INROADS

Real Analysis Exchange
Vol. 39(2), 2013/2014, pp. 385–402

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WEIGHTED INEQUALITIES FOR ONE-SIDED VECTOR-VALUED MAXIMAL OPERATORS WITH RESPECT TO A FUNCTION

Abstract

The purpose of this paper is to find necessary and sufficient conditions on the weight w for the weak type (p,p) with $1 \leq p < +\infty$ and for the strong type (p,p) with $1 < p < +\infty$ respect to a measure wdx of the vector-valued one-sided maximal operator $\left(M_g^+\right)_r$.

1 Introduction and main results

Since Sawyer established a necessary and sufficient conditions for the boundedness of the unweighted one-sided Hardy-Littlewood operators (see [12]), using the Muckenhoupt's A_p^+ and A_p^- classes, many generalizations have been found. For instance in [9] Martin-Reyes, Ortega Salvador and de La Torre obtained an analogous result for the weighted Hardy-Littlewood operator that implies Sawyer's result when $g \equiv 1$. Another way of extension is to consider the corresponding inequalities for the vector-valued maximal operator (see [2]). Here we will simultaneously consider both generalizations by considering, for suitable $A_p^+(g)$ and $A_p^-(g)$ classes: the inequalities for the vector-valued maximal operator that includes the result of [2] if $g \equiv 1$; the result of [9] if we take countable copies of the same function; and Sawyer's result with $g \equiv 1$ and also taking countable copies of the same function.

Mathematical Reviews subject classification: Primary: 26A33; Secondary: 42B25
Key words: one-sided operators, weighted inequalities
Received by the editors March 23, 2013
Communicated by: Alexander Olevskii

Let $1 \le r < \infty$. If $f = (f_i)_{i=1}^{\infty}$ is a sequence of locally integrable functions on $\mathbb{R}$, we define

$$|f(x)|_r = \left(\sum_{i=1}^{\infty} |f_i(x)|^r \right)^{1/r}.$$

For any measurable set E, χ_E will denote the characteristic function of E, and $|E|$ its Lebesgue measure.

A weight w will be a positive and locally integrable function defined on $\mathbb{R}$. If E is a measurable set, we will denote $w(E) = \int_E w(x)dx$.

Let $1 \le p, r < \infty$ and let w be a weight. The space $L^p_{\ell^r}(w)$ is the set of all the sequences $f = (f_i)_{i=1}^{\infty}$ such that

$$\|f\|_{L^p_{\ell^r}(w)} = \left(\int_{-\infty}^{\infty} |f(x)|_r^p w(x)dx \right)^{1/p} < \infty.$$

The linear space S_0 of all sequences $f = (f_i)_{i=1}^{\infty}$, where f_i is a simple function with compact support and $f_i \equiv 0$ for all sufficiently large i, is dense in $L^p_{\ell^r}(w)$, see [3].

Let g be a positive and locally integrable function defined on $\mathbb{R}$. For each locally integrable function f, the one-sided maximal Hardy-Littlewood operators of f with respect to g are

$$M_g^+(f)(x) = \sup_{h>0} \frac{\int_x^{x+h} |f(y)|g(y)dy}{\int_x^{x+h} g(y)dy},$$

and

$$M_g^-(f)(x) = \sup_{h>0} \frac{\int_{x-h}^{x} |f(y)|g(y)dy}{\int_{x-h}^{x} g(y)dy}.$$

If $f = (f_i)_{i=1}^{\infty}$ is a sequence of locally integrable functions defined on $\mathbb{R}$, we define

$$M_g^+(f) = (M_g^+(f_i))_{i=1}^{\infty}.$$

Similarly, we introduce the sequence $M_g^-(f)$. In the case $g \equiv 1$ we will denote $M^+(f)$ and $M^-(f)$, respectively.

The pair of weights (u, v) satisfies the condition $A_p^+(g)$, $p > 1$, if there exists a constant $C > 0$ such that for every real number a and every $h > 0$

$$\left(\int_{a-h}^{a} u \right) \left(\int_a^{a+h} g^{p'} v^{-1/(p-1)} \right)^{p-1} \le C \left(\int_{a-h}^{a+h} g \right)^p,$$

where $p' = \frac{p}{p-1}$, and (u,v) satisfies the condition $A_1^+(g)$ if

$$M_g^-(g^{-1}u)(x) \leq Cg(x)^{-1}v(x) \ a.e.$$

In the case $u = v$, we will denote $u \in A_p^+(g)$.

A weight w satisfies the condition $A_\infty^+(g)$ if there exists positive numbers K and η such that

$$\frac{g(E)}{g((a,c))} \leq K \left(\frac{w(E)}{w((a,b))} \right)^\eta,$$

for every $a < b < c$ and every measurable set $E \subseteq (b,c)$.

Analogously, the classes $A_p^-(g)$ are defined. If $g \equiv 1$, we will write A_p^+ or A_p^- which are the usual Muckenhoupt classes.

