X has the Radon-Nikodym property, then any $f \in RP([a,b],X)$ has a Bochner-integrable equivalent.*

PROOF. Since $f \in RP([a,b],X)$, f is bounded. Hence, the induced vector measure m_f of f has bounded average range, and hence is countably additive and $\lambda-$continuous. Since X has the Radon-Nikodym property, there exists a Bochner integrable function $g \in X^{[a,b]}$ such that $m_f(E) = \text{Bochner-}\int_E g d\lambda$, for all $E \in \Sigma$. Thus, $RP - \int_E f d\lambda = \text{Bochner-}\int_E g d\lambda$. Hence, by definition g is a Bochner integrable equivalent of f. $\square$

Corollary 27. Let X have an unconditional basis and contains no copy of c_0. If $f \in RD([a,b],X)$, then $f \in RP([a,b],X)$ and f has a Bochner-integrable equivalent.

PROOF. It follows from [34, p. 22, Theorem 1.c.10] that X is weakly sequentially complete and the basis of X is boundedly complete which implies that X has Radon-Nikodym property [16, p. 64, Theorem III.1.6]. Hence, from [26, p. 944, Theorem 31] it follows that $f \in RP([a,b],X)$. The second part follows from Theorem 26. $\square$

The following result is analogous to [31, p. 58, Theorem 3.6] whose proof is same as that of the said result and so we omit it.

Theorem 28. *Let X be a Banach space such that every $f \in RP([a,b],X)$ has a Bochner-integrable equivalent. Let there exist a countable set of bounded linear functionals on X separating the points of X. Then every $f \in RP([a,b],X)$ is Bochner integrable.*

We have already noted that a bounded function that is weakly continuous almost everywhere on $[a,b]$ is Riemann-Pettis integrable, measurable, and hence Bochner integrable. However we have the following result:

Theorem 29. *If $f \in X^{[a,b]}$ is such that x^*f is continuous almost everywhere on $[a,b]$ for each $x^* \in X^*$ and f has a relatively weakly compact range, then $f \in RP([a,b],X)$ and has a Bochner-integrable equivalent.*

PROOF. Since f has a relatively weakly compact range, f is bounded and hence x^*f is bounded for each $x^* \in X^*$. Since x^*f is continuous almost everywhere, x^*f is Riemann integrable for each $x^* \in X^*$. Hence, f is scalarly measurable. Therefore, it follows from [14, p. 259, Corollary 19] that f is Pettis integrable, and hence, $f \in RP([a,b],X)$ and f has a Bochner integrable equivalent. $\square$

The following result is analogous to [2, p. 130, Theorem 4](vide [26, p. 945, Theorem 32]), which follows from Theorem 29.

Corollary 30. If $f \in RD([a,b],X)$ and has a relatively weakly compact range, then $f \in RP([a,b],X)$ and has a Bochner-integrable equivalent.

Corollary 31. If $f \in X^{[a,b]}$ is such that $x^* f$ is continuous almost everywhere for each $x^* \in X^*$ and *weak**-closure of $f([a,b])$ is contained in X (when considered as a subset of X^{**}), then $f \in RP([a,b],X)$ and has a Bochner-integrable equivalent.

PROOF. The result follows from [12, p. 147, Exercise 3] and Theorem 29. □

Corollary 32. Let X be weakly sequentially complete and contain no subspace isomorphic to l^1. Then $f \in RD([a,b],X)$ if and only if $f \in RP([a,b],X)$ and has a Bochner-integrable equivalent.

PROOF. Follows from [29, p. 351, Theorem $((b) \Leftrightarrow (d))$] and Corollary 31. □

Recall that a subset A of a Banach space X is said to be limited if $\lim_{n\to\infty} \sup_{x\in A} |x_n^*(x)| = 0$, for every weak* null sequence $\{x_n^*\}$ in X^*.

Corollary 33. Let $f \in X^{[a,b]}$ be such that $x^* f$ is continuous almost everywhere on $[a,b]$ and $f([a,b])$ is a limited set in X. If X contains no copy of l^1, then $f \in RP([a,b],X)$ and has a Bochner-integrable equivalent.

PROOF. Since X contains no copy of l^1 and $f([a,b])$ is a limited set in X, $f([a,b])$ is relatively weakly compact [9, p. 55, Proposition (7)]. Hence, the result follows from Theorem 29. □

Since every bounded subset of a reflexive Banach space is relatively weakly compact, we have the following result which follows from Corollary 30 and [1, p. 193].

Corollary 34. Let X be a reflexive Banach space. Then, $f \in RD([a,b],X)$ if and only if $f \in RP([a,b],X)$ and has a Bochner-integrable equivalent. In this case, the indefinite integral of f is differentiable a.e. on $[a,b]$.

Corollary 35. Let X be a Hilbert space and let $f \in X^{[a,b]}$. Then the following statements are equivalent:

(a) $f \in RD([a,b],X)$.

(b) $< x, f > \in R[a,b]$ for each $x \in X$.

(c) $f \in RP([a,b],X)$ and has a Bochner-integrable equivalent.

Proof. $(a) \Leftrightarrow (b)$ Follows from the fact that a Hilbert space is self-dual.

$(a) \Leftrightarrow (c)$ Since a Hilbert space is reflexive, the result follows from Corollary 34. □

Let us recall that a Banach space X has the BD-property if every limited set in X is relatively weakly compact [22, p. 494, Definition 4(j)]. The following result follows from the very definition of BD-property and Theorem 29.

