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INVESTIGATIONS OF STRONG RIGHT UPPER POROSITY AT A POINT

Abstract

We define and study, for subsets of $[0, \infty)$, several types of strong right upper porosity at the point 0. Some characterizations of these types of porosity are obtained, including a characterization in terms of a universal property and a characterization in terms of a structural property.

1 Introduction

The basic ideas concerning the notion of set porosity for the first time appeared in some early works of Denjoy [6], [7] and Khintchine [16] and then arose independently in the study of cluster sets in 1967 (Dolženko [8]). Denjoy was interested in obtaining a classification of perfect sets on the real line in terms of the relative sizes of the complementary intervals. Khintchine had required a convenient way of describing certain arguments that use density considerations. The notion of a set of σ-porosity was defined by E. P. Dolženko [8]. The basic structure of porous sets and σ-porous sets has been studied in [11], [13] and [24]. A useful collection of facts related to the notion of porosity can be found in [23]. A number of theorems exists in the theory of cluster sets which use the notion of σ-porosity (see, for example, [27],[28], [29], [30]). No less important is a question about the relationship between porosity and dimension. In many applications the information on the dimension of certain sets is obtained via porosity. Porosity has also found interesting applications

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in connection with free boundaries [14], generalized subharmonic functions [9] and complex dynamics [21]. Estimates of dimension in terms of porosity were obtained for a wide variety of notions of porosity (and dimension) in [2], [10], [17], [18], [19], [20], [22], etc. The porosity (in an appropriate sense) of many natural sets and measures was investigated in [2], [5], [17], [25]. Moreover, the relationship between porosity and other geometric concepts such as conical densities and singular integrals was explored in [5], [15], [19]. Porosity is also a property which is preserved, for example, under quasisymmetric maps [26]. These papers show that the notion of set porosity plays a diverse role in different questions of analysis.

Many nontrivial modifications of the notion of porosity are used at present. A comparison of different definitions, and surveys of results can be found in [31] and [32]. Our paper is also a contribution to this line of study and we introduce a new subclass of subsets of $\mathbb{R}^+ = [0, +\infty)$ that are strongly porous at 0.

Let us recall the definition of the right upper porosity at a point. Let E be a subset of $\mathbb{R}^+$.

Definition 1. *The right upper porosity of E at 0 is the nonnegative number*

$$p^+(E,0) := \limsup_{h \to 0^+} \frac{\lambda(E,0,h)}{h} \tag{1}$$

where $\lambda(E,0,h)$ is the length of the largest open subinterval of $(0,h)$, which could be the empty set $\varnothing$, that contains no point of E. The set E is strongly porous on the right at 0 if $p^+(E,0) = 1$.

For the remaining of the paper, when the porosity is considered, it will always be assumed to be the right upper porosity at 0.

Let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We shall say that $\tilde{\tau}$ is *eventually decreasing* and *eventually strictly decreasing*, if the inequalities $\tau_{n+1} \le \tau_n$ and, respectively, $\tau_{n+1} < \tau_n$ hold for all sufficiently large n. Write $\tilde{E}^d$ for the set of eventually decreasing sequences $\tilde{\tau}$ with $\lim\limits_{n\to\infty} \tau_n = 0$ and having $\tau_n \in E \setminus \{0\}$ for all $n \in \mathbb{N}$.

For a set $E \subseteq \mathbb{R}^+$, we use the symbols $ExtE$ and acE to denote the exterior of E and, respectively, the set of its accumulation points (relative to the space $\mathbb{R}^+$ with the standard topology).

Remark 2. The set $\tilde{E}^d$ is empty if and only if $0 \notin acE$.

Define $\tilde{I}_E$ to be the set of sequences $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ of open intervals $(a_n, b_n) \subseteq \mathbb{R}^+$ meeting the following conditions.

• *$a_n > 0$ for each n.*

• *Every interval (a_n, b_n) is a connected component of $ExtE$, i.e., $(a_n, b_n) \cap E = \varnothing$ but for every $(a, b) \supseteq (a_n, b_n)$ we have*

$$((a, b) \neq (a_n, b_n)) \Rightarrow ((a, b) \cap E \neq \varnothing).$$

• *The limit relations $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \frac{b_n - a_n}{b_n} = 1$ hold.*

Remark 3. In other words, if $0 \notin acE$, put $\tilde{I}_E = \varnothing$. Otherwise, let $\tilde{I}_E$ be the set (possibly the empty set) of all sequences of open intervals, each interval being maximal and disjoint from E, that can be used to witness the strong right upper porosity of E at 0.

Define also an *equivalence relation* $\asymp$ on the set of sequences of positive numbers as follows. Let $\tilde{a} = \{a_n\}_{n\in\mathbb{N}}$ and $\tilde{\gamma} = \{\gamma_n\}_{n\in\mathbb{N}}$. Then $\tilde{a} \asymp \tilde{\gamma}$ if there are positive constants c_1 and $c_2 > 0$ such that

$$c_1 a_n \leq \gamma_n \leq c_2 a_n \tag{2}$$

for all $n \in \mathbb{N}$.

Equivalently, $\tilde{a} \asymp \tilde{\gamma}$ if the ratios $\frac{a_n}{\gamma_n}$ are bounded away from both 0 and ∞, i. e.

$$0 < \liminf_{n\to\infty} \frac{a_n}{\gamma_n} \leq \limsup_{n\to\infty} \frac{a_n}{\gamma_n} < \infty.$$

Definition 4. *Let $E \subseteq \mathbb{R}^+$ and $\tilde{\gamma} \in \tilde{E}^d$. The set E is $\tilde{\gamma}$-strongly porous at 0 if there is a sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that*

$$\tilde{\gamma} \asymp \tilde{a} \tag{3}$$

where $\tilde{a} = \{a_n\}_{n\in\mathbb{N}}$. The set E is completely strongly porous at 0 if E is $\tilde{\gamma}$-strongly porous for every $\tilde{\gamma} \in \tilde{E}^d$.

Remark 5. If $0 \notin acE$, then E is completely strongly porous at 0 because $\tilde{E}^d = \varnothing$.

In what follows the set of all completely strongly porous at 0 subsets of $\mathbb{R}^+$ will be denoted by $\boldsymbol{CSP}(0)$.

The main results of the paper can be informally described by the following way.

• $\boldsymbol{CSP}(0)$ - sets are uniformly strongly porous (Theorem 27), in the sense that the constants in (2) can be chosen independently of $\tilde{\gamma} \in \tilde{E}^d$ if $E \in \boldsymbol{CSP}(0)$.

• If $E \in \boldsymbol{CSP}(0)$, then there is an universal $\tilde{L} \in \tilde{I}_E$ such that for every $\tilde{A} \in \tilde{I}_E$ the members of a tail of $\tilde{A}$ are members of $\tilde{L}$ (Theorem 27).
• A description of the structure of strongly porous on the right at 0 sets $E \subseteq \mathbb{R}^+$ having a universal $\tilde{L} \in \tilde{I}_E$ (Theorem 34).
• An explicit design generating all $\boldsymbol{CSP}(0)$ - sets (Theorem 42).

Remark 6. Olli Martio's question concerning interconnections between the infinitesimal structure of a metric space (X, d) at a point $p \in X$ and the porosity of the distance set $\{d(x,p) : x \in X\}$ was a starting point in our studies of $\boldsymbol{CSP}(0)$ - sets. Some results in this direction can be found in [1], [3] and [4].

2 The $\boldsymbol{CSP}(0)$ - sets

We start with the lemma which helps to prove the membership $E \in \boldsymbol{CSP}(0)$.

Lemma 7. *Let $E \subseteq \mathbb{R}^+$, let $\tilde{\gamma} = \{\gamma_n\}_{n\in\mathbb{N}}$ and $\tilde{\tau} = \{\tau_m\}_{m\in\mathbb{N}}$ belong to $\tilde{E}^d$ and let $c_1, c_2 \in (0,\infty)$. If E is $\tilde{\gamma}$-strongly porous at* 0 *and for every $m \in \mathbb{N}$ there is $n = n(m)$ such that*

$$c_1\gamma_n \le \tau_m \le c_2\gamma_n,$$

then E is $\tilde{\tau}$-strongly porous at 0.

A simple proof is omitted here.

Using Lemma 7, we can easily construct examples of $\boldsymbol{CSP}(0)$ - sets.

Example 8. Let $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive real numbers with $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$. Define a set $W = \{0\} \cup \{x_n : n \in \mathbb{N}\}$. It is evident that the sequence $\{(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$ belongs to $\tilde{I}_W$ and W is $\tilde{x}$-strongly porous at 0. Every sequence $\tilde{\tau} \in \tilde{W}^d$ satisfies the condition of Lemma 7 with $W = E$, $\tilde{\gamma} = \tilde{x}$ and $c_1 = c_2 = 1$. Hence, W is $\tilde{\tau}$-strongly porous at 0 for every $\tilde{\tau} \in \tilde{W}^d$. Thus, by definition, $W \in \boldsymbol{CSP}(0)$.