For the well known vector-valued maximal Hardy-Littlewood operator M, weighted inequalities were obtained by K. J. Andersen and R. T. John in [2]. Our following theorem provides a version of this boundedness for the one-sided maximal operator M_g^+.

Theorem 1. *Let g and w be positive and locally integrable functions defined on $\mathbb{R}$. Let $1 < r < \infty$.*

(a) If $1 \leq p < \infty$, then there exists a constant $C_{r,p} > 0$ such that the inequality

$$w\left(\{x \in \mathbb{R} : |M_g^+(f)(x)|_r > \alpha\}\right) \leq \frac{C_{r,p}}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|_r^p w(x)dx, \quad (1)$$

holds, if and only if, $w \in A_p^+(g)$.

(b) If $1 < p < \infty$, then there exists a constant $C_{r,p} > 0$ such that the inequality

$$\int_{-\infty}^{\infty} |M_g^+(f)(x)|_r^p w(x)dx \leq C_{r,p} \int_{-\infty}^{\infty} |f(x)|_r^p w(x)dx, \quad (2)$$

holds, if and only if, $w \in A_p^+(g)$.

Of course the corresponding inequalities for M_g^- hold if and only if $w \in A_p^-(g)$.

If $g \equiv 1$, in the scalar-valued case, for every $1 < p < \infty$, the inequality

$$\int_{-\infty}^{\infty} M^+(f)(x)^p w(x)dx \leq C_{r,p} \int_{-\infty}^{\infty} f(x)^p M^-(w)(x)dx,$$

is obtained as an immediate consequence of the maximal theorem (see Proposition 6 below) which implies (2) if $w \in A_p^+$.

In the scalar case a very smart way to proceed for weigthed one-sided operators is to use the Riesz's sunrise lemma (see, for instance [5]), with respect to wdx measure, to obtain the weak-$(1,1)$ boundedness, and using this and weak-(∞,∞) it is possible to obtain strong-(p,p) by means of Marcinkiewicz Interpolation. Unfortunately in the vector-valued case it seems that there is not an obvious way to deal with all the functions of the sequences $f = (f_i)_{i=1}^{\infty}$ to write $\{x \in \mathbb{R} : |M_g^+(f)(x)|_r > \alpha\}$ as the set of shadow points of a suitable function.

For the vector-valued operator the inequality for $p > 1$, (2), is proved in [7] by Guan and Sawyer for $g = \chi_I$ that is the characteristic function of an interval. Also for the vector-valued operator and for $g \equiv 1$, the case $p > 1$ –that is inequality: (2)– can be obtained as in the article of de Rosa and Segovia Fernández, see [4], by extrapolation from the easy case $p = r$ (see lemma 7).

Here we get the general case with any g, for $p > 1$, (2) (strong-(p,p)), and for $p = 1$, (1) (weak-$(1,1)$). The technique is similar of the one used in [2] for the two-sided vector-valued maximal Hardy-Littlewood operator (with $g \equiv 1$). The key to overcome the need of a left interval I^- with equal weight respect to g of a given interval I is to first consider the case with g such that $g((-\infty,b)) = \infty$ for every real number b -Proposition 8- and afterwards we obtain the result for an arbitary g by taking a suitable sequence of functions $(g_n)_{n=1}^{\infty}$ with the above property -that is Theorem 1-.

The problem for the two-sided vector-valued maximal Hardy-Littlewood operator M with respect the $g \equiv w$ was consider by C. Pérez in [11]. In the mentioned work C. Pérez makes the following conjecture:

$$w(\{x \in \mathbb{R}^n : |M_w^c(f)(x)|_r > \alpha\}) \leq \frac{C_r}{\alpha} \int_{-\infty}^{\infty} |f(x)|_r w(x)dx$$

holds for some constant C_r, where $f = (f_i)_{i=1}^{\infty}$, and where $M_w^c(f)(x) = \sup\limits_{h>0} \frac{\int_{B_h(x)} |f(y)|w(y)dy}{w(B_h(x))}$ is the maximal centered operator with respect to w. He showed that this result, if true, can be use as a vector valued version of the classical Besicovitch lemma. This would yield easily vector valued analogs of inequalities that in the classical version for functions are consequences of Besicovitch lemma. For instance, the theorem 1.4 in [11] could be obtained almost immediately.

Now, for $n = 1$, the mentioned conjecture of C. Pérez is true, as a consequence of Theorem 1, taking $g = w$ and using that $M_w^c(f)(x) \leq M_g^+(f)(x) + M_g^-(f)(x)$, $A_p^+(w) \cap A_p^-(w) = A_p(w)$, and $w \in A_p(w)$.

In the sequel, the letter C will denote a positive constant, not necessarily the same at each occurrence.

2 Preliminaries

A pair of weights (u,v) satisfies the condition $S_p^+(g), 1<p<\infty$, if there exists a constant $C>0$ such that for every interval $I=(a,b)$ with $u((-\infty,a))>0$,

$$\int_a^b M_g^+(\chi_I g^{\frac{1}{p-1}} v^{-\frac{1}{p-1}})(x)^p u(x)dx \le C\int_a^b g(x)^{p'} v(x)^{-\frac{1}{p-1}}dx<\infty.$$

If $u=v$, we will denote $u\in S_p^+(g)$.