Corollary 36. Let $f \in X^{[a,b]}$ be such that for each $x^* \in X^*$, x^*f is continuous almost everywhere and $f([a,b])$ is a limited set in X. If X has the BD-property, then $f \in RP([a,b],X)$ and f has a Bochner integrable equivalent.

Since Mazur's property implies BD-property [8, p. 386, Proposition 3.1], the following result follows from the above Corollary.

Corollary 37. Let $f \in X^{[a,b]}$ be such that for each $x^* \in X^*$, x^*f is continuous almost everywhere and $f([a,b])$ is a limited set in X. If X has Mazur's-property, then $f \in RP([a,b],X)$ and f has a Bochner integrable equivalent.

Theorem 38. *If X is a Gelfand space, then every $f \in RP([0,1],X)$ has a Bochner-integrable equivalent g which is Borel measurable.*

Proof. Follows from [16, p. 107, Theorem IV.3.2] and Theorem 26. □

From an unpublished work of D. R. Lewis [16, p. 88], we have the following result:

Theorem 39. *Let X be weakly compactly generated. Then $RD([a,b],X) = RP([a,b],X)$ and every $f \in RP([a,b],X)$ has a bounded Bochner-integrable equivalent.*

Recall that a bounded subset K in X^* is a V-set if $\lim_n \sup_{x^* \in K} |x^*(x_n)| = 0$ for every weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ in X [15, Definition 3]. A Banach space X has the property (V) of Pelczynski if every unconditionally converging operator defined on it with values in a Banach space Y is weakly compact [15, Definition 2].

Theorem 40. *Let X be a Banach space with property(V) of Pelczynski. Let $f \in (X^*)^{[a,b]}$ be such that for each $x^{**} \in X^{**}$, $x^{**}f$ is continuous almost everywhere on $[a,b]$. If $f([a,b])$ is a V-set, then $f \in RP([a,b],X^*)$ and has a Bochner integrable equivalent.*

Proof. Follows from [15, p. 3, Proposition 4] and Theorem 29. □

Definition 41. A function $f \in X^{[a,b]}$ is said to be Darboux integrable on $[a,b]$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\omega(f, \mathcal{P}) < \epsilon$ whenever $\mathcal{P}$ is a partition of $[a,b]$ that satisfies $|\mathcal{P}| < \delta$.

The collection of all Darboux integrable functions of $X^{[a,b]}$ will be denoted by $D([a,b],X)$.

If $X = \mathbb{R}$, then $D([a,b],X)$ is denoted by $D[a,b]$.

From definition, it can be verified that

$D([a,b],X) \subset R([a,b],X) \subset RP([a,b],X) \subset RD([a,b],X) \subset l^{\infty}([a,b],X)$.

In a finite-dimensional space X,

$D([a,b],X) = R([a,b],X) = RP([a,b],X) = RD([a,b],X) \subset l^{\infty}([a,b],X)$.

From above considerations, the Darboux integral of an $f \in D([a,b],X)$ is denoted as $D - \int_a^b f(t)dt$ or simply as $D - \int_a^b f dt$ and is defined as $D - \int_a^b f(t)dt = R - \int_a^b f(t)dt = RP - \int_a^b f(t)dt$.

It can be shown that if $f \in D([a,b],X)$, then $f \in D([c,d],X)$ for any $[c,d] \subset [a,b]$.

Note. From above, it is noted that a real valued function is Riemann integrable if and only if it is Riemann-Dunford integrable. It is known that the Dirichlet's function is bounded and Lebesgue integrable but not Riemann integrable. Hence we arrive at the conclusion that a bounded Bochner integrable function may not be Riemann-Dunford integrable even in a finite dimensional space.

Theorem 42. *If $f \in X^{[a,b]}$ is of bounded variation, then $f \in D([a,b],X)$.*

PROOF. Let ϵ be an arbitrary positive real number. Since f is of bounded variation, f is bounded and there exists a positive real number M such that $\sum_{i=1}^{n} \|f(c_i) - f(d_i)\| \leq M$ for all finite collections $\{[c_i, d_i]\}$ of non-overlapping intervals in $[a,b]$. Let $\delta = \frac{\epsilon}{2M}$ and let $\mathcal{P} = \{t_i : 0 \leq i \leq n\}$ be a partition of $[a,b]$ with $|\mathcal{P}| < \delta$. Let

$$M_i = \omega(f, [t_{i-1}, t_i]) = \sup\{\|f(v) - f(u)\| : u, v \in [t_{i-1}, t_i]\}.$$

Then, for each $i = 1, 2, \cdots n$, there exists $c_i, d_i \in [t_{i-1}, t_i]$ such that $M_i < \|f(c_i) - f(d_i)\| + \frac{\epsilon}{2(b-a)}$.

Now,

$$\begin{aligned}
\omega(f,\mathcal{P}) &= \sum_{i=1}^{n}\omega(f,[t_{i-1},t_i])(t_i-t_{i-1})\\
&= \sum_{i=1}^{n} M_i(t_i-t_{i-1})\\
&< \sum_{i=1}^{n}\left\{\|f(c_i)-f(d_i)\|+\frac{\epsilon}{2(b-a)}\right\}(t_i-t_{i-1})\\
&= \sum_{i=1}^{n}\|f(c_i)-f(d_i)\|(t_i-t_{i-1})+\sum_{i=1}^{n}\frac{\epsilon}{2(b-a)}(t_i-t_{i-1})\\
&\le \sum_{i=1}^{n}\|f(c_i)-f(d_i)\||\mathcal{P}|+\frac{\epsilon}{2(b-a)}(b-a)\\
&< \sum_{i=1}^{n}\|f(c_i)-f(d_i)\|\delta+\frac{\epsilon}{2}\\
&\le M\delta+\frac{\epsilon}{2}\\
&= M\frac{\epsilon}{2M}+\frac{\epsilon}{2}\\
&= \frac{\epsilon}{2}+\frac{\epsilon}{2}\\
&= \epsilon.
\end{aligned}$$

Hence, $f \in D([a,b],X)$. □

Note. The converse of the above theorem is not necessarily true even in $\mathbb{R}$, as a continuous function is Darboux integrable but may not be of bounded variation.

