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**Real Analysis Exchange**
Vol. 39(2), 2013/2014, pp. 345–362

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# WEIGHTED A PRIORI ESTIMATES FOR THE SOLUTION OF THE DIRICHLET PROBLEM IN POLYGONAL DOMAINS IN $\mathbb{R}^2$

**Abstract**

Let $\Omega$ be a polygonal domain in $\mathbb{R}^2$ and let $U$ be a weak solution of $-\Delta u = f$ in $\Omega$ with Dirichlet boundary condition, where $f \in L^p_\omega(\Omega)$ and $\omega$ is a weight in $A_p(\mathbb{R}^2)$, $1 < p < \infty$. We give some estimates of the Green function associated to this problem involving some functions of the distance to the vertices and the angles of $\Omega$. As a consequence, we can prove an a priori estimate for the solution $u$ on the weighted Sobolev spaces $W^{2,p}_\omega(\Omega)$, $1 < p < \infty$.

## 1 Introduction

Given a polygonal domain $\Omega$ in $\mathbb{R}^2$, we consider the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Mathematical Reviews subject classification: Primary: 35A23, 35J05; Secondary: 35J08
Key words: Dirichlet problem, Green function, Weighted Sobolev spaces
Received by the editors July 2, 2013
Communicated by: V. I. Kolyada
*The research for this paper was supported by UNLP.
†The research for this paper was supported by CONICET, CAI+D (UNL) trought grants 50020110100009, 5002110100048, 50120110100382 and ANPCyT trought grant PICT-2008-2057.

where $f \in L^p_\omega(\Omega)$ and $\omega$ is a weight in the Muckenhoupt class $A_p(\mathbb{R}^2)$.

Estimates for this solution in the classical Sobolev spaces were given by Grisvard in [6] where we can see a dependence of the angles of $\Omega$. Therefore, it is a natural question whether weighted a priori estimates are valid also for the solution of the Dirichlet problem (1). In this paper we give a positive answer to this question, namely, we prove that for $1 < p < \infty$,

$$\|u\|_{L^p_\omega(\Omega)} + \sum_{|\beta|=1} \|\rho(x) D^\beta_x u\|_{L^p_\omega(\Omega)} + \sum_{|\alpha|=2} \|\sigma(x) D^\alpha_x u\|_{L^p_\omega(\Omega)} \leq C\|f\|_{L^p_\omega(\Omega)}, \quad (2)$$

where $\rho(x)$ and $\sigma(x)$ are suitable functions depending on the distance from $x$ to the nearest vertex of $\Omega$ and the corresponding angle, and $C$ is a constant depending only on $\Omega$.

The paper is organized as follows: In Section 2 we remind the already known estimates for the Green function (and its derivatives) of the problem (1) when $\Omega$ is a disk, and we define the Schwarz-Christoffel mapping. These will be the main tools for the proof of our main result (2). In Section 3 and Section 4 we state the estimates for the Green function and its derivatives when $\Omega$ is a convex and a non-convex polygon respectively. Finally, in Section 5 we give the proof of the estimate in (2).

## 2 Preliminaries

The solution of (1) is given by

$$u(x) = \int_\Omega G_\Omega(x, y)\, f(y)\, dy, \quad (3)$$

where $G_\Omega$ is the Green function for $\Omega$ which can be written as

$$G_\Omega(x, y) = \Gamma(x - y) + H_\Omega(x, y), \quad (4)$$

where

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

and $H_\Omega(x, y)$ satisfies, for each fixed $y \in \Omega$,

$$\begin{cases} \Delta_x H_\Omega(x, y) = 0 & \text{in } \Omega \\ H_\Omega(x, y) = -\Gamma(x - y) & \text{on } \partial\Omega. \end{cases}$$

For a conformal mapping $h$ from the unit disc $B$ to $\Omega$ it holds that $\Delta(u \circ h) = |h'|^2(\Delta u) \circ h$, where $|h'|^2$ is the Jacobian of $h$. Then, $u \circ h$ satisfies

$$\begin{cases} -\Delta(u \circ h) = |h'|^2(f \circ h) & \text{in } B \\ u \circ h = 0 & \text{on } \partial B, \end{cases}$$

and for $\xi \in B$ we have

$$(u \circ h)(\xi) = \int_B G_B(\xi, \eta)\, (f \circ h)(\eta)\, |h'|^2 d\eta,$$

where

$$G_B(\xi, \eta) = \frac{1}{2\pi} \log |\eta - \xi|^{-1} - \frac{1}{2\pi} \log \left( |\xi| \left| \eta - \frac{\xi}{|\xi|^2} \right| \right)^{-1}$$

is the Green function in $B$.

Let $g : \Omega \to B$ be the inverse mapping of $h$, then

$$G_\Omega(x, y) = G_B(\xi, \eta), \tag{5}$$

and

$$H_\Omega(x, y) = H_B(\xi, \eta), \tag{6}$$

where $\xi = g(x)$ and $\eta = g(y)$.

From the known estimates

$$|D_\xi^\alpha G_B(\xi, \eta)| \leq C\, |\xi - \eta|^{-|\alpha|}\ \min \left\{ 1, \frac{d_B(\eta)}{|\xi - \eta|} \right\} \qquad \text{for } |\alpha| = 1, 2,$$

(see for example [4]), where $d_B(\eta)$ denotes the distance from $\eta$ to the boundary of $B$ we have

$$|D_\xi^\alpha G_B(\xi, \eta)| \leq C\, |\xi - \eta|^{-|\alpha|}\ \min \left\{ 1, \frac{d_B(\xi)}{|\xi - \eta|} \right\} \qquad \text{for } |\alpha| = 1, 2. \tag{7}$$

Observe that the letter $C$ denotes a generic constant not necessarily the same at each occurrence. We will write $f \preceq g$ if there exists a constant $C > 0$ such that $f \leq C\, g$.