Lemma 2. *Let g and w be positive and locally integrable functions defined on $\mathbb{R}$.*

(i) For $1<p<\infty, w\in A_p^+(g)$ if and only if $w^{1-p'}g^{p'}\in A_{p'}^-(g)$.

(ii) If $w\in A_p^+(g)$, then for all $q>p$ we have that $w\in A_q^+(g)$.

(iii) For $1<p<\infty$, if $w\in A_p^+(g)$, then there exists $q<p$ such that $w\in A_q^+(g)$.

(iv) $w\in A_\infty^+(g)$ if and only if there exists $p\ge 1$ such that $w\in A_p^+(g)$.

(v) For $1<p<\infty$, the conditions $w\in A_p^+(g)$ and $w\in S_p^+(g)$ are equivalent.

In the case $g\equiv 1$, the proof of these results can be found in [8] and [12]. For any g, see [9] and [10].

Lemma 3. *Let g and $\widetilde{g}$ be positive and locally integrable functions defined on $\mathbb{R}$, such that $g(x)\le\widetilde{g}(x)$ a.e. Let w belong to $A_p^+(g)$.*

(i) If $1<p<\infty$, then $(w, g^{-p}\widetilde{g}^p w)$ satisfies the condition $S_p^+(\widetilde{g})$ with the same constant $C_w>0$ for which holds $w\in S_p^+(g)$.

(ii) If $p=1$, then $(w, g^{-1}\widetilde{g}w)$ satisfies the condition $\in A_1^+(\widetilde{g})$ with the same constant $C_w>0$ for which holds $w\in A_1^+(g)$.

Proof.

(i) By Lemma 2 part (v), the conditions $w\in A_p^+(g)$ and $w\in S_p^+(g)$ are equivalent. Then, there exists a constant $C_w>0$ such that for every interval $I=(a,b)$ with $w((-\infty,a))>0$

$$\int_a^b M_g^+(\chi_I g^{\frac{1}{p-1}} w^{-\frac{1}{p-1}})^p w\le C_w\int_a^b g^{p'} w^{-\frac{1}{p-1}}. \tag{3}$$

We observe that

$$M^+_{\widetilde{g}}(\chi_I \widetilde{g}^{\frac{1}{p-1}}(g^{-p}\widetilde{g}^p w)^{-\frac{1}{p-1}})(t) = M^+_{\widetilde{g}}(\chi_I g^{\frac{1}{p-1}} w^{-\frac{1}{p-1}} g\widetilde{g}^{-1})(t)$$

$$= \sup_{h>0} \frac{\int_t^{t+h} \chi_I g^{\frac{1}{p-1}} w^{-\frac{1}{p-1}} g}{\int_t^{t+h} \widetilde{g}} \leq \sup_{h>0} \frac{\int_t^{t+h} \chi_I g^{\frac{1}{p-1}} w^{-\frac{1}{p-1}} g}{\int_t^{t+h} g}$$

$$= M^+_g(\chi_I g^{\frac{1}{p-1}} w^{-\frac{1}{p-1}})(t),$$

where in the last inequality we use the hypothesis $g(x) \leq \widetilde{g}(x)$ *a.e.* Then, for every t

$$M^+_{\widetilde{g}}(\chi_I \widetilde{g}^{\frac{1}{p-1}}(g^{-p}\widetilde{g}^p w)^{-\frac{1}{p-1}})(t) \leq M^+_g(\chi_I g^{\frac{1}{p-1}} w^{-\frac{1}{p-1}})(t).$$

This inequality, (3) and the hypothesis $g(x) \leq \widetilde{g}(x)$ *a.e.* imply that

$$\int_a^b M^+_{\widetilde{g}}(\chi_I \widetilde{g}^{\frac{1}{p-1}}(g^{-p}\widetilde{g}^p w)^{-\frac{1}{p-1}})^p w \leq \int_a^b M^+_g(\chi_I g^{\frac{1}{p-1}} w^{-\frac{1}{p-1}})^p w$$

$$\leq C_w \int_a^b g^{p'} w^{-\frac{1}{p-1}} \leq C_w \int_a^b \widetilde{g}^{p'} w^{-\frac{1}{p-1}}.$$

Therefore, $(w, g^{-p}\widetilde{g}^p w)$ satisfies the condition $S^+_p(\widetilde{g})$ with the constant C_w.