Example 9. Let $q \in [1,\infty)$ and let W be the set from the previous example. Write

$$W(q) = \bigcup_{x\in W} [x, qx] = \{0\} \cup \{[x_n, qx_n] : n \in \mathbb{N}\}.$$

Let $m_0 \in \mathbb{N}$ be a number such that $qx_{n+1} < x_n$ for every $n \ge m_0$. The sequence $\{(qx_{m_0+n+1}, x_{m_0+n})\}_{n\in\mathbb{N}}$ belongs to $\tilde{I}_{W(q)}$. Write $q\tilde{x} = \{qx_n\}_{n\in\mathbb{N}}$.

Then $W(q)$ is $q\tilde{x}$- strongly porous at 0. Let $\tilde{\tau} = \{\tau_m\}_{m\in\mathbb{N}} \in \tilde{W}^d(q)$. It is clear that for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that

$$x_n \leq \tau_m \leq qx_n. \tag{4}$$

Reasoning as in Example 8, we obtain that (4) implies the membership $W(q) \in \boldsymbol{CSP}(0)$.

Lemma 10. *Let $E \subseteq \mathbb{R}^+$, $\tilde{\gamma} \in \tilde{E}^d$, $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ and let $\tilde{a} := \{a_n\}_{n\in\mathbb{N}}$. The following conditions are equivalent.*

(i) *The equivalence $\tilde{\gamma} \asymp \tilde{a}$ holds.*

(ii) *The chain of inequalities*

$$1 \leq \liminf_{n\to\infty} \frac{a_n}{\gamma_n} \leq \limsup_{n\to\infty} \frac{a_n}{\gamma_n} < \infty$$

hold.

(iii) *We have*

$$\limsup_{n\to\infty} \frac{a_n}{\gamma_n} < \infty \quad \text{and} \quad \gamma_n \leq a_n$$

for all sufficiently large n.

PROOF. The implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are trivial. Suppose that $\tilde{\gamma} \asymp \tilde{a}$. Then the inequality $\limsup\limits_{n\to\infty} \frac{a_n}{\gamma_n} < \infty$ follows. The membership $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ yields $\frac{b_n}{a_n} \to \infty$ with $n \to \infty$. Since $\tilde{\gamma} \asymp \tilde{a}$, the ratios $\frac{a_n}{\gamma_n}$ are bounded, and thus for all sufficiently large values of n we have $\frac{\gamma_n}{a_n} < \frac{b_n}{a_n}$, and hence $\gamma_n < b_n$. From this, and the fact that $\gamma_n \in E$ and $(a_n, b_n) \cap E = \varnothing$ for each n, it follows that $\gamma_n \leq a_n$ for all sufficiently large values of n. □

Corollary 11. *Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$. The following statements are equivalent.*

(i) *E is $\tilde{\tau}$-strongly porous at 0.*

(ii) *There exists a sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that*

$$1 \leq \liminf_{n\to\infty} \frac{a_n}{\gamma_n} \leq \limsup_{n\to\infty} \frac{a_n}{\gamma_n} < \infty.$$

(iii) *There exists a sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that*

$$\limsup_{n\to\infty} \frac{a_n}{\tau_n} < \infty \text{ and } \tau_n \leq a_n$$

for all sufficiently large n.

Using Corollary 11, it is easy to find a set $E \subseteq \mathbb{R}^+$ such that E is strongly porous on the right at 0 but $E \notin \boldsymbol{CSP}(0)$.

Example 12. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence from Example 8. Write

$$E = \{0\} \cup \{[x_{2n+1}, x_{2n}] : n \in \mathbb{N}\}.$$

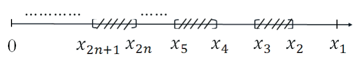


Fig. 1. The set E is shaded here

The sequence $\{(x_{2n+2}, x_{2n+1})\}_{n\in\mathbb{N}}$ belongs to $\tilde{I}_E$ and $\lim\limits_{n\to\infty} \frac{x_{2n+1}}{x_{2n+2}} = \infty$. Hence, E is strongly porous on the right at 0. Let us consider the sequence $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n = \sqrt{x_{2n+1}x_{2n}}$, $n \in \mathbb{N}$. It is clear that $\tilde{\tau} \in \tilde{E}^d$. Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be an arbitrary element of $\tilde{I}_E$ and let $n \in \mathbb{N}$ be such that $\tau_n \le a_n$. Since $\tau_n \in [x_{2n+1}, x_{2n}] \subseteq E$, we have $\tau_n \le x_{2n} \le a_n$. If $E \in \boldsymbol{CSP}(0)$, then E is $\tilde{\tau}$-strongly porous at 0. Hence, by Corollary 11, we may take $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ such that $\tau_n \le a_n$ for all sufficiently large n. Consequently,

$$\limsup_{n\to\infty} \frac{a_n}{\tau_n} \ge \limsup_{n\to\infty} \frac{x_n}{\tau_n} = \limsup_{n\to\infty} \sqrt{\frac{x_{2n}}{x_{2n+1}}} = \infty.$$

Now Corollary 11 implies that E is not $\tilde{\tau}$-strongly porous at 0, contrary to the supposition. Thus $E \notin \boldsymbol{CSP}(0)$.

The following proposition does not have any applications in the paper but is used in [4] to describe the structure of bounded tangent spaces to general metric spaces.

Note that if $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence of open intervals that witness the strong right porosity of E at 0, then $\frac{b_n}{a_n} \to \infty$. Hence, for each $K > 1$ we have $(a_n, Ka_n) \cap E = \varnothing$ for all sufficiently large n. Indeed, it is even the case that for each $k > 1$ and each $K > k$ we have $(ka_n, Ka_n) \cap E = \varnothing$ for all sufficiently large n. Although the strength of this last statement is essentially illusionary (simply choose the former value of K to be kK), this last statement allows for a formulation that we can apply to $\tilde{\tau}$-strong porosity.

Proposition 13. *Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$. The following statements are equivalent.*

(i) *E is $\tilde{\tau}$-strongly porous.*

(ii) *There is a constant* $k \in (1, \infty)$ *such that for every* $K \in (k, \infty)$ *there exists* $N_1(K) \in \mathbb{N}$ *such that*

$$(k\tau_n, K\tau_n) \cap E = \varnothing \tag{5}$$

if $n \geq N_1(K)$.

PROOF. Suppose that E is $\tilde{\tau}$-strongly porous. By Corollary 11 there is a sequence

$$\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E \tag{6}$$

such that $\limsup\limits_{n\to\infty} \frac{a_n}{\tau_n} < \infty$ and $\tau_n \leq a_n$ for all sufficiently large n. Write $k = 1 + \limsup\limits_{n\to\infty} \frac{a_n}{\tau_n}$. Then $\infty > k \geq 2$ and there is $N_0 \in \mathbb{N}$ such that

$$\tau_n \leq a_n < k\tau_n \tag{7}$$

for $n \geq N_0$. Let $K \in (k, \infty)$. Membership (6) implies the equality $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = \infty$. The last equality and (7) show that there is $N_1 \geq N_0$ such that

$$a_n < k\tau_n < K\tau_n \leq b_n$$

if $n \geq N_1$. Hence the inclusion

$$(k\tau_n, K\tau_n) \subseteq (a_n, b_n) \tag{8}$$

holds if $n \geq N_1$. Since

$$E \cap (a_n, b_n) = \varnothing, \tag{9}$$

(8) implies (5). Thus (ii) follows from (i).