**The Schwarz-Christoffel mapping.** Given a polygonal domain $\Omega$ with $N$ sides, for $j = 1, \cdots, N$, we denote by $z_j$ and $\theta_j$ its vertices and corresponding interior angles respectively. Let $k_j \in \mathbb{R}$ be such that $k_j \pi + \theta_j = \pi$. Observe that $0 < k_j < 1$ corresponds to $0 < \theta_j < \pi$ while $-1 < k_j \leq 0$ to $\pi \leq \theta_j < 2\pi$. In particular, if $\Omega$ is convex, all the $k_j$ are positive numbers.

Given complex numbers $w_j$ such that $|w_j| = 1$ for $j = 1, \cdots, N$, we define, for $\xi \in B$,

$$h'(\xi) := (\xi - w_1)^{-k_1} (\xi - w_2)^{-k_2} ... (\xi - w_N)^{-k_N},$$

which is analytic in the interior of $B$.

Then, the Schwarz-Christoffel mapping $h : B \to \Omega$ is defined as

$$h(\xi) = \int_{\xi_0}^{\xi} h'(s)\, ds,$$

where the integral is taken over the segment from a fixed $\xi_0 \in B$ to $\xi$. Note that $h$ is analytic on the same region as $h'$, continuous on $B$ and maps the points inside the unit disk $B$ to the points inside the simple closed polygon with vertex at $z_j = h(w_j)$. We will say that $w_j$ are the pre-vertices of $\Omega$. For more details about this mapping see, for example, [2].

We introduce $d_m = \min_{i\neq j} |w_i - w_j|$ and define $B_j = \overline{B(w_j, \frac{d_m}{4})} \cap B$ , for $j = 1, \cdots, N$, and $B_{N+1} = B \setminus \cup_{j=1}^{N} B_j$. Then, $\Omega_j = h(B_j)$ is a neighborhood of $z_j$ and $\Omega = \cup_{j=1}^{N+1} \Omega_j$. We will analyze the behavior of the Green function $G_\Omega$ near each vertex $z_j$. The following remark outlines some useful observations.

**Remark 1.** *For $\xi \in B_j$, with $j = 1, \cdots, N$, we have*

1. *If $\eta \in B_j$ and $s$ is in the segment from $\xi$ to $\eta$, then $|s - w_i| > \frac{d_m}{4}$ when $i \neq j$.*
2. *If $\eta \in B_i$ with $i \neq j$ and $i \neq N+1$, then $|\xi - \eta| > \frac{d_m}{4}$.*
3. *If $\eta \in B_{N+1}$ and $s$ is in the segment from $\xi$ to $\eta$, then, either $|\xi - \eta| > \frac{d_m}{8}$ or $|s - w_i| > \frac{d_m}{8}$, for all $i = 1, \cdots, N$.*

   *For $\xi \in B_{N+1}$, we have*

4. *$|\xi - w_i| > \frac{d_m}{4}$, for all $i = 1, \cdots, N$.*

## 3 The convex case

In this section we assume that $\Omega$ is a convex polygon. In this case the exponents defining the Schwarz-Christoffel mapping satisfy $0 < k_j < 1$.

**Lemma 2.** *Let $\xi, \eta \in B_j$, with $j = 1, \cdots, N$. Then if $k_j > 0$*

$$|x - y| \preceq |\xi - w_j|^{-k_j} |\xi - \eta|.$$

PROOF. By definition

$$h(\xi) - h(\eta) = \int_\eta^\xi h'(s)\, ds, \tag{8}$$

where $h'(s) = (s - w_j)^{-k_j} \phi(s)$ for

$$\phi(s) = (s - w_1)^{-k_1} ... (s - w_{j-1})^{-k_{j-1}} (s - w_{j+1})^{-k_{j+1}} ... (s - w_N)^{-k_N}.$$

$\phi$ is analytic in $w_j$ and $|\phi(s)| \preceq 1$. Moreover we can write

$$h'(s) = (s - w_j)^{-k_j} \phi(w_j) + (s - w_j)^{1-k_j} \psi(s),$$

where $\psi$ is analytic in $B_j$ and $|\psi(s)| \preceq 1$.

Then

$$|h(\xi) - h(\eta)| \preceq |\eta - w_j|^{1-k_j} + |\xi - w_j|^{1-k_j} + |\xi - \eta|.$$

When $|\xi - w_j| \leq \frac{1}{2}|\eta - w_j|$ we have $\frac{1}{2}|\eta - w_j| \leq |\xi - \eta|$ and

$$\begin{aligned} |h(\xi) - h(\eta)| &\preceq |\eta - w_j|^{1-k_j} + |\xi - \eta| \\ &\preceq |\xi - w_j|^{-k_j}|\xi - \eta|. \end{aligned}$$

When $|\xi - w_j| > \frac{1}{2}|\eta - w_j|$ and $|\xi - \eta| > \frac{1}{2}|\xi - w_j|$ we have

$$\begin{aligned} |h(\xi) - h(\eta)| &\preceq |\xi - w_j|^{1-k_j} + |\xi - \eta| \\ &\preceq |\xi - w_j|^{-k_j}|\xi - \eta|. \end{aligned}$$

If $|\xi - \eta| \leq \frac{1}{2}|\xi - w_j|$ we use that $|\xi - w_j| \leq 2|s - w_j|$ for all $s$ in the segment from $\xi$ to $\eta$ and then

$$|h(\xi) - h(\eta)| \preceq \int_\eta^\xi |s - w_j|^{-k_j}\, ds \preceq |\xi - w_j|^{-k_j}|\xi - \eta|$$

as we desire. □

**Remark 3.** *As a particular case of the previous lemma we obtain for* $\xi \in B_j$ *that*

$$|x - z_j| \preceq |\xi - w_j|^{1-k_j}, \tag{9}$$

*with* $j = 1, \cdots, N$ *and* $k_j > 0$.

If $\Omega$ is a bounded domain, it was proved in [7] that

$$G_\Omega(x, y) \preceq \log\left(1 + \frac{\min\{d_\Omega(x), d_\Omega(y)\}}{|x - y|}\right) \preceq |x - y|^{-1}, \tag{10}$$

where $d_\Omega(x)$ denotes the distance from $x$ to the boundary of $\Omega$.