(ii) Since $w \in A^+_1(g)$ then,

$$M^-_g(g^{-1}w)(x) \leq C_w g(x)^{-1} w(x) \ a.e.$$

and using the hypothesis $g(x) \leq \widetilde{g}(x)$ *a.e.* we have that

$$M^-_{\widetilde{g}}(\widetilde{g}^{-1}w)(t) = \sup_{h>0} \frac{\int_{t-h}^t \widetilde{g}^{-1} w \widetilde{g}}{\int_{t-h}^t \widetilde{g}} \leq \sup_{h>0} \frac{\int_{t-h}^t g^{-1} w g}{\int_{t-h}^t g}$$

$$= M^-_g(g^{-1}w)(t) \leq C_w g(t)^{-1} w(t)$$

$$= C_w \widetilde{g}(t)^{-1}(g^{-1}\widetilde{g}w)(t) \ a.e.$$

That is, $(w, g^{-1}\widetilde{g}w)$ satisfies the condition $A^+_1(\widetilde{g})$ with the constant C_w. □

Theorem 4. *Let g and w be positive and locally integrable functions defined on $\mathbb{R}$.*

(a) *If $1 \leq p < \infty$, then there exists a constant $C_p > 0$ such that*

$$u\left(\{x \in \mathbb{R} : M_g^+(f)(x) > \alpha\}\right) \leq \frac{C_p}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|^p v(x)dx$$

if, and only if, (u, v) satisfies the condition $A_p^+(g)$.

(b) *If $1 < p < \infty$, then there exists a constant $C_p > 0$ such that*

$$\int_{-\infty}^{\infty} M_g^+(f)(x)^p u(x)dx \leq C_p \int_{-\infty}^{\infty} |f(x)|^p v(x)dx$$

if, and only if, (u, v) satisfies the condition $S_p^+(g)$.

In the case $g \equiv 1$ and $u = v$, the proof can be found in [8] and [12]. For any g, see [9].

Lemma 5. *Let g be a positive and locally integrable function defined on $\mathbb{R}$. Assume that $f \geq 0$ is integrable with compact support. If $I = (a, b)$ is a component interval of the open set $\{x \in \mathbb{R} : M_g^+(f)(x) > \alpha\}, \alpha > 0$, then*

$$\alpha \leq \frac{\int_x^b fg}{\int_x^b g}, \quad a \leq x < b.$$

The proof is similar to the one of Lema 2.1 in [12] and it will be omitted. Also the latter lemma may be obtained with using the Riesz's rising sun lemma.

Proposition 6. *Let g and w be positive and locally integrable functions defined on $\mathbb{R}$.*

(a) *If $1 < p < \infty$, then there exists a constant $C_p > 0$ such that the inequality*

$$\int_{-\infty}^{\infty} M_g^+(f)^p(x)w(x)dx \leq C_p \int_{-\infty}^{\infty} |f(x)|^p g(x) M_g^-(g^{-1}w)(x)dx$$

holds.

(b) *Let f be an integrable function with compact support and let $I = (a, b)$ be a component interval of the open set $\{x \in \mathbb{R} : M_g^+(f)(x) > \alpha\}, \alpha > 0$. Then,*

$$w(I) \leq \frac{4}{\alpha} \int_I |f(x)| g(x) M_g^-(g^{-1}w)(x)dx.$$

PROOF. (a) By Theorem 1 in [9], since the pair $(w, gM_g^-(g^{-1}w))$ satisfies the condition $A_1^+(g)$, it follows that there exists a constant $C>0$ such that for every $\alpha>0$

$$w(\{x\in\mathbb{R} : M_g^+(f)(x)>\alpha\}) \leq \frac{C}{\alpha}\int_{-\infty}^{\infty}|f(x)|g(x)M_g^-(g^{-1}w)(x)dx.$$

On the other hand, M_g^+ is a bounded operator from $L^\infty(gM^-(g^{-1}w))$ to $L^\infty(w)$. Then, applying the Marcinkiewicz interpolation theorem, we obtain the part (a) of this proposition.

(b) Let $(x_s)_{s=0}^\infty$ be a sequence in I defined as $x_0=a$ and for every $s\geq 1$

$$\int_{x_s}^b |f(y)|g(y)dy = \frac{1}{2^s}\int_a^b |f(y)|g(y)dy.$$

Then $(x_s)_{s=0}^\infty$ is an increasing sequence and $\lim_{s\to\infty} x_s = b$. Applying Lemma 5 we have that

$$\alpha \leq \frac{4}{g([x_s,x_{s+2}])}\int_{x_{s+1}}^{x_{s+2}}|f(y)|g(y)dy.$$

Thus,

$$\begin{aligned}
w(I) &= \sum_{s=0}^\infty \int_{x_s}^{x_{s+1}} w(x)dx \\
&\leq \frac{1}{\alpha}\sum_{s=0}^\infty \frac{4}{g([x_s,x_{s+2}])}\int_{x_{s+1}}^{x_{s+2}}|f(y)|g(y)dy\int_{x_s}^{x_{s+1}}w(x)dx \\
&\leq \frac{4}{\alpha}\sum_{s=0}^\infty \int_{x_{s+1}}^{x_{s+2}}|f(y)|g(y)\frac{1}{g([x_s,y])}\int_{x_s}^{y}w(x)dxdy \\
&\leq \frac{4}{\alpha}\sum_{s=0}^\infty \int_{x_{s+1}}^{x_{s+2}}|f(y)|g(y)M_g^-(g^{-1}w)(y)dy \\
&\leq \frac{4}{\alpha}\int_a^b |f(y)|g(y)M_g^-(g^{-1}w)(y)dy,
\end{aligned}$$