Note. It is well known that a function of bounded variation on $[a,b]$ is not necessarily continuous on $[a,b]$. However we have the following result which follows from the above theorem and [26, p. 933, Theorem 18]:

Corollary 43. A function $f \in X^{[a,b]}$ of bounded variation on $[a,b]$ is continuous almost everywhere on $[a,b]$.

Theorem 44. *Let $C_0([a,b],X)$ be the space of all functions $f \in X^{[a,b]}$ such that for any $\epsilon > 0$ the set $\{t \in [a,b] : \|f(t)\| > \epsilon\}$ is finite. Then $C_0([a,b],X) \subset D([a,b],X)$.*

PROOF. Let $f \in C_0([a,b],X)$. Let $\epsilon > 0$ be arbitrary. Let $\epsilon' = \frac{\epsilon}{4(b-a)}$. So there are finite number of points $t_1, t_2, \cdots, t_k$ in $[a,b]$ such $\|f(t)\| \leq \epsilon'$ for all $t \neq t_i, i = 1, 2, \cdots, k$. Let $T = \{t_1, t_2, \cdots, t_k\}$. It is clear that f is bounded. Let $M > 0$ be such that $\|f(t)\| \leq M$ for all $t \in [a,b]$. Let $\delta = \min\left\{\epsilon', \frac{\epsilon}{4kM}\right\}$. Let $\mathcal{P} = \{s_1, s_2, \cdots, s_n\}$ be a partition of $[a,b]$ with $|\mathcal{P}| < \delta$.

Now it is noted that

$\omega(f, [s_{i-1}, s_i]) \leq 2\epsilon'$ if $[s_{i-1}, s_i] \cap T = \phi$ and $\omega(f, [s_{i-1}, s_i]) \leq 2M$ if $[s_{i-1}, s_i] \cap T \neq \phi$.

Therefore,

$$\begin{aligned}
&\omega(f, \mathcal{P}) \\
&= \sum_{[s_{i-1}, s_i] \cap T = \phi} \omega(f, [s_{i-1}, s_i])(s_i - s_{i-1}) \\
&\qquad + \sum_{[s_{i-1}, s_i] \cap T \neq \phi} \omega(f, [s_{i-1}, s_i])(s_i - s_{i-1}) \\
&\leq 2\epsilon' \sum_{[s_{i-1}, s_i] \cap T = \phi} (s_i - s_{i-1}) + 2M \sum_{[s_{i-1}, s_i] \cap T \neq \phi} (s_i - s_{i-1}) \\
&< 2\epsilon'(b-a) + 2Mk\delta \\
&\leq 2\frac{\epsilon}{4(b-a)}(b-a) + 2Mk\frac{\epsilon}{4kM} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, $f \in D([a,b],X)$ and the result follows. □

Theorem 45. *Let $f \in X^{[a,b]}$. Let us consider the following statements:*

(a) $f \in X^{[a,b]}$ is bounded and $f = 0$ almost everywhere on $[a,b]$ with respect to Jordan content.

(b) $f \in X^{[a,b]}$ is bounded and $f = 0$ weakly almost everywhere on $[a,b]$ with respect to the Jordan content [the exceptional set does not depend upon $x^* \in X^*$].

(c) $f \in D([a,b],X)$ and $D - \int_c^d f dt = 0$ for all $[c,d] \subset [a,b]$.

Then $(a) \Leftrightarrow (b) \Rightarrow (c)$.

PROOF. $(a) \Rightarrow (b)$ Trivial.

$(b) \Rightarrow (a)$ There exists a set $E \subset [a,b]$ (not depending upon $x^* \in X^*$) such that Jordan content of E is zero and $x^* f(t) = 0$ for all $t \in [a,b] \setminus E$ and for all $x^* \in X^*$.

Hence, $f(t) = 0$ for all $t \in [a,b] \setminus E$ i.e. $f(t) = 0$ almost everywhere on $[a,b]$ with respect to Jordan content and (a) follows.

$(a) \Rightarrow (c)$ Let $f(t) \neq 0$ for all $t \in E \subset [a,b]$. Then, Jordan content of E is zero and $f(t) = 0$ for $t \in [a,b] \setminus E$. Since E is Jordan null, its Jordan content is zero and hence its Jordan outer content is zero. So for any $\epsilon > 0$ there exists a partition $\mathcal{P}_\epsilon$ such that the sum of the lengths of the subintervals of $\mathcal{P}_\epsilon$ which contains points of $E \cup \delta E$ is less than ϵ, where δE is the boundary of E; i.e.,

$$\overline{J}(\mathcal{P}_\epsilon, E) < \frac{\epsilon}{2M+1} \text{ where } M = \sup_{t\in[a,b]} \|f(t)\|.$$