In order to have some estimates for the first and second order derivatives of $G_\Omega(x, y)$, using (5) we obtain

$$|D_x^\alpha G_\Omega(x, y)| \preceq |D_\xi^\alpha G_B(\xi, \eta)|\, |g'(x)| \quad \text{for } |\alpha| = 1, \tag{11}$$

and

$$|D_x^\alpha G_\Omega(x, y)| \preceq |D_\xi^\alpha G_B(\xi, \eta)|\, |g'(x)|^2 + |D_\xi^\beta G_B(\xi, \eta)|\, |g''(x)| \quad \text{for } |\alpha| = 2, \tag{12}$$

where $|\beta| = 1$. We will use the following estimates for $g$:

$$|g^{'}(x)| = \frac{1}{|h^{'}(\xi)|} \preceq |\xi - w_1|^{k_1}|\xi - w_2|^{k_2}...|\xi - w_N|^{k_N} \preceq |\xi - w_j|^{k_j} \tag{13}$$

and

$$|g^{''}(x)| \preceq |\xi - w_1|^{k_j-1}|g^{'}(x)| \preceq |\xi - w_j|^{2k_j-1}, \tag{14}$$

for $x \in \Omega_j$, with $j = 1, ..., N$.

**Lemma 4.** *Let $x, y \in \Omega$ and $|\alpha| = 1$. Then we have*

$$|D_x^\alpha G_\Omega(x,y)| \preceq |x-y|^{-1}.$$

PROOF. Consider first $x \in \Omega_j$, with $j = 1, ..., N$. For $y \in \Omega_j$ we have that

$$|D_x^\alpha G_\Omega(x,y)| \preceq |D_\xi^\alpha G_B(\xi,\eta)|\,|g^{'}(x)| \preceq |\xi-\eta|^{-1}\,|\xi - w_j|^{k_j} \preceq |x-y|^{-1},$$

by (11), (7), (13) and Lemma 2.

For $y \in (\Omega_j \cup \Omega_{N+1})^c$, recalling that $|\xi - \eta| > \frac{d_m}{4}$, we have

$$|D_x^\alpha G_\Omega(x,y)| \preceq |\xi-\eta|^{-1}|\xi - w_j|^{k_j} \preceq 1.$$

For $y \in \Omega_{N+1}$, it only remains to see the case when $\frac{d_m}{8} < |s - w_i| \leq 1$, for $i = 1, ..., N$ and $s$ is in the segment from $\xi$ to $\eta$. But there $|g^{'}(x)| \preceq 1$ and $|h^{'}(x)| \preceq 1$, then

$$|D_x^\alpha G_\Omega(x,y)| \preceq |\xi-\eta|^{-1} \preceq |x-y|^{-1}.$$

Finally, if $x \in \Omega_{N+1}$, we have $\frac{d_m}{4} < |\xi - w_i| \leq 1$ for all $i = 1, ..., N$. Therefore $|x - y| \preceq |\xi - \eta|$ and

$$|D_x^\alpha G_\Omega(x,y)| \preceq |D_\xi^\alpha G_B(\xi,\eta)|\,|g^{'}(x) \preceq |x-y|^{-1}.$$

□

In the following two lemmas we analyze separately each term of (12) to obtain estimates for the second order derivatives of $G_\Omega(x, y)$.

**Lemma 5.** *Let $x \in \Omega_j$, with $j = 1, ..., N$ and $|\beta| = 1$. Then we have:*

1. $|x - z_j|^{1-a}\,|D_\xi^\beta G_B(\xi,\eta)|\,|g^{''}(x)| \preceq |x-y|^{-1-a}$, *if $y \in \Omega_j$ and $0 \leq a < 1$.*
2. $|D_\xi^\beta G_B(\xi,\eta)|\,|g^{''}(x)| \preceq 1$, *if $y \in (\Omega_j \cup \Omega_{N+1})^c$.*

*3.* $|D^\beta_\xi G_B(\xi,\eta)|\,|g^{''}(x)| \preceq |x-y|^{-1}$, *if* $y \in \Omega_{N+1}$.

PROOF. (1) If $y \in \Omega_j$ and $|x-y| \leq |x-z_j|$, we have for any $a \geq 0$

$$\begin{aligned}|D^\beta_\xi G_B(\xi,\eta)|\,|g^{''}(x)| &\preceq |\xi-\eta|^{-1}|\xi-w_j|^{2k_j-1}\\ &\preceq |x-y|^{-1-a}|x-z_j|^a|\xi-w_j|^{k_j-1}\\ &\preceq |x-y|^{-1-a}|x-z_j|^{a-1},\end{aligned}$$

by (7), (14), Lemma 2 and (9).

On the other hand, if $|x-y| > |x-z_j|$, we have for $0 \leq a < 1$

$$\begin{aligned}|D^\beta_\xi G_B(\xi,\eta)|\,|g^{''}(x)| &\preceq \frac{d_B(\xi)}{|\xi-\eta|^2}|\xi-w_j|^{2k_j-1}\\ &\preceq |x-y|^{-2}\\ &\preceq |x-z_j|^{-1+a}|x-y|^{-1-a},\end{aligned}$$

by (7), (14) and Lemma 2.

(2) If $y \in (\Omega_j \cup \Omega_{N+1})^c$, since $|\xi-\eta| > \frac{d_m}{4}$, we obtain

$$|D^\beta_\xi G_B(\xi,\eta)|\,|g^{''}(x)| \preceq \frac{d_B(\xi)}{|\xi-\eta|^2}|\xi-w_j|^{2k_j-1} \preceq |\xi-w_j|^{2k_j} \preceq 1.$$

(3) For $y \in \Omega_{N+1}$, it remains to consider the case when $\frac{d_m}{8} < |s-w_i| \leq 1$, for $i = 1,...,N$ and $s$ is in the segment from $\xi$ to $\eta$. But there $|g^{''}(x)| \preceq 1$ and

$$|D^\beta_\xi G_B(\xi,\eta)|\,|g^{''}(x)| \preceq |\xi-\eta|^{-1} \preceq |x-y|^{-1}.$$

□

**Lemma 6.** *Let* $x \in \Omega_j$, *with* $j = 1,...,N$ *and* $|\alpha| = 2$. *Then we have:*

*1.* $|D^\alpha_\xi G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq \dfrac{d_\Omega(x)}{|x-y|^3}$, *if* $y \in \Omega_j \cup \Omega_{N+1}$.