which proves part (b) of our proposition. □

3 Proof of Theorem 1

Lemma 7. *Let g be a positive and locally function defined on $\mathbb{R}$. Let $1 < r < \infty$. There exists a constant $C_r > 0$ such that the inequality*

$$\int_{-\infty}^{\infty} |M_g^+(f)(x)|_r^r u(x)dx \leq C_r \int_{-\infty}^{\infty} |f(x)|_r^r v(x)dx, \tag{4}$$

holds if, and only if, the pair (u, v) satisfies the condition $S_r^+(g)$. Moreover, in the case $u = v$, the inequality (4) holds if, and only if, $u \in A_r^+(g)$.

PROOF. With respect to the necessity, there is nothing to prove since, using Theorem 4 part (b), this condition is necessary in the scalar-valued case, that is, $f = (f_i)_{i=1}^{\infty}$ with $f_i \equiv 0$ for every $i \geq 2$. As a consequence of the same result, if (u, v) satisfies $S_r^+(g)$, then

$$\int_{-\infty}^{\infty} |M_g^+(f)(x)|_r^r u(x)dx = \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} |M_g^+(f_i)(x)|^r u(x)dx$$

$$\leq C_r \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} |f_i(x)|^r v(x)dx = C_r \int_{-\infty}^{\infty} |f(x)|_r^r v(x)dx.$$

In the case $u = v$, by Lemma 2 part (v), the condition $u \in S_r^+(g)$ is equivalent to $u \in A_r^+(g)$. □

Proposition 8. *Let g be a positive and locally integrable function defined on $\mathbb{R}$. Assume that $g((-\infty, b)) = \infty$ for every real number b.*

(i) Let $1 < p \leq r < \infty$. If (u, v) satisfies the condition $S_p^+(g)$, $u \in A_p^+(g)$ and $u(x) \leq v(x)$ a.e., then there exists a constant $C_{r,p} > 0$ such that the inequality

$$u\left(\{x \in \mathbb{R} : |M_g^+(f)(x)|_r > \alpha\}\right) \leq \frac{C_{r,p}}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|_r^p v(x)dx$$

holds.

(ii) Let $1 = p < r < \infty$. If $u \in A_1^+(g)$ and $u(x) \leq v(x)$ a.e., then there exists a constant $C_r > 0$ such that the inequality

$$u\left(\{x \in \mathbb{R} : |M_g^+(f)(x)|_r > \alpha\}\right) \leq \frac{C_r}{\alpha} \int_{-\infty}^{\infty} |f(x)|_r v(x)dx$$

holds.

PROOF. (i) The case $p = r$ is an immediate consequence of Lemma 7.

Assume that $1 < p < r < \infty$, the pair (u, v) satisfies the condition $S_p^+(g), u \in A_p^+(g)$ and $\alpha > 0$. Without loss of generality, we can suppose that f belongs to the linear subspace S_0 then, the general case follows by a standard argument of density. Let $\Omega = \{x \in \mathbb{R} : M_g^+(|f|_r)(x) > \alpha\}$. Since $|f|_r \in L^1$, by Theorem 4 part (a), then Ω is an open set of finite measure. Thus,

$$\Omega = \bigcup_j I_j$$

where $(I_j)_{j=1}^\infty$ are its component intervals. By Lebesgue differentiation theorem we have that

$$|f(x)|_r \leq \alpha\ ,\ \ a.e.\ x \notin \Omega, \tag{5}$$

and Lemma 5 implies that

$$\frac{1}{\int_{I_j} g(y)dy} \int_{I_j} |f(y)|_r g(y)dy = \alpha \quad , \quad j \geq 1. \tag{6}$$

Let $f = f' + f''$ where $f' = (f_i')_{i=1}^\infty, f_i'(x) = f_i(x)\chi_{\mathbb{R}-\Omega}(x)$. By Minkowski's inequality we have

$$|M_g^+(f)|_r \leq |M_g^+(f')|_r + |M_g^+(f'')|_r.$$

Then, (i) will follow if we prove that

$$u\left(\{x \in \mathbb{R} : |M_g^+(f')(x)|_r > \alpha\}\right) \leq \frac{C_{r,p}}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|_r^p v(x)dx \tag{7}$$

and

$$u\left(\{x \in \mathbb{R} : |M_g^+(f'')(x)|_r > \alpha\}\right) \leq \frac{C_{r,p}}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|_r^p v(x)dx. \tag{8}$$

Since $u \in A_p^+(g)$ and $r > p$, by Lemma 2 part (ii), then $u \in A_r^+(g)$. Applying Chebyshev's inequality and Lemma 7 we obtain that

$$\begin{aligned} & u\left(\{x \in \mathbb{R} : |M_g^+(f')(x)|_r > \alpha\}\right) \\ & \leq \frac{1}{\alpha^r} \int_{-\infty}^{\infty} |M_g^+(f')(x)|_r^r u(x)dx \\ & \leq \frac{C_r}{\alpha^r} \int_{-\infty}^{\infty} |f'(x)|_r^r u(x)dx. \end{aligned}$$

By (5), it follows that $|f'(x)|_r^r \leq \alpha^{r-p}|f(x)|_r^p$. Taking into account the hypothesis $u(x) \leq v(x)$ *a.e.*, we obtain (7).