Let $I_1, I_2, \cdots, I_n$ be the subintervals of $\mathcal{P}_\epsilon$ and let $I'_1, I'_2, \cdots, I'_k$ be those subintervals of $\mathcal{P}_\epsilon$ which contains points of $E \cup \delta E$ and let $I''_1, I''_2, \cdots, I''_{n-k}$ be those subintervals which contain no points of $E \cup \delta E$. Then,

$$I_1 \cup I_2 \cup \cdots \cup I_n = (I'_1 \cup I'_2 \cup \cdots \cup I'_k) \cup (I''_1 \cup I''_2 \cup \cdots \cup I''_{n-k})$$

and

$$\sum_{i=1}^{k} \delta I'_i = \overline{J}(\mathcal{P}_\epsilon, E) < \frac{\epsilon}{2M+1}.$$

Therefore,

$$\begin{aligned}
\omega(f, \mathcal{P}_\epsilon) &= \sum_{i=1}^{n} \omega(f, I_i)\delta I_i \\
&= \sum_{i=1}^{k} \omega(f, I'_i)\delta I'_i + \sum_{i=1}^{n-k} \omega(f, I''_i)\delta I''_i \\
&= \sum_{i=1}^{k} \sup\{\|f(u) - f(v)\| : u, v \in I'_i\}\delta I'_i \\
&\qquad\qquad + \sum_{i=1}^{n-k} \sup\{\|f(u) - f(v)\| : u, v \in I''_i\}\delta I''_i \\
&\le \sum_{i=1}^{k} \sup\{\|f(u)\| + \|f(v)\| : u, v \in I'_i\}\delta I'_i \\
&\qquad\qquad + \sum_{i=1}^{n-k} \sup\{\|0 - 0\| : u, v \in I''_i\}\delta I''_i \\
&\text{(since } f(t) = 0 \text{ for all } t \in [a,b] \setminus E \text{ and } I''_i \text{ contains} \\
&\text{ elements of } [a,b] \setminus E \text{ only).}
\end{aligned}$$

$$\leq 2M \sum_{i=1}^{k} \delta I_i'$$
$$< 2M \frac{\epsilon}{2M+1} < \epsilon.$$

Now, let $\mathcal{P}$ be any partition of $[a,b]$ that refines $\mathcal{P}_\epsilon$. Then, it is easy to verify that

$$\omega(f,\mathcal{P}) \leq \omega(f,\mathcal{P}_\epsilon) < \epsilon$$

This implies that $f \in D([a,b],X)$ [26, p. 933, Definition 17].

For the second part, let $[c,d] \subset [a,b]$. Let $E_1 = [c,d] \cap E$ and $E_2 = [c,d] \setminus E$. Then, $f = 0$ on E_2 and Jordan content of $E_1 = 0$ and so $\lambda(E_1) = 0$.

Now, $E_1 \cup E_2 = [c,d]$ and $E_1 \cap E_2 = \phi$. So $D - \int_c^d f dt = RP - \int_c^d f dt =$ Pettis $- \int_c^d f d\lambda =$ Pettis $- \int_{E_1} f d\lambda +$ Pettis $- \int_{E_2} f d\lambda = 0$, since Pettis $- \int_{E_2} f d\lambda = 0$ for $f = 0$ on E_2 and Pettis $- \int_{E_1} f d\lambda = 0$ for $\lambda(E_1) = 0$. □

Note. In the above theorem, $(c) \Rightarrow (a)$ is not necessarily true even in the case of real valued functions.

For example, let

$$f(t) = \begin{cases} 0 & \text{for all t irrational or } t = 0 \text{ in } [0,1] \\ \frac{1}{q} & \text{if } t = \frac{p}{q} \neq 0 \text{ rational in } [0,1] \end{cases}$$

Then, $f \in D[a,b]$ and $D - \int_0^1 f(t)dt = 0$ and so $D - \int_c^d f(t)dt = 0$ for all subintervals $[c,d]$ of $[0,1]$. But, the outer Jordan content of $Q - \{0\}$ is $1 \neq 0$.

Corollary 46. Let $f,\ g \in X^{[a,b]}$ be bounded and $f = g$ almost everywhere on $[a,b]$ with respect to Jordan content. Then,

(a) $f \in R([a,b],X)$ if and only if $g \in R([a,b],X)$.

(b) $f \in D([a,b],X)$ if and only if $g \in D([a,b],X)$.

PROOF. Let $h(t) = f(t) - g(t)$ so that $g(t) = f(t) - h(t)$ for all $t \in [a,b]$. Then, $h(t) = 0$ almost everywhere on $[a,b]$ with respect to Jordan content. Hence, $h \in D([a,b],X)$ by Theorem 45.

(a) Let $f \in R([a,b],X)$. Since $h \in D([a,b],X)$, we have $h \in R([a,b],X)$. So $g = f - h \in R([a,b],X)$.

Similarly, $g \in R([a,b],X)$ implies that $f \in R([a,b],X)$.

(b) Let $f \in D([a,b],X)$. Since $h \in D([a,b],X)$, $g = f - h \in D([a,b],X)$.

Similarly, $g \in D([a,b],X)$ implies that $f \in D([a,b],X)$. □

Corollary 47. If $f \in X^{[a,b]}$ is bounded and $f =$ constant almost everywhere on $[a,b]$ with respect to Jordan content, then $f \in D([a,b],X)$.

Note. If we replace Jordan content by Lebesgue measure, then Theorem 45 may not be true. For example, let us consider the function χ_E where $E = Q \cap [0,1]$ denotes the set of all rational numbers in $[0,1]$. Then, $\chi_E = 0$ almost everywhere on $[0,1]$ with respect to Lebesgue measure. But $\chi_E \notin D([0,1])$.