*2.* $|D^\alpha_\xi G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq 1$, *if* $y \in (\Omega_j \cup \Omega_{N+1})^c$.

PROOF. (1) If $y \in \Omega_j$ we have that

$$|D^\alpha_\xi G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq \frac{d_B(\xi)}{|\xi-\eta|^3}|\xi-w_j|^{2k_j} \preceq \frac{d_B(\xi)}{|x-y|^3}|\xi-w_j|^{-k_j},$$

by (7), (13) and Lemma 2.

Let now $X_0 \in \partial\Omega$ such that $d_\Omega(x) = |x - X_0|$ and $Q_0 \in \partial B$ with $g(X_0) = Q_0$. Then there exists $\xi_0$ in the segment from $x$ to $X_0$ and $\eta_0 = g(\xi_0)$ such that

$$d_B(\xi) \leq |g^{'}(\xi_0)||x - X_0| \preceq |\eta_0 - w_j|^{k_j} d_\Omega(x). \tag{15}$$

Therefore

$$|D^\alpha_\xi G_B(\xi,\eta)|\, |g^{'}(x)|^2 \preceq \frac{d_\Omega(x)}{|x-y|^3} |\eta_0 - w_j|^{k_j} |\xi - w_j|^{-k_j},$$

and for each $1 \leq i \leq M$, there exists $\xi_i$ in the segment from $\xi_{i-1}$ to $z_j$ such that

$$|\eta_{i-1} - w_j|^{k_j} = |g(\xi_{i-1}) - g(z_j)| \leq |g^{'}(\xi_i)||\xi_i - z_j| \preceq |\eta_i - w_j|^{k_j} |\xi_i - z_j|.$$

By iterating, we have

$$\begin{aligned}
|\eta - w_j| &\preceq |\eta_1 - w_j|^{k_j} |\xi_0 - z_j| \\
&\preceq |\eta_2 - w_j|^{k_j^2} |\xi_1 - z_j|^{k_j} |\xi_0 - z_j| \\
&\preceq |\eta_3 - w_j|^{k_j^3} |\xi_2 - z_j|^{k_j^2} |\xi_1 - z_j|^{k_j} |\xi_0 - z_j| \\
&\dots \\
&\preceq |\eta_M - w_j|^{k_j^M} \dots |\xi_2 - z_j|^{k_j^2} |\xi_1 - z_j|^{k_j} |\xi_0 - z_j| \\
&\preceq |x - z_j|^{k_j^M} \dots |x - z_j|^{k_j^2} |x - z_j|^{k_j} |x - z_j|,
\end{aligned}$$

where we used that $|\xi_i - z_j| \preceq |x - z_j|$ and $|\eta_i - w_j| \preceq |x - z_j|$.

Note that the implicit constant involved in $\preceq$ above does not depend on $M$. In fact, by (13) and (9)

$$|g^{'}(\xi_i)| \preceq |\eta_i - w_j|^{k_j} \left(\frac{d_m}{4}\right)^p,$$

where $p = \sum_{k_j<0} k_j$ and we have that

$$\left(\frac{d_m}{4}\right)^{p \sum_{n=0}^{M} k_j^n} \leq \left(\frac{d_m}{4}\right)^{p \sum_{n=0}^{\infty} k_j^n} < \infty.$$

Therefore

$$|D^\alpha_\xi G_B(\xi,\eta)|\, |g^{'}(x)|^2 \preceq \frac{d_\Omega(x)}{|x-y|^3} |x - z_j|^\beta |\xi - w_j|^{-k_j},$$

where $\beta = \sum_{n=1}^{M+1} k_j^n = k_j \left( \frac{1-k_j^{M+2}}{1-k_j} \right)$. Taking $\gamma = \frac{k_j}{1-k_j}$, by (9), it follows that

$$|x-z_j|^{-\beta+\gamma} |D_\xi^\alpha G_B(\xi,\eta)|\, |g^{'}(x)|^2 \preceq \frac{d_\Omega(x)}{|x-y|^3}.$$

Then, given $\varepsilon > 0$ there exists $M$ large enough such that $-\beta+\gamma < \varepsilon$ and taking $\varepsilon$ tending to zero

$$|D_\xi^\alpha G_B(\xi,\eta)|\, |g^{'}(x)|^2 \preceq \frac{d_\Omega(x)}{|x-y|^3}.$$

For $y \in \Omega_{N+1}$, we consider only the case when $\frac{d_m}{8} < |s-w_i| \leq 1$, for $i = 1, ..., N$ and $s$ is in the segment from $\xi$ to $\eta$ (the other case will be considered in (2)). In this case, $|x-y| \preceq |\xi-\eta|$ and

$$|D_\xi^\alpha G_B(\xi,\eta)|\, |g^{'}(x)|^2 \preceq \frac{d_B(\xi)}{|\xi-\eta|^3}|\xi-w_j|^{2k_j} \preceq \frac{d_\Omega(x)}{|x-y|^3}|\eta-w_j|^{k_j} \preceq \frac{d_\Omega(x)}{|x-y|^3}.$$

(2) If $y \in (\Omega_j \cup \Omega_{N+1})^c$, since $|\xi-\eta| > \frac{d_m}{4}$, we obtain

$$|D_\xi^\alpha G_B(\xi,\eta)|\, |g^{'}(x)|^2 \preceq |\xi-\eta|^{-2}|\xi-w_j|^{2k_j} \preceq |\xi-w_j|^{2k_j} \preceq 1.$$

□

## 4 The non-convex case

In this section we assume that $\Omega$ is a nonconvex polygon. In this case the exponents defining the Schwarz-Christoffel mapping can be negative, i.e. there exists at least one $j = 1, ..., N$ such that $-1 < k_j \leq 0$.