In order to prove (8), we define $\overline{f} = (\overline{f}_i)_{i=1}^{\infty}$ as

$$\overline{f}_i(x) = \begin{cases} \frac{1}{\int_{I_j} g(y)dy} \int_{I_j} |f_i(y)| g(y)dy & \text{if } x \in I_j, j \geq 1 \\ \\ 0 & \text{otherwise,} \end{cases}$$

and let $\overline{I}_j = I_j^- \cup I_j$ where $g(I_j^-) = g(I_j)$. Our hypothesis $g((-\infty, b)) = \infty$ for every real number b, allows us to obtain these intervals I_j^-.

We will see that for every $i \geq 1$

$$M_g^+(f_i'')(x) \leq C M_g^+(\overline{f}_i)(x) \quad , \quad x \notin \overline{\Omega} = \cup_j \overline{I}_j. \tag{9}$$

In fact, let $x \notin \overline{\Omega}$ and $z > x$. Then,

$$\frac{1}{g([x,z])} \int_x^z |f_i''(t)| g(t)dt = \frac{1}{g([x,z])} \sum_{j \in J_{x,z}} \int_{[x,z] \cap I_j} |f_i''(t)| g(t)dt,$$

where $J_{x,z} = \{j : [x,z] \cap I_j \neq \emptyset\}$.

There exists two possibilities.

(a) $z \notin I_j$ for every j.

(b) $z \in I_{j_0}$ for some j_0.

In the case (a), since $x \notin \overline{\Omega} \supset \Omega = \cup I_j$ then, for every j we have that $x \notin I_j$ and $z \notin I_j$. In consequence, for every j we obtain $[x,z] \cap I_j = I_j$ or $[x,z] \cap I_j = \emptyset$. Taking into account that $(I_j)_{j=1}^{\infty}$ are disjoint intervals, and according with the definition of $\overline{f}_i$, it follows that

$$\begin{aligned} \frac{1}{g([x,z])} \int_x^z |f_i''(t)| g(t)dt &= \frac{1}{g([x,z])} \sum_{j \in J_{x,z}} \int_{I_j} |f_i''(t)| g(t)dt \\ &= \frac{\sum_{j \in J_{x,z}} \int_{I_j} |\overline{f}_i(t)| g(t)dt}{g([x,z])} \\ &\leq \frac{\int_x^z |\overline{f}_i(t)| g(t)dt}{g([x,z])} \\ &\leq M_g^+(\overline{f}_i)(x). \end{aligned}$$

In the case (b), since $I_j = (a_j, b_j)$ are disjoint intervals we have that if $j \in J_{x,z}$ and $j \neq j_0$, then $b_j < z$. Thus, we can decompose,

$$\begin{aligned}\frac{\int_x^z |f_i''(t)|g(t)dt}{g([x,z])} &\leq \frac{\sum_{j\in J_{x,z}-\{j_0\}} \int_{I_j} |f_i''(t)|g(t)dt}{g([x,z])} + \frac{\int_{a_{j_0}}^{b_{j_0}} |f_i''(t)|g(t)dt}{g([x,z])} \\ &= \frac{\sum_{j\in J_{x,z}-\{j_0\}} \int_{I_j} |\overline{f}_i(t)|g(t)dt}{g([x,z])} + \frac{\int_{a_{j_0}}^{b_{j_0}} |\overline{f}_i(t)|g(t)dt}{g([x,z])}.\end{aligned}$$

Let $b = \sup\left(\cup_{j\in J_{x,z}-\{j_0\}} I_j\right) \leq a_{j_0} < z$. Then,

$$\begin{aligned}\frac{\int_x^z |f_i''(t)|g(t)dt}{g([x,z])} &\leq \frac{\int_x^b |\overline{f}_i(t)|g(t)dt}{g([x,b])} + \frac{\int_x^{b_{j_0}} |\overline{f}_i(t)|g(t)dt}{g([x,z])} \\ &\leq M_g^+(\overline{f}_i)(x) + \frac{\int_x^{b_{j_0}} |\overline{f}_i(t)|g(t)dt}{g([x,z])}.\end{aligned}$$