However, if we replace Jordan content by Lebesgue measure in Theorem 45, then it will be valid provided that the exceptional set is closed. To prove this result we need help of the following Lemma:

Lemma 48. *Let X and Y be two topological spaces and let G be an open set in X. Let $f : X \to Y$ be a function such that $f(x) =$ constant for all $x \in G$. Then f is continuous at every point of G.*

PROOF. Let $f(x) = y \in Y$, for all $x \in G$. Let $z \in G$ be arbitrary. Then $f(z) = y$. Let H be an arbitrary open set in Y such that $y \in H$. Then, we see that $f(x) \in H$, for all $x \in G$. Thus, for any open set H in Y with $f(z) = y \in H, \exists$ an open set G in X such that $z \in G$ and $f(x) \in H$, for all $x \in G$. Hence, f is continuous at $z \in G$. But $z \in G$ is arbitrary. Hence, f is continuous at every point of G. □

Theorem 49. *If $f \in X^{[a,b]}$ is bounded and $f =$ constant almost everywhere on $[a,b]$ and the exceptional set is closed, then $f \in D([a,b],X)$.*

PROOF. Let D be the set of points of $[a,b]$ such that f is not constant on D. Then, D is closed with $\lambda(D) = 0$ and f is constant on $[a,b] \setminus D$ which is an open set in $[a,b]$ and hence by the Lemma 48, f is continuous on $[a,b] \setminus D$. Thus, f is continuous almost everywhere on $[a,b]$ and therefore $f \in D([a,b],X)$. □

Alexiewicz and Orlicz have shown with an example that weak continuity of a Banach space valued function need not imply Riemann integrability [2, p. 130–132] (vide [26, p. 947, Example 35]).

However, the following result is valid which follows from [41, p. 206, Theorem 7.3.7] and [26, p. 933, Theorem 18].

Theorem 50. *Let $f \in X^{[a,b]}$ be bounded and weakly differentiable almost everywhere on $[a,b]$. Then, $f \in D([a,b],X)$. In particular, every weakly differentiable function $f \in X^{[a,b]}$ is Darboux integrable.*

Corollary 51. Let X be weakly sequentially complete. If $f \in X^{[a,b]}$ is bounded and if there exists a Lebesgue null set $E \subset [a,b]$ such that for each $x^* \in X^*$, x^*f is differentiable at each point of $[a,b] \setminus E$, then f is Darboux integrable on $[a,b]$.

PROOF. Follows from [41, p. 203, Theorem 7.3.3] and Theorem 50. □

Let us recall that a Banach space X is said to have Gelfand-Phillips property if limited subsets of X are relatively compact. If a Banach X space has the Gelfand-Phillips property, then X is called a Gelfand-Phillips space.

From the very definition of a Gelfand-Phillips space and [26, p. 945, Theorem 32], the following result follows.

Theorem 52. *Let X be a Gelfand-Phillips space and let $f \in X^{[a,b]}$ be such that for each $x^* \in X^*$, x^*f is continuous almost everywhere on $[a,b]$ and $f([a,b])$ is a limited set in X. Then, $f \in D([a,b],X)$.*

Let us recall that a Banach space X has property SK if and only if B_{X^*} is *weak** sequentially compact in X^* [46, p. 230].

Theorem 53. *In each of the following cases X is a Gelfand-Phillips space, and hence the above result is valid in X:*

(a) X is separable.

(b) X is reflexive.

(c) X^* contains no copy of l^1.

(d) X is a subspace of a weakly compactly generated Banach space.

(e) X^* has the Radon-Nikodym property.

(f) X has property SK.

(g) X is a weak Asplund space.

(h) X is a closed subspace of a dual Banach space Y^* with Y not containing l^1.

(i) X is a Schur space.

PROOF. (a) and (b) follows from [9, p. 55, Proposition (5) and (6)](vide [4, p. 5, Corollary 3.2]).

(c) to (e) Follow from [4, p. 5, Corollary 3.2].

(f) Follows from [46, p. 230] and [17, p. 238, Exercise 4(i)].

(g) Since X is a weak Asplund space, X has property SK [17, p. 239]. Hence, the result follows from (f).

(h) Follows from [21, p. 157, Corollary 5].

(i) Follows from [40, p. 85]. □

Theorem 54. *Let $f \in X^{[a,b]}$ be such that $x^* f$ is continuous almost everywhere on $[a, b]$ for each $x^* \in X^*$ and f has a relatively weakly compact range. If X has the Schur property, then $f \in D([a, b], X)$.*

PROOF. Follows from [23, p. 37, Theorem 2.3.7] and [26, p. 945, Theorem 32]. □

Alexiewicz and Orlicz have shown that a weakly continuous function may not be Riemann integrable [2, p. 130]. However, we have the following result which follows from the above theorem.

Corollary 55. Let $f \in X^{[a,b]}$ be weakly continuous and X have the Schur property, then $f \in D([a, b], X)$.

The following result is analogous to [27, p. 171, Theorem 4] which is generally known as Fundamental Theorem of Calculus.

Theorem 56. *Let $f \in X^{[a,b]}$ be weakly differentiable with weak derivative f'_w.*

(a) If $f'_w \in RD([a, b], X)$, then $RD - \int_a^b f'_w(t)dt = f(b) - f(a)$.

(b) If $f'_w \in RP([a, b], X)$, then $RP - \int_a^b f'_w(t)dt = f(b) - f(a)$.

(c) If $f'_w \in R([a, b], X)$, then $R - \int_a^b f'_w(t)dt = f(b) - f(a)$.

(d) If $f'_w \in D([a, b], X)$, then $D - \int_a^b f'_w(t)dt = f(b) - f(a)$.