**Lemma 7.** *Let $\xi, \eta \in B_j$, with $j = 1, ..., N$. Then if $k_j \leq 0$*

$$|x-y| \preceq |u-v|.$$

Proof. As $k_j \leq 0$ we have $|s-w_j|^{-k_j} \leq 1$ and by (8)

$$\begin{aligned} |h(u)-h(v)| &\leq \int_v^u |s-w_j|^{-k_j} |\phi(s)|\, ds \\ &\preceq |u-v|. \end{aligned}$$

□

To complete the study of the first and second order derivatives of $G_\Omega(x,y)$ for the non-convex case we need to obtain estimates when $-1 < k_j \le 0$. To do this, we use (11), (12), (13) and (14) as in the convex case.

**Lemma 8.** *Let $x \in \Omega_j$, with $j = 1, ..., N$, $y \in \Omega$ and $|\alpha| = 1$. Then we have*

$$|x - z_j|^{1-\frac{\pi}{\theta_j}} |D_x^\alpha G_\Omega(x,y)| \preceq |x - y|^{-1}.$$

PROOF. For $y \in \Omega_j$ we have that

$$|D_x^\alpha G_\Omega(x,y)| \preceq |g^{'}(x)||\xi - \eta|^{-1} \preceq |\xi - w_j|^{k_j} |\xi - \eta|^{-1} \preceq |\xi - w_j|^{k_j} |x - y|^{-1},$$

by (11), (7), (13) and Lemma 7. Taking $\gamma := \frac{-k_j}{(1-k_j)} = 1 - \frac{\pi}{\theta_j} > 0$ it follows from (9) that $|x - z_j|^\gamma \preceq |\xi - w_j|^{(1-k_j)\gamma}$ and

$$|x - z_j|^\gamma |D_x^\alpha G_\Omega(x,y)| \preceq |x - y|^{-1},$$

as we wanted to prove.

For $y \in (\Omega_j \cup \Omega_{N+1})^c$ we have

$$|D_\xi^\alpha G_B(\xi,\eta)|\, |g^{'}(x)| \preceq |\xi - w_j|^{k_j}$$

and we obtain the desired inequality as before.

For $y \in \Omega_{N+1}$ the proof is analogous to the case $0 < \theta_j < \pi$. □

Analogously to the convex case, we analyze separately each term of (12) to obtain estimates for the second order derivatives of $G_\Omega(x,y)$.

**Lemma 9.** *Let $x \in \Omega_j$ with $j = 1, ..., N$ and $|\beta| = 1$. Then we have:*

1. $|x - z_j|^{2-\frac{\pi}{\theta_j}} |D_\xi^\beta G_B(\xi,\eta)|\, |g^{''}(x)| \preceq |x - y|^{-1}$, *if* $y \in \Omega_j$.
2. $|x - z_j|^{2-\frac{\pi}{\theta_j}} |D_\xi^\beta G_B(\xi,\eta)|\, |g^{''}(x)| \preceq 1$, *if* $y \in (\Omega_j \cup \Omega_{N+1})^c$.
3. $|D_\xi^\beta G_B(\xi,\eta)|\, |g^{''}(x)| \preceq |x - y|^{-1}$, *if* $y \in \Omega_{N+1}$.

PROOF. (1) If $y \in \Omega_j$ we have that

$$|D_\xi^\beta G_B(\xi,\eta)|\, |g^{''}(x)| \preceq |\xi - \eta|^{-1} |\xi - w_j|^{2k_j - 1} \preceq |x - y|^{-1} |\xi - w_j|^{2k_j - 1},$$

by (7), (14) and Lemma 7. Taking $\gamma = \frac{1-2k_j}{(1-k_j)} = 2 - \frac{\pi}{\theta_j}$ it follows from (9) that $|x - z_j|^\gamma \preceq |\xi - w_j|^{-2k_j+1}$ and

$$|x - z_j|^\gamma |D_\xi^\beta G_B(\xi,\eta)|\, |g^{''}(x)| \preceq |x - y|^{-1}.$$

(2) If $y \in (\Omega_j \cup \Omega_{N+1})^c$,

$$|D_\xi^\beta G_B(\xi,\eta)|\,|g^{''}(x)| \preceq |\xi-\eta|^{-1}|\xi-w_j|^{2k_j-1} \preceq |\xi-w_j|^{2k_j-1}$$

and the result follows in the same way as above.

(3) For $y \in \Omega_{N+1}$ and $\xi$ and $\eta$ are at a distance from the pre-vertex of $\Omega$ greater than $\frac{d_m}{8}$, $|x-y| \preceq |\xi-\eta|$ and

$$|D_\xi^\beta G_B(\xi,\eta)|\,|g^{''}(x)| \preceq |\xi-\eta|^{-1}|\xi-w_j|^{2k_j-1} \preceq |x-y|^{-1}.$$

□

**Lemma 10.** *Let $x \in \Omega_j$ with $j = 1, ..., N$ and $|\alpha| = 2$. Then we have:*

1. $|x-z_j|^{2-\frac{\pi}{\theta_j}}\,|D_\xi^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq \dfrac{d_\Omega(x)}{|x-y|^3}$, *if $y \in \Omega_j \cup \Omega_{N+1}$ and $x$ such that $d_\Omega(x) \le \frac{1}{2}|x-z_j|$.*
2. $|x-z_j|^{a+2-2\frac{\pi}{\theta_j}}\,|D_\xi^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq |x-y|^{-2+a}$, *if $y \in \Omega_j \cup \Omega_{N+1}$, $x$ such that $\frac{1}{2}|x-z_j| < d_\Omega(x) \le |x-y|$ and $a > 0$.*
3. $|x-z_j|^{2-\frac{\pi}{\theta_j}}\,|D_\xi^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq 1$, *if $y \in (\Omega_j \cup \Omega_{N+1})^c$.*

PROOF. (1) If $y \in \Omega_j$ and $d_\Omega(x) \le \frac{1}{2}|x-z_j|$, we have that

$$|D_\xi^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq \frac{d_B(\xi)}{|\xi-\eta|^3}|\xi-w_j|^{2k_j} \preceq \frac{d_\Omega(x)}{|x-y|^3}|\eta-w_j|^{k_j}|\xi-w_j|^{2k_j}, \tag{16}$$

by (7), (13), Lemma 7 and (15), where $h(\eta) = \xi$ is in the segment form $x$ to $X_0$.