Since $x \notin \overline{I}_{j_0} = I_{j_0}^- \cup I_{j_0}$ and $z \in I_{j_0}$, then $I_{j_0}^- \subset [x,z]$, and if we denote $I_{j_0}^- = (c_{j_0}, a_{j_0}]$ it follows that

$$\begin{aligned}g([x,z]) &= g([x,c_{j_0}]) + g(I_{j_0}^-) + g((a_{j_0}, z]) \\ &\geq g([x,c_{j_0}]) + g(I_{j_0}^-) \\ &= g([x,c_{j_0}]) + \frac{1}{2}g(I_{j_0}^-) + \frac{1}{2}g(I_{j_0}) \\ &\geq \frac{1}{2}g([x,b_{j_0}]).\end{aligned}$$

In consequence,

$$\frac{\int_x^z |f_i''(t)|g(t)dt}{g([x,z])} \leq M_g^+(\overline{f}_i)(x) + 2\frac{\int_x^{b_{j_0}} |\overline{f}_i(t)|g(t)dt}{g([x,b_{j_0}])} \leq 3M_g^+(\overline{f}_i)(x).$$

Since the last inequality holds for every $z > x$ we obtain (9).

Therefore, we will have (8) if we are able to prove

$$u(\overline{\Omega}) \leq \frac{C_{r,p}}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|_r^p v(x)dx \tag{10}$$

and

$$u\left(\{x \in \mathbb{R} : |M_g^+(\overline{f})(x)|_r > \alpha\}\right) \leq \frac{C_{r,p}}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|_r^p v(x)dx. \tag{11}$$

In [9] (page 520 for $p = 1$ and page 522 for $p > 1$), it is proved that for each $j \geq 1$

$$u(I_j) \leq \frac{C_p}{\alpha^p} \int_{I_j} |f(x)|_r^p v(x)dx. \tag{12}$$

Since $u \in A_p^+(g)$, by Lemma 2 part (iv), $u \in A_\infty^+(g)$ and, recalling that $g(I_j^-) = g(I_j)$, there exists a constant $C > 0$ satisfying

$$u(I_j^-) \leq Cu(I_j).$$

Then, collecting these inequalities and summing on $j \geq 1$, we obtain (10).

On the other hand, by Minkowski's inequality and (6), if $x \in \Omega$ we have that

$$\begin{aligned} |\overline{f}(x)|_r &= \left(\sum_{i=1}^{\infty} |\overline{f}_i(x)|^r\right)^{1/r} \\ &= \left(\sum_{i=1}^{\infty} \left[\frac{1}{\int_{I_j} g(y)dy} \int_{I_j} |f_i(y)|g(y)dy\right]^r\right)^{1/r} \\ &\leq \frac{1}{\int_{I_j} g(y)dy} \int_{I_j} \left(\sum_{i=1}^{\infty} |f_i(y)|^r\right)^{1/r} g(y)dy \\ &= \alpha. \end{aligned}$$

Thus, $|\overline{f}(x)|_r$ is supported in Ω and $|\overline{f}(x)|_r \leq \alpha$. Then applying Chebyshev's inequality and Lemma 7 it follows that

$$\begin{aligned} u\left(\{x \in \mathbb{R} : |M_g^+(\overline{f})(x)|_r > \alpha\}\right) &\leq \frac{1}{\alpha^r} \int_{-\infty}^{\infty} |M_g^+(\overline{f})(x)|_r^r u(x)dx \\ &\leq \frac{C_r}{\alpha^r} \int_{-\infty}^{\infty} |\overline{f}(x)|_r^r u(x)dx \\ &\leq C_r \int_{\Omega} u(x)dx. \end{aligned}$$

Now, using (12) we obtain (11).

Part (ii) follows the same arguments and it will be omitted. Then, the proposition is proved. □

PROOF OF THEOREM 1. With respect to the necessity, there is nothing to prove since, using Theorem 4 part (b), this condition is necessary in the scalar-valued case, that is, $f = (f_i)_{i=1}^{\infty}$ with $f_i \equiv 0$ for every $i \geq 2$.

The case $r = p$ is proved in Lemma 7.

Let $r > p \geq 1$. If g satisfies the condition $g((-\infty, b)) = \infty$ for every real number b, since $w \in A_p^+(g)$ implies that $w \in S_p^+(g)$ (see Lemma 2 part (v)), using Proposition 8, we have the part (a) of this theorem.

For any g we define the sequence of functions $(g_n)_{n=1}^{\infty}$ as

$$g_n(x) = \begin{cases} g(x) & \text{if } x \geq -n \\ \\ \max\{1, g(x)\} & \text{if } x < -n. \end{cases}$$

We observe that $g(x) \leq g_n(x)$ for every $n \geq 1$ and every real number x. Then, since $w \in A_p^+(g)$, by Lemma 3, we have that the pair $(w, g^{-p} g_n^p w)$ satisfies the condition $S_p^+(g_n)$ with the same constant $C_w > 0$ for which holds $w \in S_p^+(g)$ in the case $p > 1$, and $w \in A_1^+(g_n)$ with the same constant $C_w > 0$ for which holds $w \in A_1^+(g)$ in the case $p = 1$.