PROOF. (a) Let $f'_w \in RD([a, b], X)$. Then, for each $x^* \in X^*$, $x^* f'_w \in R[a, b]$, and hence $x^* f'_w$ is bounded and continuous almost everywhere on $[a, b]$. From [32, p. 47, Theorem 66], it follows that

$$\int_a^b x^* f'_w(t)dt = x^* f(b) - x^* f(a), \text{ for all } x^* \in X^*.$$

Hence,

$$RD - \int_a^b f'_w(t)dt = f(b) - f(a).$$

The proofs of $(b), (c)$ and (d) are same as (a). □

Corollary 57. Let $f \in X^{[a,b]}$ be weakly differentiable and if $f'_w \in RD([a, b], X)$, then $f'_w \in RP([a, b], X)$.

PROOF. From the above theorem it follows that for any $t \in [a, b]$, $RD - \int_a^t f'_w(t)dt = f(t) - f(a) \in X$. Hence, the result follows from Corollary 13($(a) \Leftrightarrow (d)$). □

Definition 58. A function $f \in (X^*)^{[a,b]}$ is said to be weak*-scalarly Riemann integrable on $[a,b]$ if xf is Riemann integrable on $[a,b]$ for each $x \in X$.

It can be shown that every weak*-scalarly Riemann integrable function is bounded and Gelfand integrable, and hence, it is also said to be Riemann-Gelfand integrable. Thus, if $f \in (X^*)^{[a,b]}$ is weak*-scalarly Riemann integrable on $[a,b]$, then for each $E \in \Sigma$, there exists an element $x_E^* \in X^*$ such that $x_E^*(x) = \int_E xf d\lambda$, for all $x \in X$. The element x_E^* is called the Riemann-Gelfand integral of f over E and is denoted by $RG - \int_E f d\lambda$. Thus $RG - \int_E f d\lambda \in X^*$ and $x(RG - \int_E f d\lambda) = \int_E xf d\lambda$ for all $x \in X$. The collection of all weak*-scalarly Riemann integrable functions of $(X^*)^{[a,b]}$ will be denoted by $RG([a,b], X^*)$.

From the very definition we have the following result:

Lemma 59. *For any Banach space X, $RP([a,b], X^*) \subset RD([a,b], X^*) \subset RG([a,b], X^*)$.*

Lemma 60. *Let $f \in RD([a,b], X^*)$. Then $f \in RG([a,b], X^*)$. If for each $x^{**} \in X^{**}$, $x^{**}f$ is Borel measurable and $\|RD - \int_a^b f dt\| = \|RG - \int_a^b f dt\|$, then $f \in RP([a,b], X^*)$.*

PROOF. First part follows from the above Lemma.

Since Lebesgue measure on the sigma algebra of Borel subsets of $[a,b]$ is a perfect measure, the second part follows by proceeding in the same way as in the proof of [3, p. 268, Theorem 4]. □

Lemma 61. *Let $f \in RD([a,b], X^*)$. Then, $f \in RG([a,b], X^*)$. Moreover, $f \in RP([a,b], X^*)$ if and only if $RD - \int_c^d f dt = RG - \int_c^d f dt$ for any subinterval $[c,d]$ of $[a,b]$.*

PROOF. First part follows from Lemma 59.

For the second part, if $f \in RP([a,b], X^*)$, then for any subinterval $[c,d]$ of $[a,b]$, $\int_c^d x^{**} f dt = x^{**}(RP - \int_c^d f dt)$ for all $x^{**} \in X^{**}$. In particular, $\int_c^d xf dt = x(RP - \int_c^d f dt)$ for all $x \in X$. Also as $f \in RG([a,b], X^*)$, we have $\int_c^d xf dt = x(RG - \int_c^d f dt)$ for all $x \in X$. Hence $x(RP - \int_c^d f dt) = x(RG - \int_c^d f dt)$ for all $x \in X$ which implies that $RP - \int_c^d f dt = RG - \int_c^d f dt$. Also, $RP - \int_c^d f dt = RD - \int_c^d f dt$. Hence, $RD - \int_c^d f dt = RG - \int_c^d f dt$.

Conversely, let $RD - \int_c^d f dt = RG - \int_c^d f dt$ for any subinterval $[c,d]$ of $[a,b]$. Hence, $\int_c^d x^{**} f dt = x^{**}(RD - \int_c^d f dt) = x^{**}(RG - \int_c^d f dt)$ for all $x^{**} \in X^{**}$. Since $RG - \int_c^d f dt \in X^*$, it follows that $RG - \int_c^d f dt \in J_f([c,d])$ which implies that $J_f([c,d]) \neq \phi$ and the result follows from Theorem 12($(a) \Leftrightarrow (e)$). □

Theorem 62. *If X is a Grothendieck space, then*

$$RP([a,b],X^*) = RD([a,b],X^*).$$

PROOF. Let $f \in RD([a,b],X^*)$. Then, $f \in RG([a,b],X^*)$. Let $[c,d]$ be any subinterval of $[a,b]$. For each positive integer n, let $\mathcal{P}_n$ be a tagged partition of $[c,d]$ with points $\{c + (\frac{k}{n})(d-c) : 0 \le k \le n\}$. Since $f \in RD([a,b],X^*)$, $x^{**}f \in R[a,b]$, and hence $x^{**}f \in R[c,d]$ for each $x^{**} \in X^{**}$. Hence, the sequence $\{x^{**}f(\mathcal{P}_n)\}$ converges to $\int_c^d x^{**}f dt = x^{**}(RD - \int_c^d f dt)$ for each $x^{**} \in X^{**}$. Again, $xf \in R[c,d]$ for each $x \in X$. Therefore, the sequence $\{xf(\mathcal{P}_n)\}$ converges to $\int_c^d xf dt = x(RG - \int_c^d f dt)$ for all $x \in X$. Thus, the sequence $\{f(\mathcal{P}_n)\}$ is weak* convergent to $RG - \int_c^d f dt$. Since X is a Grothendieck space, $\{f(\mathcal{P}_n)\}$ must be weakly convergent to $RG - \int_c^d f dt$. Therefore, $\{x^{**}f(\mathcal{P}_n)\}$ converges to $x^{**}(RG - \int_c^d f dt)$ for all $x^{**} \in X^{**}$. Hence, $x^{**}(RD - \int_c^d f dt) = x^{**}(RG - \int_c^d f dt)$ for all $x^{**} \in X^{**}$. Therefore, $RD - \int_c^d f dt = RG - \int_c^d f dt$. Hence, the result follows from above lemma. □