Taking $\gamma = \frac{-2k_j}{1-k_j}$ and $\beta = \frac{-k_j}{1-k_j}$, by (9) it follows that

$$|\xi-z_j|^\beta|x-z_j|^\gamma|D_\xi^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq \frac{d_\Omega(x)}{|x-y|^3}.$$

Since $\gamma+\beta < 2-\frac{\pi}{\theta_j}$ it is enough to prove that $|x-z_j| \preceq |\xi-z_j|$ provided that $d_\Omega(x) \le \frac{1}{2}|x-z_j|$.

We will consider the following two cases:

If $|x-\xi| \le \frac{1}{4}|x-z_j|$ the result follows directly.

If $|x-\xi| > \frac{1}{4}|x-z_j|$ we also have that $\frac{1}{2}|x-z_j| \le |X_0-z_j|$. Then $\frac{1}{2}|x-z_j| \le |X_0-z_j| \le d_B(\xi)+|\xi-z_j| \le 2|\xi-z_j|$ as we desire.

(2) If $y \in \Omega_j$ and $\frac{1}{2}|x-z_j| < d_\Omega(x) < |x-y|$, we have for any $a > 0$

$$|D_u^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq |\xi-\eta|^{-2}|\xi-w_j|^{2k_j} \preceq |x-y|^{-2+a}|x-z_j|^{-a}|\xi-w_j|^{2k_j}, \tag{17}$$

by (7), (13) and Lemma 7. Taking $\gamma = \frac{-2k_j}{1-k_j}$, by (9) it follows that

$$|x-z_j|^{a+\gamma}|D_u^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq |x-y|^{-2+a}.$$

For $y \in \Omega_{N+1}$ and $\xi$ and $\eta$ at a distance from the pre-vertex of $\Omega$ greater than $\frac{d_m}{8}$ (the other case will be considered in (3)), $|x-y| \preceq |\xi-\eta|$ and consider again the previous two cases using that $|\xi-w_j|^{2k_j}$ in (16) and (17) is bounded.

(3) Since $|\xi-\eta| > \frac{d_m}{4}$ we obtain

$$|D_u^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq |\xi-\eta|^{-2}|\xi-w_j|^{2k_j} \preceq |\xi-w_j|^{2k_j}$$

and the result follows in the same way that (16). □

To complete the study of the behavior of the second order derivatives of the Green function $G_\Omega$, it suffices to consider $x \in \Omega_{N+1}$. In this case there is no relation to the vertex of $\Omega$ as we prove in the following lemma:

**Lemma 11.** *Let $x \in \Omega_{N+1}$ and $y \in \Omega$. Then we have:*

1. *For $|\beta| = 1$*

$$|D_u^\beta G_B(\xi,\eta)|\,|g^{''}(x)| \preceq |x-y|^{-1}.$$

2. *For $|\alpha| = 2$ and $d_\Omega(x) \le |x-y|$*

$$|D_u^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq \frac{d_\Omega(x)}{|x-y|^3}.$$

PROOF. (1) For $|\beta| = 1$ we have that

$$|D_u^\beta G_B(\xi,\eta)|\,|g^{''}(x)| \preceq |\xi-\eta|^{-1},$$

by (14) and using that $\frac{d_m}{4} < |\xi-w_i| \le 1$ for $i = 1,...,N$. Moreover, we have by Lemma 2 that $|x-y| \preceq |\xi-\eta|$ and the result follows directly.

(2) For $|\alpha| = 2$ we have that

$$|D_u^\alpha G_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq \frac{d_B(\xi)}{|\xi-\eta|^3}|\xi-w_j|^{k_j} \preceq \frac{d_\Omega(x)}{|x-y|^3}|\eta-w_j|^{k_j},$$

where we are assuming as in (15) that $w_j$ is the pre-vertex closest to $\eta$.

If $k_j > 0$ we have $|\eta-w_j|^{k_j} \preceq 1$ as we desired.

If $k_j \le 0$ we can follow the proof of (2) of Lemma 6 and we consider the cases $d_\Omega(x) \le \frac{1}{2}|x-z_j|$ and $\frac{1}{2}|x-z_j| < d_\Omega(x) < |x-y|$ respectively using also that, when $x \in \Omega_{N+1}$, $|x-z_j| > \frac{d_m}{4}$. □

## 5 Main result

Let us recall that the solution of the problem (1) is given by

$$u(x) = \int_\Omega G_\Omega(x,y)\, f(y)\, dy,$$

where $G_\Omega(x,y) = \Gamma(x,y) + H_\Omega(x,y)$.

In order to prove our main result (2) we also need to establish some estimates for the second order derivatives of $H_\Omega$ and $\Gamma$.