Conversely, assume that statement (ii) holds. Let $K > 1$. Then for $K = 2k$ there is $N_0 \in \mathbb{N}$ such that

$$(k\tau_n, 2k\tau_n) \cap E = \varnothing$$

if $n \geq N_0$. Consequently, for every $n \geq N_0$, we can find a connected component (a_n, b_n) of $Ext E$ meeting the inclusion

$$(k\tau_n, 2k\tau_n) \subseteq (a_n, b_n). \tag{10}$$

Write $(a_n, b_n) = (a_{N_0}, b_{N_0})$ for $n < N_0$. Since, for $n \geq N_0$, we have

$$\tau_n \in E,\ \tau_n < k\tau_n \text{ and } (a_n, k\tau_n) \cap E = \varnothing,$$

the double inequality $\tau_n \leq a_n < k\tau_n$ holds for such n. To prove (i) it is sufficient to show that

$$\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E.$$

All intervals (a_n, b_n) are connected components of $ExtE$ and $\lim\limits_{n\to\infty} a_n = 0$ because $\lim\limits_{n\to\infty} \tau_n = 0$, so that $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ if and only if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \infty. \tag{11}$$

Let K be an arbitrary point of (k, ∞). Applying (5) we can find $N_1(K) \in \mathbb{N}$ such that

$$(k\tau_n, K\tau_n) \subseteq (a_n, b_n)$$

for $n \geq N_1(K)$. Consequently, for such n, we have

$$\frac{b_n}{a_n} \geq \frac{K\tau_n}{k\tau_n} = \frac{K}{k}.$$

Letting $K \to \infty$ we see that (11) follows. □

It is clear that, if there is $\tilde{\tau} \in \tilde{E}^d$ such that E is $\tilde{\tau}$-strongly porous, then E is strongly porous on the right at 0. Conversely we have the following

Proposition 14. *Let $E \subseteq \mathbb{R}^+$ and $0 \in acE$. If E is strongly porous on the right at 0, then there is $\tilde{\tau} \in \tilde{E}^d$ for which E is $\tilde{\tau}$-strongly porous.*

The proof is immediate and can be omitted.

Remark 15. If $0 \notin acE$, then E is strongly porous on the right at 0 but there are no $\tilde{\tau} \in \tilde{E}^d$ because $\tilde{E}^d = \varnothing$.

Definition 16. *Let $E \subseteq \mathbb{R}^+$. The set E is uniformly strongly porous at 0 if there exists a constant $c > 0$ such that for every $\tilde{\tau} \in \tilde{E}^d$ there is $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that*

$$1 \leq \liminf_{n\to\infty} \frac{a_n}{\tau_n} \leq \limsup_{n\to\infty} \frac{a_n}{\tau_n} \leq c$$

for all sufficiently large n.

Remark 17. If $0 \notin acE$, then E is uniformly strongly porous at 0 since $\tilde{E}^d = \varnothing$.

If E is uniformly strongly porous at 0, then $E \in \boldsymbol{CSP}(0)$. The converse is also true and we prove this in Theorem 27 given below.

Define, for $\tilde{\tau} \in \tilde{E}^d$, a subset $\tilde{I}_E(\tilde{\tau})$ of the set $\tilde{I}_E$ by the rule:

$$(\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E(\tilde{\tau})) \Leftrightarrow$$

$$(\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E \,\text{and}\, \tau_n \le a_n \,\text{for all sufficiently large}\, n \in \mathbb{N}).$$

Write

$$C(\tilde{\tau}) := \inf(\limsup_{n\to\infty} \frac{a_n}{\tau_n}) \quad \text{and} \quad C(E) := \sup_{\tilde{\tau}\in\tilde{E}_0^d} C(\tilde{\tau}) \tag{12}$$

where the infimum in the left formula is taken over all $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E(\tilde{\tau})$.

Remark 18. Let $E \subseteq \mathbb{R}^+$ and let $0 \in acE$. The set E is strongly porous at 0 if and only if

$$\tilde{I}_E(\tilde{\tau}) \neq \varnothing \tag{13}$$

for every $\tilde{\tau} \in \tilde{E}^d$. The set E is completely strongly porous at 0 if and only if $C(\tilde{\tau}) < \infty$ for every $\tilde{\tau} \in \tilde{E}^d$. The set E is uniformly strongly porous at 0 if and only if $C(E) < \infty$.

Lemma 19. *Let $E \subseteq \mathbb{R}^+$. If $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ and $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ are sequences such that $\tilde{a} \asymp \tilde{\tau}$, then $\tilde{a} := \{a_n\}_{n\in\mathbb{N}}$ and $\tilde{b} := \{b_n\}_{n\in\mathbb{N}}$ are eventually decreasing.*

PROOF. It suffices to show that $\tilde{a}$ is eventually decreasing. If $\tilde{a}$ is not eventually decreasing, then there is an infinite $A \subseteq \mathbb{N}$ such that

$$a_{n+1} > a_n \tag{14}$$

for every $n \in A$. Since $(a_n, b_n) \cap E = \varnothing$, inequality (14) implies that $a_{n+1} \ge b_n > a_n$. By Lemma 10 we have $a_n \ge \tau_n$ for all sufficiently large n. In addition, for such n, we may suppose also $\tau_n \ge \tau_{n+1}$ because $\tilde{\tau}$ is eventually decreasing. Consequently, we obtain

$$a_{n+1} \ge b_n > a_n \ge \tau_n \ge \tau_{n+1} \tag{15}$$

for all sufficiently large $n \in A$. Inequalities (15) imply

$$\frac{b_n}{a_n} \le \frac{a_{n+1}}{\tau_{n+1}}.$$

Hence

$$\infty = \lim_{n\to\infty, n\in A} \frac{b_n}{a_n} \le \limsup_{n\to\infty, n\in A} \frac{a_{n+1}}{\tau_{n+1}} \le \limsup_{n\to\infty} \frac{a_{n+1}}{\tau_{n+1}},$$

contrary to Lemma 10. $\square$

Proposition 20. *Let $E \subseteq \mathbb{R}^+$, $\tilde{\tau} \in \tilde{E}^d$, and let $\{(a_n^{(1)}, b_n^{(1)})\}_{n\in\mathbb{N}}$, $\{(a_n^{(2)}, b_n^{(2)})\}_{n\in\mathbb{N}}$ be two sequences belonging to $\tilde{I}_E$. If $\tilde{a}^1 \asymp \tilde{\tau}$ and $\tilde{a}^2 \asymp \tilde{\tau}$, where $\tilde{a}^i := \{a_n^{(i)}\}_{n\in\mathbb{N}}$, $i = 1, 2$, then there is $N_0 \in \mathbb{N}$ such that*

$$(a_n^{(2)}, b_n^{(2)}) = (a_n^{(1)}, b_n^{(1)}) \tag{16}$$

for every $n \geq N_0$.

PROOF. Let us denote by E_c the closure of E in $\mathbb{R}^+$. Using Remark 2 we see that $0 \in acE_c$ and $\tilde{\tau} \in \tilde{E}_c^d$. Since the sequences $\{(a_n^{(i)}, b_n^{(i)})\}_{n\in\mathbb{N}}$, $i = 1, 2$, belong to $\tilde{I}_E$, they also belong to $\tilde{I}_{E_c}$. By Lemma 19, we obtain $\tilde{a}^i \in \tilde{E}_c^d$, $i = 1, 2$. We also have $\tilde{\tau} \asymp \tilde{a}^1$, and $\tilde{\tau} \asymp \tilde{a}^2$. Consequently the equivalence $\tilde{a}^1 \asymp \tilde{a}^2$ holds. Applying Lemma 10 we can find $N_0 \in \mathbb{N}$ such that $a_n^{(1)} \leq a_n^{(2)}$ and $a_n^{(2)} \leq a_n^{(1)}$ for $n \geq N_0$. Thus $a_n^{(1)} = a_n^{(2)}$ for $n \geq N_0$ which implies (16) for such n. □

Define the set $\tilde{I}_E^d \subseteq \tilde{I}_E$ by the rule

$$(\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d) \Leftrightarrow$$

$$(\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E \text{ and } \{a_n\}_{n\in\mathbb{N}} \text{ is eventually decreasing}).$$

Remark 21. Let $E \subseteq \mathbb{R}^+$. If $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$, then there are $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ and $\tilde{\beta} = \{\beta_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ such that

$$\lim_{n\to\infty} \frac{\tau_n}{a_n} = \lim_{n\to\infty} \frac{\beta_n}{b_n} = 1. \tag{17}$$

Definition 22. *Let $\tilde{A} := \{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$ and $\tilde{L} := \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$. We write $\tilde{A} \preceq \tilde{L}$ if there are a natural number $N_1 = N_1(\tilde{A}, \tilde{L})$ and a function $f : \mathbb{N}_{N_1} \to \mathbb{N}$, where $\mathbb{N}_{N_1} := \{N_1, N_1 + 1, ...\}$, such that*

$$a_n = l_{f(n)} \tag{18}$$

for every $n \in \mathbb{N}_{N_1}$. We say that $\tilde{L} \in \tilde{I}_E^d$ is universal if $\tilde{A} \preceq \tilde{L}$ for every $\tilde{A} \in \tilde{I}_E^d$.

In other words, $\tilde{A} \preceq \tilde{L}$ means that there is $N_1 \in \mathbb{N}$ such that the range of the mapping $\mathbb{N}_{N_1} \ni n \mapsto (a_n, b_n) \in Com$ is a subset of the range of the mapping $\mathbb{N} \ni n \mapsto (l_n, m_n) \in Com$ where Com is the set of all connected components of $ExtE$. (See also Proposition 24 and Remark 25 below for other reformulations of Definition 22.)