Corollary 63. If X is reflexive, then $RG([a,b],X^*) = RP([a,b],X^*) = RD([a,b],X^*)$.

PROOF. Follows from the very definition of a reflexive space and Lemma 59. □

Theorem 64. *Let $f \in (X^*)^{[a,b]}$ be weak* differentiable and $f'_{w^*} \in RG([a,b],X^*)$. Then,*

$$RG - \int_a^b f'_{w^*}(t)dt = f(b) - f(a).$$

PROOF. Similar to Theorem 56. □

Theorem 65. *If B_X be weak*-sequentially dense in $B_{X^{**}}$, then $RD([a,b],X^*) = RP([a,b],X^*)$.*

PROOF. Let $f \in RD([a,b],X^*)$ so that $f \in RG([a,b],X^*)$ and $x^{**} \in X^{**}$ be arbitrary so that $\frac{x^{**}}{\|x^{**}\|} \in B_{X^{**}}$. Then, there exists a sequence $\{x_n\}$ in B_X which converges to $\frac{x^{**}}{\|x^{**}\|}$ in *weak**-topology. Therefore, $\{x_n f(t)\}$ converges to $\frac{x^{**}}{\|x^{**}\|}f(t)$ for all $t \in [a,b]$. So $\{\|x^{**}\|x_n f(t)\}$ converges to $x^{**}f(t)$ for all $t \in [a,b]$. Also, $\{\|x^{**}\|x_n(RG - \int_a^b f dt)\}$ converges to $x^{**}(RG - \int_a^b f dt)$. Clearly, $\{\|x^{**}\|x_n\}$ is bounded in X. Hence, f is Pettis integrable [3, p. 269, Theorem 5] and the result follows. □

4 A comparative study with Henstock-Kurzweil-type integration in Banach spaces.

In this section we shall present a comparative study of Riemann-type integration with Henstock-Kurzweil-type integration in Banach spaces. There are so many similarities as well as dissimilarities between the two types of integrals. The fundamental results which hold for both the two types of integration are not listed here.

Henstock-Kurzweil-type integration is a generalization of Riemann-type integration. The necessity to introduce the notion of Henstock-Kurzweil integration originates from the fact that the derivative of a differentiable function on $[a, b]$ with values in a Banach space (even in $\mathbb{R}$) may not be Riemann integrable, but is Henstock-Kurzweil integrable [41, p. 209, Theorem 7.3.10].

For definitions and various properties of Henstock-Kurzweil, Henstock-Kurzweil-Pettis and Henstock-Kurzweil-Dunford (in short HK, HKP and HKD respectively) integrable functions, we refer to [6], [11], [18], [37].

From the very definitions the following results follow obviously :

A Banach space-valued Riemann integrable function is Henstock-Kurzweil integrable, a Riemann-Pettis integrable function is Henstock-Kurzweil-Pettis integrable and a Riemann-Dunford integrable function is Henstock-Kurzweil-Dunford integrable.

Since a RD-integrable (and consequently a Riemann and RP-integrable) function is bounded which is not necessarily true for HK-type integrable functions, the converse of none of the above statements is true. Even a bounded real valued HK-integrable function may not be Riemann integrable [44, p. 88, Example 14.3].

There is a similarity between the interrelations among the two types of integrals :

A Riemann integrable function is RP-integrable and a RP-integrable function is RD-integrable. Similarly, an HK- integrable function is HKP-integrable and an HKP-integrable function is HKD-integrable.

However, the reverse implications do not hold in general (e.g., see [18, p. 545, Remark 1], [7, p. 589, Example 1]).

In a weakly sequentially complete space, an RD-integrable function is RP-integrable [26, p. 944, Theorem 31]. This result is valid for an HKD-integrable function to be HKP-integrable if the function is measurable [47, p. 1241, Theorem 3.4].

In a Schur space, RD-integrability, RP-integrability and Riemann integrability of a function coincide. This result is valid in a Schur space for HK-type integrable functions provided that the functions are measurable and the condition (C) is satisfied [48, p. 221, Theorem 4.5].

An RP-integrable function is Pettis integrable and an RD-integrable function is Dunford integrable, whereas an HKP-integrable function (even an HK-integrable function) may not be Pettis integrable [18, p. 545, Remark 1] and an HKD-integrable function may not be Dunford integrable as, in $\mathbb{R}$, an HKD-integrable function is HK-integrable and a Dunford integrable function is Lebesgue integrable, and an HK-integrable function is not necessarily Lebesgue integrable.

A measurable Pettis integrable function is HK-integrable [41, p. 173, Theorem 6.2.1 and p. 47, Theorem 3.2.3], but a measurable Pettis integrable function (even a Bochner integrable function) may not be Riemann integrable (even for a real valued function)[26, p. 947, Example 35].