Applying the same ideas used in the previous section for $G_\Omega$, by (6), we have for $|\beta| = 1$ and $|\alpha| = 2$

$$|D_x^\alpha H_\Omega(x,y)| \preceq |D_\xi^\alpha H_B(\xi,\eta)|\, |g^{'}(x)|^2 + |D_\xi^\beta H_B(\xi,\eta)|\, |g^{''}(x)|.$$

Moreover, as $B$ is a smooth bounded domain, we have

$$|D_\xi^\beta H_B(\xi,\eta)| \le C\, d_B(\xi)^{-1} \quad \text{and} \quad |D_\xi^\alpha H_B(\xi,\eta)| \le C\, d_B(\xi)^{-2}, \tag{18}$$

(see Lemma 2.1 in [5]) and we have the following lemma:

**Lemma 12.** *Let $y \in \Omega$ and $|\alpha| = 2$. Then*

1. *For $x \in \Omega_j$, with $j = 1, ..., N$ we have:*
   (a) $|D_x^\alpha H_\Omega(x,y)| \preceq d_\Omega(x)^{-2}$, *if* $0 < k_j < 1$.
   (b) $|x - z_j|^{2-\frac{\pi}{\theta_j}} |D_x^\alpha H_\Omega(x,y)| \preceq d_\Omega(x)^{-2}$, *if* $-1 < k_j \le 0$.
2. *For $x \in \Omega_{N+1}$ we have:*

$$|D_x^\alpha H_\Omega(x,y)| \preceq d_\Omega(x)^{-2}.$$

PROOF. (1) For $x \in \Omega_j$, let $X_0 \in \partial\Omega$ such that $g(X_0) = \xi_0$, with $d_B(\xi) = |\xi - \xi_0|$. Then there exists $\eta$ in the segment from $\xi$ to $\xi_0$ such that

$$d_\Omega(x) \le |h^{'}(\eta)||\xi - \xi_0| \preceq |\eta - w_j|^{-k_j}\, d_B(\xi). \tag{19}$$

Consider first $0 < k_j < 1$. It follows from (18), (13), (14) and (19) that

$$\begin{aligned} |D_\xi^\alpha H_B(\xi,\eta)|\, |g^{'}(x)|^2 &\preceq d_B(\xi)^{-2} |\xi - w_j|^{2k_j} \\ &\preceq |\eta - w_j|^{-2k_j}\, d_\Omega(x)^{-2} |\xi - w_j|^{2k_j} \end{aligned} \tag{20}$$

and

$$|D_\xi^\beta H_B(\xi,\eta)|\,|g^{''}(x)| \preceq d_B(\xi)^{-1}|\xi - w_j|^{2k_j-1} \\ \preceq |\eta - w_j|^{-k_j} d_\Omega(x)^{-1}|\xi - w_j|^{2k_j-1}. \tag{21}$$

If we also consider $|\eta - w_j| > \frac{1}{2}|\xi - w_j|$ we have

$$|D_\xi^\alpha H_B(\xi,\eta)|\,|g^{'}(x)|^2 + |D_\xi^\beta H_B(\xi,\eta)|\,|g^{''}(x)| \preceq d_\Omega(x)^{-2} + d_\Omega(x)^{-1}|x - z_j|^{-1} \\ \preceq d_\Omega(x)^{-2},$$

by (20), (21) and (9).

If $|\eta - w_j| \le \frac{1}{2}|\xi - w_j|$, we can see that $d_B(\xi) \ge \frac{1}{2}|\xi - w_1|$. Then we have

$$|D_\xi^\alpha H_B(\xi,\eta)|\,|g^{'}(x)|^2 + |D_\xi^\beta H_B(\xi,\eta)|\,|g^{''}(x)| \preceq |x - z_j|^{-2} \preceq d_\Omega(x)^{-2},$$

by (20), (21) and (9).

Now, consider $-1 < k_j \le 0$. By (19) we obtain $d_\Omega(x) \preceq d_B(\xi)$ and taking $\gamma_1 := \frac{-2k_j}{1-k_j} < 2 - \frac{\pi}{\theta}{}_j$ and $\gamma_2 := \frac{1-2k_j}{1-k_j} = 2 - \frac{\pi}{\theta}{}_j$, it follows from (20), (21) and (9) that

$$|x - z_j|^{\gamma_1}|D_\xi^\alpha H_B(\xi,\eta)|\,|g^{'}(x)|^2 \preceq d_\Omega(x)^{-2}$$

and

$$|x - z_j|^{\gamma_2}|D_\xi^\beta H_B(\xi,\eta)|\,|g^{''}(x)| \preceq d_\Omega(x)^{-1},$$

as we desired.

(2) If $x \in \Omega_{N+1}$ we have $|g^{'}(x)| \preceq 1$, $|g^{''}(x)| \preceq 1$ and

$$|D_x^\alpha H_B(x,y)| \preceq d_B(\xi)^{-2} + d_B(\xi)^{-1}.$$

In order to prove that $d_\Omega(x) \preceq d_B(\xi)$ we consider two cases depending on the $k_j$ associated with the pre-vertex $w_j$ closest to $\eta$ given by (19).

If $k_j \le 0$ the proof follows directly and if $k_j > 0$ we have to consider $|\eta - w_j| > \frac{1}{2}|\xi - w_j| > \frac{d_m}{8}$ and $|\eta - w_j| \le \frac{1}{2}|\xi - w_j|$ as above. □

With respect to $\Gamma$, since $|D_x^\beta \Gamma(x)| \le C|x|^{1-n}$ for $|\beta| = 1$, we have

$$D_x^\beta \int_\Omega \Gamma(x-y)\, f(y)\, dy = \int_\Omega D_x^\beta \Gamma(x-y)\, f(y)\, dy.$$

However, for $|\alpha| = 2$, $D_x^\alpha \Gamma$ is not an integrable function and we cannot interchange the order between second derivatives and integration. A known standard argument shows that for $|\delta| = |\beta| = 1$

$$D_x^\delta \int_\Omega D_x^\beta \Gamma(x-y)\, f(y)\, dy = Kf(x) + c(x)f(x),$$

where $c$ is a bounded function and

$$Kf(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} D_x^\alpha \Gamma(x-y)\, f(y)\, dy$$

is a Calderón-Zygmund operator. Indeed, since $D_x^\beta \Gamma \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and it is a homogeneous function of degree $-1$, it follows that $D_x^\alpha \Gamma_\Omega(x-y)$ is homogeneous of degree $-2$ and has vanishing average on the unit sphere (see Lemma 11.1 in [1, page 152]). Then, it follows from the general theory given in [3] that $K$ is a bounded operator in $L^p$ for $1 < p < \infty$.

Moreover, the maximal operator

$$\widetilde{K}f(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} D_x^\alpha \Gamma_\Omega(x-y)\, f(y)\, dy \right|$$

is also bounded in $L^p$ for $1 < p < \infty$.