If $\tilde{A}$ is a subsequence of $\tilde{L}$, then the relation $\tilde{A} \preceq \tilde{L}$ holds. As the following example shows, the converse is, in general, not true.

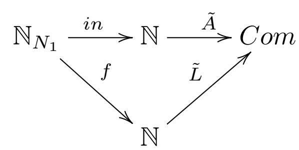
Example 23. Let $\{x_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive real numbers with $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$ and let $W = \{0\} \cup \{x_n : n \in \mathbb{N}\}$. Let us consider the sequence $\tilde{A} = \{(a_k, b_k)\}_{k\in\mathbb{N}}$ such that $(a_k, b_k) = (x_{n+1}, x_n)$ if and only if $n^2 \le k < (n+1)^2$. As was noted in Example 8, the membership $\tilde{X} \in \tilde{I}_W$ holds with $\tilde{X} = \{(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$. By Lemma 7 we obtain $\tilde{A} \in \tilde{I}_W$. Definition 22 implies that $\tilde{A} \preceq \tilde{L}$. It still remains to note that $\tilde{A}$ is not a subsequence of $\tilde{L}$.

The first part of Definition 22 can be reformulated as the following.

Proposition 24. *Let $\tilde{A} = \{(a_n, b_n)\}_{n\in\mathbb{N}}$ and $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}}$ belong to $\tilde{I}^d_E$. Then $\tilde{A} \preceq \tilde{L}$ if and only if there are $N_1 = N_1(\tilde{A}, \tilde{L}))$ and $f : \mathbb{N}_{N_1} \to \mathbb{N}$ such that*

$$b_n = m_{f(n)} \text{ for all } n \in \mathbb{N}_{N_1}.$$

Remark 25. The universality of $\tilde{L} \in \tilde{I}^d_E$ can be expressed in the language of arrows. An element $\tilde{L} \in \tilde{I}^d_E$ is universal if for every $\tilde{A} \in \tilde{I}^d_E$ there are $N_1 \in \mathbb{N}$ and $f : \mathbb{N}_{N_1} \to \mathbb{N}$ such the diagram



is commutative. Here in is the natural inclusion of $\mathbb{N}_{N_1}$ in $\mathbb{N}$ defined by $in(n) = n$ for each $n \in \mathbb{N}_{N_1}$.

Recall that a reflexive and transitive binary relation on a set X is a quasi-ordering on X. An antisymmetical quasi-ordering is a partial ordering and a poset is a set equipped with a partial ordering (see, for example, [12, p. 31–32]).

Proposition 26. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. The relation $\preceq$ is a quasi-ordering on the set $\tilde{I}^d_E$.*

PROOF. We must show that $\preceq$ is reflexive and transitive. The reflexivity of $\preceq$ is evident. To prove that $\preceq$ is transitive note that if $\tilde{A} \preceq \tilde{L}$, then there is an *increasing* function $f : \mathbb{N}_{N_1} \to \mathbb{N}$ such that (18) holds. (The existence of an increasing f meeting (18) follows because the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{l_n\}_{n\in\mathbb{N}}$ are eventually decreasing.) Suppose that $\tilde{A} \preceq \tilde{L}$ and $\tilde{L} \preceq \tilde{T}$, $\tilde{T} = \{(t_n, p_n)\}_{n\in\mathbb{N}} \in \tilde{I}^d_E$. Let $f : \mathbb{N}_{N_1} \to \mathbb{N}$ and $g : \mathbb{N}_{N_2} \to \mathbb{N}$ be two functions such that

$$a_n = l_{f(n)} \quad \text{for} \quad n \ge N_1 \quad \text{and} \quad l_n = t_{g(n)} \quad \text{for} \quad n \ge N_2.$$

Put $M := \max\{n \in \mathbb{N} : f(n) \leq N_2\}$. Since f is increasing and unbounded, we have $M < \infty$. Define

$$N_3 := \max\{M, N_1\}$$

with $N_3 := N_1$ if $\{n \in \mathbb{N} : f(n) \leq N_2\} = \varnothing$. Then the inequality $N_3 < \infty$ holds. In accordance with the construction, we have $f(n) \geq N_2$ for every $n \in \mathbb{N}_{N_3}$. Consequently we obtain

$$a_n = l_{f(n)} = t_{g(f(n))}$$

for such n. Thus $\tilde{A} \preceq \tilde{L}$ and $\tilde{L} \preceq \tilde{T}$ imply $\tilde{A} \preceq \tilde{T}$. □

Using standard facts from the theory of ordered sets we may prove that the quasi-ordering $\preceq$ generates an equivalence relation $\equiv$ on $\tilde{I}_E^d$ if we put

$$(\tilde{A} \equiv \tilde{T}) \Leftrightarrow (\tilde{A} \preceq \tilde{T} \text{ and } \tilde{T} \preceq \tilde{A}). \tag{19}$$

Passing to the quotient set induced by the equivalence relation $\equiv$ we obtain a poset. *Then $\tilde{I}_E^d$ has a universal element if and only if this poset has a largest element.*

Let $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$ be universal. Let us define the quantity

$$M(\tilde{L}) := \limsup_{n\to\infty} \frac{l_n}{m_{n+1}}. \tag{20}$$

Recall that a sequence $\tilde{a} = \{a_n\}_{n\in\mathbb{N}}$, $a_n \in \mathbb{R}$, is eventually strictly decreasing if $a_{n+1} < a_n$ for all sufficiently large n. Write $\tilde{I}_E^{sd}$ for the set of $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$ having eventually strictly decreasing $\{a_n\}_{n\in\mathbb{N}}$.

Theorem 27. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. The following conditions are equivalent.*

(i) *E is a **CSP**(0) - set.*

(ii) *$\tilde{I}_E^d$ contains a universal element $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^{sd}$ with*

$$M(\tilde{L}) < \infty. \tag{21}$$

(iii) *E is uniformly strongly porous at 0.*

To prove Theorem 27 we need some additional lemmas.

Lemma 28. *Let $E \subseteq \mathbb{R}^+$. If $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$ is universal, then there is a subsequence $\tilde{L}' = \{(l_{n_k}, m_{n_k})\}_{k\in\mathbb{N}}$ of $\tilde{L}$ such that $\tilde{L}'$ is also universal and $\tilde{L}' \in \tilde{I}_E^{sd}$.*

PROOF. We construct $\tilde{L}'$ by induction. Since $\{l_n\}_{n\in\mathbb{N}}$ is eventually decreasing, there exists $n_1 \in \mathbb{N}$ such that $l_{n+1} \le l_n$ for $n \ge n_1$. The limit relation $\lim\limits_{n\to\infty} l_n = 0$ implies that there is $n \ge n_1$ such that $l_n < l_{n_1}$. Write

$$n_2 := \min\{n \in \mathbb{N}_{n_1} : l_n < l_{n_1}\}.$$

Similarly we set

$$n_{k+1} := \min\{n \in \mathbb{N}_{n_k} : l_n < l_{n_k}\} \tag{22}$$

for $k = 2, 3, 4\ldots$. For every $n \ge n_1$ there is the unique $k \in \mathbb{N}$ such that

$$n_k \le n < n_{k+1}. \tag{23}$$

Furthermore, the decrease of the sequence $\{l_n\}_{n\in\mathbb{N}_{n_1}}$ implies that

$$l_{n_k} = l_n \tag{24}$$

if n satisfies (23). Let us define $g : \mathbb{N}_{n_1} \to \mathbb{N}$ by the rule $g(n) = k$ where k is the unique index satisfying (23). In fact, it was proved above that $\tilde{L} \preceq \tilde{L}'$. By Proposition 26 the relation $\preceq$ is transitive. Since $\tilde{L}$ is universal, we have $\tilde{T} \preceq \tilde{L}$ for every $\tilde{T} \in \tilde{I}^d_E$. Consequently $\tilde{T} \preceq \tilde{L}'$ for every $\tilde{T} \in \tilde{I}^d_E$, i.e., $\tilde{L}'$ is universal. It still remains to note that (22) implies the inequality $l_{n_k} > l_{n_{k+1}}$ for every $k \in \mathbb{N}$. Hence $\{l_{n_k}\}_{k\in\mathbb{N}}$ is a strictly decreasing sequence. Thus $\tilde{L}' \in \tilde{I}^{sd}_E$. $\square$

Remark 29. If $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}^{sd}_E$ and $\tilde{A} = \{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}^{sd}_E$, then $\tilde{L} \equiv \tilde{A}$ if and only if there exist N_1, $N_2 \in \mathbb{N}$ such that

$$(l_{n+N_1}, m_{n+N_1}) = (a_{n+N_2}, b_{n+N_2})$$

for every $n \in \mathbb{N}$, where $\equiv$ is defined by (19).

We will not use Remark 29 in the sequel and omit the proof here.