A measurable RD-integrable (and hence a measurable Riemann integrable) function is bounded and Bochner integrable [26, p. 944]. However a measurable HK-integrable function may not be Bochner integrable [37, p. 1098, Remark 4].

A Riemann integrable (and hence an HK-integrable) function with values in a Banach space is not necessarily measurable [26, p. 930, Example 12]. However, an HK-integrable function is scalarly measurable. On the other hand an RD-integrable (and hence Riemann and RP-integrable) function is scalarly integrable which is not true for HKD-integrable functions, even in $\mathbb{R}$.

Ye has shown that a weakly continuous and bounded function with values in a Banach space is HKP-integrable [47, p. 1243, Theorem 3.6]. However a stronger result exists, namely, a weakly continuous (in fact, a weakly continuous almost everywhere and bounded) function is measurable and RP-integrable and hence Bochner integrable [26, p. 944] which implies that it is HK-integrable. However, a weakly continuous function may not be Riemann integrable [26, p. 947, Example 35].

If $f = g$ a.e. in $[a, b]$ (with respect to Lebesgue measure) and if $g \in HK([a, b], X)$, then $f \in HK([a, b], X)$. However this result is not true for a Riemann integrable function (even in $\mathbb{R}$).

A function $f \in X^{[a,b]}$ is said to be absolutely Riemann(HK, Darboux) integrable if both f and $\|f\|$ are Riemann (respectively HK, Darboux) integrable. A real valued Riemann integrable function is absolutely Riemann integrable but a real valued HK-integrable function may not be absolutely HK-integrable [41, p. 79, Example in Remark]. A Banach space valued Darboux integrable function is absolutely Darboux integrable, but a Banach space valued Riemann integrable function may not be absolutely Riemann integrable. Also, a Banach space valued HK-integrable function is not necessarily absolutely HK-integrable [26, p. 931, Example 14] and [18, p. 548, Example in Remark

3]. Even though a Riemann integrable function is HK-integrable, a Riemann integrable function is not necessarily absolutely HK-integrable. An absolutely Riemann integrable function is absolutely HK-integrable, but the converse is not true. Even a Riemann integrable and absolutely HK-integrable function may not be absolutely Riemann integrable. For example, for $E = \mathbb{Q} \cap [0,1]$, the function $f : [0,1] \to l_\infty[0,1]$ defined by $f(t) = \theta$ if $t \notin E$ and $f(t) = \chi_{\{t\}}$ if $t \in E$ is Riemann integrable but $\|f\| = \chi_E$ is Lebesgue integrable, and hence HK-integrable but not Riemann integrable.

The indefinite integral of an RD-integrable function satisfies Lipschitz condition and hence is absolutely continuous. This result is not true even for an HK-integrable function. The indefinite integral for an HK-integrable function is continuous on $[a,b]$, and there is a countable partition of $[a,b]$ on each subinterval of which it is absolutely continuous [41, p. 224, Theorem 7.4.19]. However, if the indefinite integral of an HK-integrable function f is absolutely continuous, then f is Pettis integrable [36, p. 15, Definition 4]. The indefinite integral of an HKP-integrable function may not even be continuous [36, p. 17, Theorem 4].

The indefinite integral of an RP-integrable function and hence a Riemann integrable function is scalarly differentiable almost everywhere on $[a,b]$ which is also valid for an HKP-integrable function. Consequently our Theorem 5(c) follows from the corresponding result for HKP-integrable functions.

An RD-integrable function is Pettis integrable (and hence RP-integrable) if and only if the indefinite RD-integral over every closed sub-interval $[c,d]$ of $[a,b]$ belongs to X [26, p. 944]. But, this result is not true for HKD-integrable functions as an HKP-integrable function is not necessarily Pettis integrable.

Di Piazza and Musiał have shown that weak derivative of a Banach space valued function is HKP-integrable [19, p. 171, Proposition 2] but as the derivative of a differentiable function is not necessarily Riemann integrable (even in $\mathbb{R}$ where Riemann and RP-integrability coincide), the weak derivative of a weakly differentiable function may not be RP-integrable.

In [19, p. 171, Proposition 2], Di Piazza and Musiał have shown that if $f \in X^{[0,1]}$ is weakly differentiable, then its weak derivative f' is HKP-integrable and (HKP)$\int_0^s f'(t)dt = f(s) - f(0)$ for each $s \in [0,1]$. As stated earlier, the derivative of a differentiable real valued function defined on a closed interval of $\mathbb{R}$ is not necessarily Riemann integrable (even not Lebesgue integrable, for example, let $f(t) = t^2 cos(\frac{\pi}{t^2}), t \in (0,1], f(0) = 0$. Then f is differentiable on $[0,1]$ and $f'(t) = 2t cos(\frac{\pi}{t^2}) + \frac{2\pi}{t} sin(\frac{\pi}{t^2})$ for $t \in (0,1], f'(0) = 0$ and f' is not Lebesgue integrable), the same result does not hold for an RP-integrable function. Moreover, if the weak derivative f' of a weakly differentiable function f is RP-integrable, then we have (RP)$\int_a^s f'(t)dt = f(s) - f(a)$ for each $s \in [a,b]$

[Theorem 56].

Acknowledgment. The authors gratefully acknowledge the valuable comments and suggestions of the anonymous referee, which helps them to add the section "A Comparative study with Henstock-Kurzweil-type integration in Banach spaces" to include the references [1, 6, 11, 18, 19, 20, 33, 36, 37, 38, 43, 47, 48], and to prepare the revised version of the manuscript in the present form.

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