Our main result is a consequence of the next proposition which follows the same ideas of Lemma 2.3 in [5].

**Proposition 13.** *Let $u$ be a solution of (1) and let $\rho$ and $\sigma$ be the functions given by*

$$\rho(x) := \begin{cases} |x-z_j|^{1-\frac{\pi}{\theta_j}} & \text{for } x \in \Omega_j \text{ and } \pi \le \theta_j < 2\pi \\ 1 & \text{for either } x \in \Omega_j \text{ and } 0 < \theta_j < \pi \text{ or } x \in \Omega_{N+1}, \end{cases}$$

*and*

$$\sigma(x) := \begin{cases} |x-z_j|^{2-\frac{\pi}{\theta_j}} & \text{for } x \in \Omega_j \text{ and } \pi \le \theta_j < 2\pi \\ |x-z_j|^{1-a} & \text{for } x \in \Omega_j \text{ and } 0 < \theta_j < \pi \\ 1 & \text{for } x \in \Omega_{N+1}, \end{cases}$$

*with $0 \le a < 1$.*

*Then for any $x \in \Omega$, $|\beta| = 1$ and $|\alpha| = 2$ we have*

$$|u(x)| + |\rho(x) D_x^\beta u(x)| \preceq Mf(x),$$

$$|\sigma(x) D_x^\alpha u(x)| \preceq \widetilde{K}f(x) + Mf(x) + |f(x)|,$$

*where $Mf(x)$ is the ususal Hardy-Littlewood maximal function of $f$.*

Proof. Calling $\delta$ the diameter of $\Omega$

$$|u(x)| \preceq \int_{|x-y|\le\delta} \frac{|f(y)|}{|x-y|^{-1}}\, dy = \sum_{k=0}^{\infty} \int_{\{2^{-(k+1)}\delta \le |x-y| \le 2^{-k}\delta\}} \frac{|f(y)|}{|x-y|}\, dy$$

by (3) and (10). Then, it follows that

$$|u(x)| \preceq Mf(x)$$

(see Lemma 2.8.3 in [9, page 85] for details).

Analogously, from Lemma 4 and Lemma 8 we obtain

$$|\rho(x)D_x^\beta u(x)| \preceq Mf(x).$$

On the other hand, by (3) and (4) we obtain

$$\begin{aligned}\sigma(x)D_x^\alpha u(x) &= \lim_{\epsilon\to 0}\int_{\epsilon<|x-y|\le d_\Omega(x)} \sigma(x)\, D_x^\alpha\Gamma(x-y)\, f(y)\, dy + cf(x)\\ &+ \int_{|x-y|\le d_\Omega(x)} \sigma(x)\, D_x^\alpha H_\Omega(x,y)\, f(y)\, dy\\ &+ \int_{|x-y|> d_\Omega(x)} \sigma(x)\, D_x^\alpha G_\Omega(x,y)\, f(y)\, dy\\ &:= I + II + III + IV.\end{aligned}$$

Now, we have

$$|I| \le |Kf(x)| + \widetilde{K}f(x) \le 2\widetilde{K}f(x).$$

Since $c$ is a bounded function we have $|II| \preceq f(x)$. Therefore, we only need to estimate the last two terms. By Lemma 12 and as $\sigma(x) \preceq 1$ for $x \in \Omega_j$ with $0 < k_j < 1$ it holds that

$$\begin{aligned}\int_{|x-y|\le d_\Omega(x)} \sigma(x)D_x^\alpha H_\Omega(x,y)\, f(y)\, dy &\preceq d_\Omega(x)^{-2}\int_{|x-y|\le d_\Omega(x)} |f(y)|\, dy\\ &\preceq Mf(x).\end{aligned}$$

Finally, by the results given by Lemma 5, Lemma 6, Lemma 9, Lemma 10 and Lemma 11, it follows that

$$\int_{|x-y|>d_\Omega(x)} \sigma(x)D_x^\alpha G_\Omega(x,y)\, f(y)\, dy \preceq Mf(x)$$

and the proposition is proved. □

We can now state and prove our main result. First we recall the definition of the $A_p(\mathbb{R}^2)$ class for $1 < p < \infty$. A non-negative locally integrable function $\omega$ belongs to $A_p(\mathbb{R}^2)$ if there exists a constant $C$ such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

for every cube $Q \subset \mathbb{R}^2$.

For any weight $\omega$, $L^p_\omega(\Omega)$ is the space of measurable functions $f$ defined in $\Omega$ such that

$$\|f\|_{L^p_\omega(\Omega)} = \left(\int_\Omega |f(x)|^p\,\omega(x)\,dx\right)^{1/p} < \infty$$

and $W^{k,p}_\omega(\Omega)$ is the space of functions such that

$$\|f\|_{W^{k,p}_\omega(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha f\|^p_{L^p_\omega(\Omega)}\right)^{1/p} < \infty.$$

**Theorem 14.** *Let $\Omega$ be a polygonal domain in $\mathbb{R}^2$. Let $u$ be a solution of (1) with $f \in L^p_\omega(\Omega)$, $1 < p < \infty$.*

*Then, for $\omega \in A_p(\mathbb{R}^2)$, we have*

$$\|u\|_{L^p_\omega(\Omega)} + \sum_{|\beta|=1} \|\rho(x) D^\beta_x u\|_{L^p_\omega(\Omega)} + \sum_{|\alpha|=2} \|\sigma(x) D^\alpha_x u\|_{L^p_\omega(\Omega)} \preceq \|f\|_{L^p_\omega(\Omega)},$$

*where $\rho(x)$ and $\sigma(x)$ are the functions defined in Proposition 13.*

Proof. Taking $\Omega = \cup_{j=1}^{N+1}\Omega_j$, since $M$ and $\widetilde{K}$ are bounded operators in $L^p_\omega$ (see [8, Chapter V]), the proof is a consequence of Proposition 13. □

**Acknowledgment**. We would like to thank Ricardo Duran very much for his constant support.

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