Lemma 30. *Let E be a **CSP**(0) - set. If $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}^{sd}_E$ is universal, then*

$$M(\tilde{L}) = C(E) \tag{25}$$

where the quantities $M(\tilde{L})$ and $C(E)$ are defined by (20) *and* (12) *respectively.*

PROOF. Let $\tilde{L} \in \tilde{I}^{sd}_E$ be universal. We shall first prove the inequality

$$M(\tilde{L}) \ge C(E). \tag{26}$$

Inequality (26) holds if and only if

$$M(\tilde{L}) \ge C(\tilde{\tau}) \tag{27}$$

for every $\tilde{\tau} \in \tilde{E}^d$, where $C(\tilde{\tau})$ was defined in (12). Let $\tilde{\tau} \in \tilde{E}^d$. By the lemma's hypothesis, E is completely strongly porous at 0. Hence there is $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that $\tilde{\tau} \asymp \tilde{a}$. By Lemma 10 we have the inequality

$$\limsup_{n\to\infty} \frac{a_n}{\tau_n} < \infty \tag{28}$$

and, for all sufficiently large n, the inequality

$$\tau_n \le a_n. \tag{29}$$

Proposition 20 and the definition of $C(\tilde{\tau})$ imply

$$C(\tilde{\tau}) = \limsup_{n\to\infty} \frac{a_n}{\tau_n}. \tag{30}$$

Hence to prove (27) we must show that

$$M(\tilde{L}) \ge \limsup_{n\to\infty} \frac{a_n}{\tau_n}. \tag{31}$$

Be Lemma 19 we have

$$\tilde{A} := \{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d. \tag{32}$$

Since $\tilde{L}$ is universal, from (32) follows that $\tilde{A} \preceq \tilde{L}$. Consequently there are $N_1 \in \mathbb{N}$ and an increasing function $f : \mathbb{N}_{N_1} \to \mathbb{N}$ such that

$$a_n \ge a_{n+1} \quad \text{and} \quad a_n = l_{f(n)} \tag{33}$$

for $n \ge N_1$. Since $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^{sd}$, we may suppose that $\tilde{l} = \{l_n\}_{\in\mathbb{N}}$ is strictly decreasing. Replacing $\tilde{\tau}$ by a suitable subsequence we may assume that $\tilde{\tau}$ and $\tilde{a}$ are also strictly decreasing, f is strictly increasing, and that the relations

$$\tau_1 \le l_1, \quad \lim_{n\to\infty} \frac{a_n}{\tau_n} = \limsup_{n\to\infty} \frac{a_n}{\tau_n} \tag{34}$$

hold. The closed intervals $[m_{n+1}, l_n]$, $n = 1, 2, \ldots$, together with the half-open interval $[m_1, \infty)$ form a cover of the set $E_0 = E \setminus \{0\}$, i.e.

$$E_0 \subseteq [m_1, \infty) \cup \left(\bigcup_{n\in\mathbb{N}} [m_{n+1}, l_n] \right).$$

Since the elements of this cover are pairwise disjoint and $\tau_1 \le l_1$, for every $n \in \mathbb{N}$ there is a unique $k(n) \in \mathbb{N}$ such that

$$\tau_n \in [m_{k(n)+1}, l_{k(n)}]. \tag{35}$$

We claim that the equality

$$k(n) = f(n) \tag{36}$$

holds for all sufficiently large n. Indeed, using (29), (33) and (35) we obtain

$$\tau_n \leq l_{f(n)} \quad \text{and} \quad \tau_n \geq m_{k(n)+1}.$$

These inequalities and

$$m_{k(n)+1} > l_{k(n)+1} > l_{k(n)+2} > l_{k(n)+3} > \dots$$

imply

$$f(n) \leq k(n). \tag{37}$$

Suppose that the last inequality is strict for n belonging to an infinite set $A \subseteq \mathbb{N}$, i.e.

$$f(n) \leq k(n) - 1 \tag{38}$$

for $n \in A$. Since $\{a_n\}_{n\in\mathbb{N}} \asymp \{\tau_n\}_{n\in\mathbb{N}}$ and $a_n = l_{f(n)}$, we can find a constant $c \in (0, 1)$ such that

$$cl_{f(n)} \leq \tau_n \leq l_{f(n)} \tag{39}$$

for all sufficiently large n. From (35), (37) and (39) it follows that

$$cl_{f(n)} \leq \tau_n \leq l_{k(n)} \leq l_{f(n)}. \tag{40}$$

Since $\tilde{l} = \{l_n\}_{n\in\mathbb{N}}$ is strictly increasing and $(l_n, m_n) \cap (l_j, m_j) = \varnothing$ if $n \neq j$, (38) implies that

$$l_{k(n)} < m_{k(n)} \leq l_{k(n)-1} \leq l_{f(n)} < m_{f(n)}.$$

These inequalities and (40) show that

$$cl_{f(n)} \leq \tau_n \leq l_{k(n)} < m_{k(n)} \leq l_{k(n)-1} < l_{f(n)}$$

for $n \in A$. Consequently we have

$$\frac{1}{c} = \lim_{n\to\infty} \frac{l_{f(n)}}{cl_{f(n)}} \geq \limsup_{n\to\infty,\, n\in A} \frac{m_{k(n)}}{l_{k(n)}},$$

contrary to the limit relation

$$\lim_{n\to\infty} \frac{m_n}{l_n} = \infty.$$

Hence the set of $n \in \mathbb{N}$ meeting the condition $f(n) < k(n)$ is finite. Thus (36) holds for all sufficiently large n.

Now it is easy to prove (31). By (33) and (36) we have

$$a_n = l_{f(n)} = l_{k(n)}.$$

Relation (35) implies $\tau_n \geq m_{k(n)+1}$. Consequently

$$\frac{a_n}{\tau_n} \leq \frac{l_{k(n)}}{m_{k(n)+1}}.$$

Hence

$$\limsup_{n\to\infty} \frac{a_n}{\tau_n} \leq \limsup_{n\to\infty} \frac{l_{k(n)}}{m_{k(n)+1}} \leq \limsup_{n\to\infty} \frac{l_n}{m_{n+1}} = M(\tilde{L}).$$

Inequality (31) follows, so that (26) is proved.

To prove the inequality

$$M(\tilde{L}) \leq C(E) \tag{41}$$

we take a sequence $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ such that (35) holds with $k(n) = n$ and

$$\lim_{n\to\infty} \frac{m_{n+1}}{\tau_n} = 1. \tag{42}$$

A desirable $\tilde{\tau}$ can be constructed as in the proof of Proposition 14. The set E is $\tilde{\tau}$-strongly porous at 0 because this is a $\boldsymbol{CSP}(0)$ - set. Hence there is $\tilde{a} \asymp \tilde{\tau}$ such that $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$. By Lemma 19 the sequence $\tilde{a}$ is eventually decreasing. Since $\tau_n \in [m_{n+1}, l_n]$, using (36) we obtain

$$a_n = l_n$$

for all sufficiently large n. From (30) and (42) it follows that

$$C(\tilde{\tau}) = \limsup_{n\to\infty} \frac{a_n}{\tau_n} = \limsup_{n\to\infty} \frac{l_n}{m_{n+1}} \frac{m_{n+1}}{\tau_n}$$

$$= \limsup_{n\to\infty} \frac{l_n}{m_{n+1}} \lim_{n\to\infty} \frac{m_{n+1}}{\tau_n} = \limsup_{n\to\infty} \frac{l_n}{m_{n+1}} = M(\tilde{L}). \tag{43}$$

Since $C(E) \geq C(\tilde{\tau})$, inequality (41) follows.

To complete the proof, it suffices to observe that (26) and (41) imply (25). ☐

Directly from (43) we obtain

Corollary 31. *Let $E \subseteq \mathbb{R}^+$ be a $\boldsymbol{CSP}(0)$ - set. If $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^{sd}$ is universal, then $M(\tilde{L}) < \infty$.*

Remark 32. As has been shown in Lemma 30 the equality $M(\tilde{L}) = C(E)$ holds for every universal $\tilde{L} \in \tilde{I}_E^{sd}$. Suppose that $\tilde{L} \in \tilde{I}_E^d$ is universal but $\tilde{L} \notin \tilde{I}_E^{sd}$. Define the set $A \subseteq \mathbb{N}$ by the rule

$$(n \in A) \Leftrightarrow (n \in \mathbb{N} \text{ and } (l_{n+1}, m_{n+1}) = (l_n, m_n)).$$

Let $\tilde{L}' \in \tilde{I}_E^{sd}$ be the universal element of $\tilde{I}_E^d$ constructed from $\tilde{L}$ as in Lemma 28. Using the definition of the set A we obtain

$$M(\tilde{L}) = \limsup_{n\to\infty} \frac{l_{n+1}}{m_n} = \limsup_{n\to\infty,\, n\in A} \frac{l_{n+1}}{m_n} \vee \limsup_{n\to\infty,\, n\in\mathbb{N}\setminus A} \frac{l_{n+1}}{m_n}$$

$$= \limsup_{n\to\infty,\, n\in A} \frac{l_n}{m_n} \vee M(\tilde{L}') = 0 \vee M(\tilde{L}') = M(\tilde{L}').$$

Consequently if $\tilde{L}, \tilde{S} \in \tilde{I}_E^d$ are universal, then $M(\tilde{L}) = M(\tilde{S})$. Thus condition (ii) of Theorem 27 can be formulated by the following equivalent way.

