Real Analysis Exchange
Vol. 39(2), 2013/2014, pp. 363–366

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REMARKS ON A SUM INVOLVING BINOMIAL COEFFICIENTS

Abstract

Let
$$S_p(a,b;t)=\frac{1}{b}\sum_{k=0}^{p}\frac{\binom{p}{k}}{\binom{ak+b}{b}}t^k,$$
with $p\in\mathbf{N}$, $0<a\in\mathbf{R}$, $0<b\in\mathbf{R}$, $t\in\mathbf{R}$. We prove that $S_p(a,b;t)$ is completely monotonic on $(0,\infty)$ as a function of a (if $t>0$) and as a function of b (if $t\geq -1$). This extends a result of Sofo, who proved that $a\mapsto S_p(a,b;t)$ is strictly decreasing, convex, and log-convex on $[1,\infty)$.

Here, we are concerned with the sum

$$S_p(a,b;t)=\frac{1}{b}\sum_{k=0}^{p}\frac{\binom{p}{k}}{\binom{ak+b}{b}}t^k,$$

with $p\in\mathbf{N}$, $0<a\in\mathbf{R}$, $0<b\in\mathbf{R}$, and $t\in\mathbf{R}$. As usual, we use the notation

$$\binom{x}{y}=\frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}\quad (x,y\in\mathbf{R}),$$

where Γ denotes Euler's gamma function.

The work on this note has been inspired by two interesting papers published by Sofo [3,4]. He used the elegant integral representation

$$S_p(a,b;t)=\int_0^1(1-x)^{b-1}(1+tx^a)^p\,dx \tag{1}$$

Mathematical Reviews subject classification: Primary: 26A48; Secondary: 26B20, 11B65
Key words: Finite sum, binomial coefficients, completely monotonic
Received by the editors September 10, 2013
Communicated by: Brian S. Thomson

to prove that if $t > 0$, then the function $a \mapsto S_p(a, b; t)$ is strictly decreasing and convex on $[1, \infty)$. Moreover, he applied the Cauchy-Schwarz inequality for integrals to prove that $a \mapsto S_p(a, b; t)$ is even log-convex on $[1, \infty)$.

A function $f : I \to \mathbf{R}$, where $I \subset \mathbf{R}$ is an interval, is called completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x \in I \quad \text{and} \quad n = 0, 1, 2, ... \tag{2}$$

Dubourdieu [2] pointed out that if f is a non-constant completely monotonic function on $I = (c, \infty)$, then (2) holds with "$>$" instead of "$\geq$".

Since a completely monotonic function is not only decreasing and convex, but also log-convex, it is tempting to conjecture that $a \mapsto S_p(a, b; t)$ is completely monotonic. We verify that this is true. Actually, we prove a bit more. We show that the integral representation (1) can be used to provide short and simple proofs for the complete monotonicity of $S_p(a, b; t)$ as a function of a and as a function of b. The following extension of Sofo's result holds.

Theorem. *Let p be a positive integer, a, b be positive real numbers, and t be a real number.*

(i) If $t > 0$, then $a \mapsto S_p(a, b; t)$ is completely monotonic on $(0, \infty)$.

(ii) If $t \geq -1$, then $b \mapsto S_p(a, b; t)$ is completely monotonic on $(0, \infty)$.

PROOF. (i) Using the binomial formula

$$(1 + tx^a)^p = \sum_{\nu=0}^{p} \binom{p}{\nu} t^\nu x^{a\nu}$$

we obtain

$$\begin{aligned} \int_0^1 (1-x)^{b-1}(1+tx^a)^p dx &= \int_0^1 (1-x)^{b-1} \sum_{\nu=0}^{p} \binom{p}{\nu} t^\nu x^{a\nu} dx \\ &= \sum_{\nu=0}^{p} \binom{p}{\nu} t^\nu \int_0^1 (1-x)^{b-1} x^{a\nu} dx. \end{aligned} \tag{3}$$

Since

$$\frac{d^n}{da^n} x^{a\nu} = (\nu \log x)^n x^{a\nu} \quad (n = 0, 1, 2, ...),$$

we conclude from (1) and (3) that

$$(-1)^n \frac{d^n}{da^n} S_p(a, b; t) = \sum_{\nu=0}^{p} \binom{p}{\nu} t^\nu \int_0^1 (1-x)^{b-1} \nu^n (-\log x)^n x^{a\nu} dx > 0.$$

(ii) We have

$$\frac{d^n}{db^n}(1-x)^{b-1} = (\log(1-x))^n(1-x)^{b-1} \quad (n = 0, 1, 2, ...).$$

Applying (1) yields

$$(-1)^n \frac{d^n}{db^n} S_p(a, b; t) = \int_0^1 (1-x)^{b-1}(-\log(1-x))^n(1+tx^a)^p dx > 0.$$

The proof is complete. $\square$

Completely monotonic functions have interesting applications in various fields, like, for instance, probability theory and numerical analysis. The most important properties of these functions are collected in [5, Chapter IV]. In the recent past, numerous papers on completely monotonic functions appeared. The authors showed that certain functions, which are defined in terms of special functions, like, for instance, the gamma and polygamma functions, are completely monotonic and applied their results to present many new inequalities. We refer to [1] and the references therein.

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