We also have that $g_n((-\infty, b)) = \infty$ for every real number b and $w(x) \leq (g^{-p} g_n^p w)(x)$ *a.e.* for every $n \geq 1$. Then, by Proposition 8, for every $n \geq 1$ we obtain

$$w\left(\{x \in \mathbb{R} : |M_{g_n}^+(f)(x)|_r > \alpha\}\right) \leq \frac{C_{r,p}}{\alpha^p} \int_{-\infty}^{\infty} |f(x)|_r^p g(x)^{-p} g_n(x)^p w(x) dx.$$

Applying this inequality to the function $f\chi_{(-n,\infty)}$ and taking into account that $M_{g_n}^+(f)(x) = M_g^+(f)(x), x \geq -n$, we have

$$w\left(\{x \geq -n : |M_g^+(f)(x)|_r > \alpha\}\right) \leq \frac{C_{r,p}}{\alpha^p} \int_{-n}^{\infty} |f(x)|_r^p w(x) dx.$$

Then, tending n to infinite we obtain (1) in the case $p < r$.

If $r > p > 1$, then by Lemma 2 parts (ii) and (iii), it follows that $w \in A_{p_1}^+(g) \cap A_{p_2}^+(g)$ with $p_1 < p < p_2 < r$, and we have (1) with p_1 and p_2. Then, by Lemma 2.1 in [2], we obtain (2) with p.

Now, suppose that $p > r$ and $w \in A_p^+(g)$. Applying Lemma 2 part (iii), there exists $r_0 : 1 < r_0 < p$ such that $w \in A_q^+(g)$ with $q \geq p/r_0$, and by

Lemma 2 part (i), $w^{1-q'}g^{q'} \in A^-_{q'}(g)$. Using Theorem 4 part (b), for M_g^-, if $\varphi \geq 0$ belongs to the unit ball of $L^{q'}(w)$, then

$$\int_{-\infty}^{\infty} M_g^-(g^{-1}\varphi w)(x)^{q'} g(x)^{q'} w(x)^{1-q'} dx \leq C_q \int_{-\infty}^{\infty} |\varphi(x)|^{q'} w(x) dx = C_q.$$

For every $1 < s < \infty$, by Proposition 6 part (a), the inequality

$$\int_{-\infty}^{\infty} M_g^+(h)(x)^s u(x) dx \leq C_s \int_{-\infty}^{\infty} |h(x)|^s g(x) M_g^-(g^{-1}u)(x) dx$$

holds. Then, taking into account these two inequalities and applying Hölder's inequality we obtain

$$\int_{-\infty}^{\infty} |M_g^+(f)(x)|_r^r \varphi(x) w(x) dx \leq C_r \int_{-\infty}^{\infty} |f(x)|_r^r \frac{M_g^-(g^{-1}\varphi w)(x)}{w(x)} g(x) w(x) dx$$

$$\leq C_r \left(\int_{-\infty}^{\infty} |f(x)|_r^{rq} w(x) dx\right)^{1/q} \left(\int_{-\infty}^{\infty} M_g^-(g^{-1}\varphi w)(x) g(x)^{q'} w(x)^{1-q'} dx\right)^{1/q'}$$
$$\leq C_r \left(\int_{-\infty}^{\infty} |f(x)|_r^{rq} w(x) dx\right)^{1/q} C_q \left(\int_{-\infty}^{\infty} |\varphi(x)|^{q'} w(x) dx\right)^{1/q'}$$
$$= C_{r,q} \left(\int_{-\infty}^{\infty} |f(x)|_r^{rq} w(x) dx\right)^{1/q}.$$

Taking the supremum over all the functions φ in the unit ball of $L^{q'}(w)$ it follows that the inequality

$$\left(\int_{-\infty}^{\infty} |M_g^+(f)(x)|_r^{rq} w(x) dx\right)^{1/q} \leq C_{r,q} \left(\int_{-\infty}^{\infty} |f(x)|_r^{rq} w(x) dx\right)^{1/q} \qquad (13)$$

holds for every $r : 1 < r \leq r_0$.

Since $w \in A_p^+(g)$, applying Lemma 7, we have that

$$\int_{-\infty}^{\infty} |M_g^+(f)(x)|_p^p w(x) dx \leq C_p \int_{-\infty}^{\infty} |f(x)|_p^p w(x) dx.$$

By (13) with $r = r_0$ and $q = \frac{p}{r_0}$ it follows that

$$\int_{-\infty}^{\infty} |M_g^+(f)(x)|_{r_0}^p w(x) dx \leq C_{r_0,p} \int_{-\infty}^{\infty} |f(x)|_{r_0}^p w(x) dx.$$

Collecting these inequalities and using Lemma 2.2 in [2] with $p = p_i = q_i$, $s_0 = r_0$ and $s_1 = r_1 = p$ we obtain that the inequality

$$\int_{-\infty}^{\infty} |M_g^+(f)(x)|_r^p w(x)dx \leq C_{r,p} \int_{-\infty}^{\infty} |f(x)|_r^p w(x)dx$$

holds for $r_0 < r < p$. The proof of this theorem is complete. □

Acknowledgment. The author wishes to thank the anonymous referees for their advice that improved this paper.

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