• *The set of universal elements $\tilde{L} \in \tilde{I}_E^d$ is nonempty and the inequality $M(\tilde{L}) < \infty$ holds for every universal $\tilde{L}$.*

PROOF OF THEOREM 27. (i) $\Rightarrow$ (ii). Let E be a ***CSP***(0) - set. We shall first prove that there is a sequence $\tilde{u} = \{u_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ such that for every $\tilde{\tau} = \{\tau_k\}_{k\in\mathbb{N}} \in \tilde{E}^d$ can be found an eventually increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying the relation

$$\{\tau_k\}_{k\in\mathbb{N}} \asymp \{u_{f(k)}\}_{k\in\mathbb{N}}. \tag{44}$$

Let us define the sequence of sets E_j, $j \in \mathbb{N}$, by the rule

$$E_1 := E \cap [1, \infty),\ E_2 := E \cap [\frac{1}{2}, 1), ..., E_j := E \cap [\frac{1}{2^{j-1}}, \frac{1}{2^{j-2}}). \tag{45}$$

There is the unique subsequence $\{E_{j_n}\}_{n\in\mathbb{N}}$ of the sequence $\{E_j\}_{j\in\mathbb{N}}$ such that

$$E \setminus \{0\} = \bigcup_{n\in\mathbb{N}} E_{j_n} \quad \text{and} \quad E_{j_n} \neq \varnothing$$

for every $n \in \mathbb{N}$. For convenience we set $A_n := E_{j_n}$, $n \in \mathbb{N}$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers meeting the condition $u_n \in A_n$ for every $n \in \mathbb{N}$. It is clear that $\{u_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$. For every $\tilde{\tau} = \{\tau_k\}_{k\in\mathbb{N}} \in \tilde{E}^d$, define $f : \mathbb{N} \to \mathbb{N}$ by the rule

$$f(k) = n \quad \text{if and only if} \quad \tau_k \in A_n.$$

The function f is well-defined because

$$E \setminus \{0\} = \bigcup_{n\in\mathbb{N}} A_n \quad \text{and} \quad A_j \cap A_i = \varnothing \quad \text{if} \quad i \neq j.$$

It follows directly from (45) that

$$\frac{1}{2}\tau_k \le u_{f(k)} \le 2\tau_k$$

if $f(k) \ge 2$. Moreover, since $\tilde{\tau}$ and $\tilde{u}$ are eventually decreasing and $\lim\limits_{n\in\mathbb{N}} \tau_n = 0$, the function $f : \mathbb{N} \to \mathbb{N}$ is eventually increasing and the set $\{k \in \mathbb{N} : f(k) = 1\}$ is finite. Consequently there are positive constants c_1 and c_2 such that

$$c_2\tau_k \le u_{f(k)} \le c_1\tau_k$$

for all $k \in \mathbb{N}$. Thus (44) holds.

Let $\{u_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ be the sequence constructed above. Since E is a $\boldsymbol{CSP}(0)$ - set, E is $\tilde{u}$-strongly porous at 0. Hence, there is $\tilde{A} := \{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that

$$\tilde{a} \asymp \tilde{u}. \tag{46}$$

Lemma 19 implies that $\tilde{a}$ is eventually decreasing, i.e., $\tilde{A} \in \tilde{I}_E^d$. We claim that $\tilde{A}$ is universal. Indeed, as was shown for every $\tilde{\tau} = \{\tau\}_{k\in\mathbb{N}} \in \tilde{E}^d$ there is $f : \mathbb{N} \to \mathbb{N}$ such that (44) holds. The relation $\{u_n\}_{n\in\mathbb{N}} \asymp \{a_n\}_{n\in\mathbb{N}}$ implies that

$$\{u_{f(k)}\}_{k\in\mathbb{N}} \asymp \{a_{f(k)}\}_{k\in\mathbb{N}}. \tag{47}$$

Every interval $(a_{f(n)}, b_{f(n)})$ is a connected component of $ExtE$ and, in addition, $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = \infty$ implies $\lim\limits_{k\to\infty} \frac{b_{f(k)}}{a_{f(k)}} = \infty$ because $\lim\limits_{n\to\infty} f(n) = \infty$. Consequently we obtain

$$\{(a_{f(k)}, b_{f(k)})\}_{k\in\mathbb{N}} \in \tilde{I}_E. \tag{48}$$

Moreover, since f is eventually increasing and $\tilde{A} = \{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$, (48) implies

$$\{(a_{f(k)}, b_{f(k)})\}_{k\in\mathbb{N}} \in \tilde{I}_E^d. \tag{49}$$

From (44) and (47) we obtain

$$\{\tau_k\}_{k\in\mathbb{N}} \asymp \{a_{f(k)}\}_{k\in\mathbb{N}}. \tag{50}$$

Using (49), (50) and Remark 21, we can prove that $\tilde{L} \preceq \tilde{A}$ for every $\tilde{L} \in \tilde{I}_E^d$, as required.

By Lemma 28 we can find a universal element $\tilde{L} \in \tilde{I}_E^{sd}$. In accordance with Corollary 31 we have $M(\tilde{L}) < \infty$. Thus condition (i) implies (ii).

The implication (iii) $\Rightarrow$ (i) is evident. Moreover, using Lemma 30, we can simply verify that the implication ((i)&(ii)) $\Rightarrow$ (iii) is true. Consequently to complete the proof is suffices to show that (ii) $\Rightarrow$ (i). Suppose that condition (ii) holds. Let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ and let $\tilde{L} = \{(l_k, m_k)\}_{k\in\mathbb{N}} \in \tilde{I}_E^{sd}$ be universal. As in the proof of Lemma 30 we may suppose that $\{l_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence and that $\tau_1 \le l_1$. Then for every $n \in \mathbb{N}$ there is a unique $k(n) \in \mathbb{N}$ such that

$$m_{k(n)+1} \le \tau_n \le l_{k(n)}, \tag{51}$$

(see (35)). Inequality chain (51) implies

$$\limsup_{n\to\infty} \frac{l_{k(n)}}{\tau_n} \le \limsup_{n\to\infty} \frac{l_{k(n)}}{m_{k(n)+1}} \le \limsup_{k\to\infty} \frac{l_k}{m_{k+1}} = M(\tilde{L}) < \infty.$$

Since $\{(l_{k(n)}, m_{k(n)})\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$, the set E is $\tilde{\tau}$-strongly porous at 0 by Lemma 10. Thus condition (i) follows from condition (ii). □

Remark 33. Conditions (i) and (iii) of Theorem 27 are equivalent for arbitrary $E \subseteq \mathbb{R}^+$. Indeed, if $p^+(E, 0) < 1$, then both (i) and (iii) are evidently false. If $p^+(E, 0) = 1$ but $0 \notin acE$, then (i) and (iii) are true (see Remark 5 and Remark 17). In this connection it should be pointed out that condition (ii) of Theorem 27 implies $\tilde{I}_E \neq \varnothing$. Consequently, if (ii) holds, then $0 \in acE$ and $p^+(E, 0) = 1$ (see Remark 3).

Example 12 shows that the existence of a universal $\tilde{L} \in \tilde{I}_E^{sd}$ does not imply the inequality $M(\tilde{L}) < \infty$.

The next theorem describes the structure of sets $E \subseteq \mathbb{R}^+$ for which there is a universal $\tilde{L} \in \tilde{I}_E^{sd}$.

As in Remark 25 write Com for the set of all connected components of $ExtE$.

Theorem 34. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. Then $\tilde{I}_E^d$ contains a universal element if and only if there is a constant $c > 1$ such that for every $K > 1$ there is $t > 0$ for which the inequalities $t > a$ and $\frac{b}{a} > c$ imply the inequality $\frac{b}{a} > K$ for every $(a, b) \in Com$.*

Proof. Suppose that there is a universal element $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$. We must prove that

$$\begin{gathered}\exists\, c > 1\, \forall\, K > 1\, \exists\, t > 0\, \forall\, (a, b) \in Com : \\ (a < t)\ \&\left(\frac{b}{a} > c\right) \Rightarrow \left(\frac{b}{a} > K\right).\end{gathered} \tag{52}$$

By Lemma 28 we may assume that $\{l_n\}_{n\in\mathbb{N}}$ is strictly decreasing. Using the limit relations

$$\lim_{n\to\infty}\frac{m_n}{l_n}=\infty \text{ and } \lim_{n\to\infty} l_n = 0$$

and the strict decrease of $\{l_n\}_{n\in\mathbb{N}}$ we obtain that

$$\forall\, K>1\,\exists\, t>0\,\forall\, n\in\mathbb{N}:\ (l_n<t)\Rightarrow\left(\frac{m_n}{l_n}>K\right). \tag{53}$$

If (52) does not hold, then

$$\forall\, c>1\,\exists\, K=K(c)>1\,\forall\, t>0\,\exists\,(a,b)\in Com:(t>a)\ \&\left(c<\frac{b}{a}\le K(c)\right). \tag{54}$$

Using this formula with $c=j$ and $K=K(j)$, for $j=1,2,...$, we see that

$$\forall\, t>0\,\exists\,(a_j,b_j)\in Com:(a_j<t)\ \&\left(j\le\frac{b_j}{a_j}\le K(j)\right). \tag{55}$$

Formula (53) implies that

$$\forall\, n\in\mathbb{N}\ \exists\ t_j>0:\ (l_n<t_j)\Rightarrow\left(\frac{m_n}{l_n}>K(j)\right). \tag{56}$$

We can suppose also that $\lim\limits_{j\to\infty} t_j=0$ and $\{t_j\}_{j\in\mathbb{N}}$ is strictly decreasing. From (55) with $t=t_j$ it follows that

$$\forall\, j\in\mathbb{N}\,\exists\,(a_j,b_j)\in Com:(a_j<t_j)\ \&\left(j\le\frac{b_j}{a_j}\le K(j)\right). \tag{57}$$

Consequently the sequence $\tilde{A}:=\{(a_j,b_j)\}_{j\in\mathbb{N}}$ belongs to $\tilde{I}_E$. Using the limit relation $\lim\limits_{j\to\infty} t_j=0$ and passing to a suitable subsequence we may assume that $\tilde{A}\in\tilde{I}_E^d$. Formulas (56) and (57) imply that

$$(a_j,b_j)\ne(l_n,m_n)$$

for every element (l_n,m_n) of $\tilde{L}$. Consequently $\tilde{L}$ is not universal, contrary to the assumption.

Conversely, suppose that (52) holds. Let us prove that there exists a universal element in $\tilde{I}_E^d$. Let c be the constant satisfying (52). Define a subset $Com(c)$ of the set Com by the rule

$$((a,b)\in Com(c))\Leftrightarrow\left((a,b)\in Com,\ a>1 \text{ and } \frac{b}{a}>c\right).$$

We can enumerate of the intervals $(a, b) \in Com(c)$ in the sequence

$$(a_1, b_1),\ (a_2, b_2),\ ..., (a_n, b_n),\ ...$$

such that $\{a_n\}_{n\in\mathbb{N}}$ is strictly decreasing. Condition (52) implies that $\tilde{A} = \{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}^d_E$. The universality of $\tilde{A}$ follows directly from Definition 22 and (52). □

As in Remark 33 it should be noted that the existence of a universal $\tilde{L} \in \tilde{I}^d_E$ implies that $0 \in acE$ and $p^+(E, 0) = 1$.

An illustrative model for Theorem 34. Let $E \subseteq (0, 1]$ be closed and let $0 \in acE$. Write

$$W := \left\{ \ln\left(\frac{1}{x}\right) : x \in E \right\}.$$

We can consider W as "a photograph of a one-dimensional liquid" with some "gas bubbles" $\left(\ln\left(\frac{1}{b}\right), \ln\left(\frac{1}{a}\right)\right)$, where $(a, b) \in Com$, which move to $+\infty$. Theorem 34 means that there is a critical value $\ln c$ such that if the sizes of the gas bubbles are greater than $\ln c$, then these bubbles undergo an unbounded blow up during their motion.

The following simple proposition can be considered as a limit case of Theorem 34.

Proposition 35. *Let $E \subseteq \mathbb{R}^+$ and $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}^d_E$. Suppose that for every $(a, b) \in Com$ there is $n \in \mathbb{N}$ such that $(a, b) = (l_n, m_n)$. Then $\tilde{L}$ is universal.*

The proof follows directly from Definition 22.

3 Another characterizations of $\boldsymbol{CSP}(0)$ - sets

Let E be a subset of $\mathbb{R}^+$. Define the set $\tilde{H} = \tilde{H}(E)$ of the sequences $\tilde{h} = \{h_n\}_{n\in\mathbb{N}}$, $h_n > 0$, $\lim\limits_{n\to\infty} h_n = 0$ by the rule:

$$(\tilde{h} \in \tilde{H}) \Leftrightarrow \left(\frac{\lambda(E, 0, h_n)}{h_n} \to p^+(E, 0) \quad \text{with} \quad n \to \infty \right) \tag{58}$$

where the quantities $p^+(E, 0)$ and $\lambda(E, 0, h_n)$ are the same as in Definition 1.

Theorem 36. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0. Then E is a $\boldsymbol{CSP}(0)$ - set if and only if for every $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d_0$ there is $\tilde{h} = \{h_n\}_{n\in\mathbb{N}} \in \tilde{H}(E)$ such that $\tilde{\tau} \asymp \tilde{h}$.*

Proof. The necessity is easy to prove. Suppose E is a $\boldsymbol{CSP}(0)$ - set. Let $\tilde{\tau} \in \tilde{E}^d$. By Theorem 27 there is a universal element $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^{sd}$ with $M(\tilde{L}) < \infty$. Reasoning as in the proof of Theorem 27, we can find $k(n) \in \mathbb{N}$ such that

$$\tau_n \in [m_{k(n)+1}, l_{k(n)}] \tag{59}$$

for all sufficiently large n (see (35)). Membership (59) implies the inequalities

$$m_{k(n)+1} \le \tau_n \quad \text{and} \quad \frac{\tau_n}{m_{k(n)+1}} \le \frac{l_{k(n)}}{m_{k(n)+1}}.$$

Thus we have

$$\limsup_{n\to\infty} \frac{\tau_n}{m_{k(n)+1}} \le \limsup_{n\to\infty} \frac{l_{k(n)}}{m_{k(n)+1}} \le M(\tilde{L}) < \infty.$$

Consequently there are $c_1 \ge 1$ and $N_1 \in \mathbb{N}$ such that $m_{k(n)+1} \le \tau_n \le c_1 m_{k(n)+1}$ for $n \ge N_1$. If we set $m_{k(n)+1} := m_{k(N_1)+1}$ for $n < N_1$, then it is easy to see that $\{\tau_n\}_{n\in\mathbb{N}} \asymp \{m_{k(n)+1}\}_{n\in\mathbb{N}}$. To be certain that $\{m_{k(n)+1}\}_{n\in\mathbb{N}} \in \tilde{H}(E)$, it suffices to check that

$$\lim_{n\to\infty} \frac{\lambda(E, 0, m_{k(n)+1})}{m_{k(n)+1}} = 1. \tag{60}$$

(Indeed, $p^+(E, 0) = 1$ because E is strongly porous on the right at 0.) Since the quantity $\lambda(E, 0, m_{k(n)+1})$ is the length of the largest open interval in the set $(0, m_{k(n)+1}) \cap ExtE$ and

$$(l_{k(n)+1}, m_{k(n)+1}) \subseteq (0, m_{k(n)+1}) \cap ExtE,$$

we have

$$\frac{m_{k(n)+1} - l_{k(n)+1}}{m_{k(n)+1}} \le \frac{\lambda(E, 0, m_{k(n)+1})}{m_{k(n)+1}} \le 1. \tag{61}$$

The sequence $\tilde{L}$ belongs to $\tilde{I}_E^{sd}$. Hence

$$\lim_{n\to\infty} \frac{m_{k(n)+1} - l_{k(n)+1}}{m_{k(n)+1}} = 1.$$

The last relation and (61) imply (60).

The proof of the sufficiency is more awkward, so we divide it into several lemmas.

Lemma 37. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ and $\tilde{h} = \{h_n\}_{n\in\mathbb{N}} \in \tilde{H}(E)$. If $\tilde{\tau} \asymp \tilde{h}$, then there is $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that*

$$\{\tau_n\}_{n\in\mathbb{N}} \asymp \{b_n\}_{n\in\mathbb{N}}. \tag{62}$$

PROOF. Let $\tilde{\tau} \asymp \tilde{h}$. By the definition of $\tilde{H}(E)$, for every $n \in \mathbb{N}$, there is an interval $(a_n^{'}, b_n^{'}) \subseteq (0, h_n) \cap ExtE$ such that

$$\lim_{n\to\infty} \frac{b_n^{'} - a_n^{'}}{h_n} = 1. \tag{63}$$

Moreover, the relation $\tilde{\tau} \asymp \tilde{h}$ implies that there are constants $k \in (0,1)$ and $K \in (1,\infty)$ such that

$$\tau_n \in (kh_n, Kh_n) \tag{64}$$

for every $n \in \mathbb{N}$. Consequently

$$\tau_n \in (0, Kh_n) \setminus (a_n^{'}, b_n^{'}). \tag{65}$$

Using (63) we can show that

$$b_n^{'} > kh_n > a_n^{'} \tag{66}$$

for all sufficiently large n. It is clear that $Kh_n > h_n \geq b_n^{'}$. Hence (64) – (66) imply

$$\tau_n \in [b_n^{'}, Kh_n) \tag{67}$$

for all sufficiently large n (see Fig. 2 below).

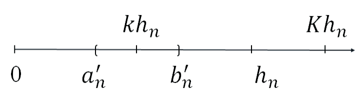


Fig. 2

Let (a_n, b_n) be the connected component of $ExtE$ meeting the inclusion $(a_n^{'}, b_n^{'}) \subseteq (a_n, b_n)$. From (67) it follows $\tau_n \geq b_n$. Hence

$$kh_n < b_n^{'} \leq b_n \leq \tau_n < Kh_n \tag{68}$$

for all sufficiently large n. Consequently $\tilde{\tau} \asymp \tilde{h}$ and $\tilde{b} \asymp \tilde{h}$, so that (62) follows. To complete the proof, it suffices to show the membership $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$. The last relation holds if and only if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0. \tag{69}$$

Inequalities $a_n \le a'_n < b'_n \le b_n$ imply that

$$0 \le \frac{a_n}{b_n} \le \frac{a_n^{'}}{b_n^{'}}. \tag{70}$$

Moreover, since

$$\frac{b_n^{'} - a_n^{'}}{h_n} \le \frac{b_n^{'} - a_n^{'}}{b_n^{'}} \le 1,$$

limit relation (63) yields

$$\lim_{n\to\infty} \frac{a_n^{'}}{b_n^{'}} = 0.$$

Thus (70) follows from (69). □

Remark 38. It is clear that $\{b_n\}_{n\in\mathbb{N}} \in \tilde{H}(E)$ for each $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$.

The following lemmas are analogs of Lemma 19 and Proposition 20, and have similar proofs.

Lemma 39. *Let $E \subseteq \mathbb{R}^+$. If $\tilde{\tau} \in \tilde{E}^d$ and $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$, then the equivalence $\tilde{b} \asymp \tilde{\tau}$ implies that $\tilde{b}$ and $\tilde{a}$ are eventually decreasing.*

Lemma 40. *Let $E \subseteq \mathbb{R}^+$, $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$, and let $\{(a_n^{(i)}, b_n^{(i)})\}_{n\in\mathbb{N}} \in \tilde{I}_E$, $i = 1, 2$. If*

$$\tilde{b}^1 \asymp \tilde{\tau} \asymp \tilde{b}^2$$

where $\tilde{b}^i = \{b_n^{(i)}\}_{n\in\mathbb{N}}$, $i = 1, 2$, then there is $N_0 \in \mathbb{N}$ such that

$$(a_n^{(1)}, b_n^{(1)}) = (a_n^{(2)}, b_n^{(2)})$$

for every $n \ge N_0$.

The next lemma is closely related to the implication (i) ⇒ (ii) from Theorem 27.

Lemma 41. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. If for every $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ there is $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that $\{\tau_n\}_{n\in\mathbb{N}} \asymp \{b_n\}_{n\in\mathbb{N}}$, then there is a universal $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^{sd}$ with*

$$M(\tilde{L}) < \infty. \tag{71}$$

The following proof is a modification of the corresponding part of the proof of Theorem 27.

Proof of Lemma 41. Suppose that for every $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ there is

$\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that $\tilde{\tau} \asymp \tilde{b} = \{b_n\}_{n\in\mathbb{N}}$. In the proof of Theorem 27 we have found a sequence $\tilde{u} = \{u_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ such that for every $\tilde{\tau} \in \tilde{E}^d$ there is an eventually increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying the relation

$$\{\tau_k\}_{k\in\mathbb{N}} \asymp \{u_{f(k)}\}_{k\in\mathbb{N}}. \tag{72}$$

By the supposition there is $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that

$$\tilde{u} \asymp \tilde{b}. \tag{73}$$

Since $\tilde{u} \in \tilde{E}^d$, Lemma 39 implies that $\tilde{b}$ and $\tilde{u}$ are eventually decreasing. Consequently $\tilde{A} := \{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^d$. We shall show that $\tilde{A}$ is universal. Let $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}}$ be an arbitrary element of $\tilde{I}_E^d$. Using Definition 24 we see that $\tilde{A}$ is universal if and only if there are $N_1 \in \mathbb{N}$ and $f : \mathbb{N}_{N_1} \to \mathbb{N}$ such that

$$m_n = b_{f(n)} \tag{74}$$

for $n \in \mathbb{N}_{N_1}$. It is easy to show that there is $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ such that

$$\lim_{n\to\infty} \frac{\tau_n}{m_n} = 1. \tag{75}$$

The last limit relation implies that $\{m_n\}_{n\in\mathbb{N}} = \tilde{m} \asymp \tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}}$. This equivalence, (72) and (73) give us

$$\{m_k\}_{k\in\mathbb{N}} \asymp \{b_{f(k)}\}_{k\in\mathbb{N}}.$$

It is clear that $\{(a_{f(k)}, b_{f(k)})\}_{k\in\mathbb{N}} \in \tilde{I}_E^d$. Consequently, by Lemma 40, there is $N_0 \in \mathbb{N}$ such that

$$(l_k, m_k) = (a_{f(k)}, b_{f(k)})$$

for all $k \geq N_0$. Equality (74) follows for all sufficiently large n. Hence $\tilde{A} \in \tilde{I}_E^d$ is universal. Using Lemma 28 we can assume that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are strictly decreasing. To complete the proof it suffices to show that $M(\tilde{A}) < \infty$. As in the proof of Lemma 30 we may consider the closed intervals $[b_{n+1}, a_n]$, $n = 1, 2, ...$, that together with the half-open interval $[b_1, \infty)$ form a disjoint cover of the set $E \setminus \{0\}$,

$$E \setminus \{0\} \subseteq [b_1, \infty) \cup \left(\bigcup_{n\in\mathbb{N}} [b_{n+1}, a_n] \right).$$

We can find a sequence $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ such that

$$\lim_{n\to\infty} \frac{\tau_n}{a_n} = 1 \quad \text{and} \quad \tau_n \in [b_{n+1}, a_n] \tag{76}$$

for every $n \in \mathbb{N}$. Reasoning as in the proof of equality (36) we can see that

$$\{\tau_n\}_{n\in\mathbb{N}} \asymp \{b_{n+1}\}_{n\in\mathbb{N}},$$

i.e., there are positive constants c_1, c_2 such that

$$c_1 b_{n+1} \le \tau_n \le c_2 b_{n+1}.$$

The last inequality and (76) imply

$$\infty > c_2 \ge \limsup_{n\to\infty} \frac{\tau_n}{b_{n+1}} = \limsup_{n\to\infty} \frac{\tau_n}{a_n}\frac{a_n}{b_{n+1}} = \limsup_{n\to\infty} \frac{a_n}{b_{n+1}} = M(\tilde{A}),$$

and so the lemma is proved. □

It is now simple finish the proof of Theorem 36.

Proof of Theorem 36. *The sufficiency.* Suppose for every $\tilde{\tau} = \{\tau_n\}_{n\in\mathbb{N}} \in \tilde{E}^d$ there is $\tilde{h} = \{h_n\}_{n\in\mathbb{N}} \in \tilde{H}(E)$ such that $\tilde{\tau} \asymp \tilde{h}$. Then, by Lemma 37, for every $\tilde{\tau} \in \tilde{E}^d$ there is $\{(a_n, b_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E$ such that $\tilde{\tau} \asymp \tilde{b}$. Consequently, by Lemma 41, the set $\tilde{I}_E^d$ has a universal element $\tilde{L} \in \tilde{I}_E^{sd}$ satisfying the inequality $M(\tilde{L}) < \infty$. By Theorem 27 E is a $\boldsymbol{CSP}(0)$ - set. □

Let A and B be subsets of $\mathbb{R}^+$. We shall write $A \sqsubseteq B$ if there is $t = t(A, B) > 0$ such that

$$A \cap (0, t) \subseteq B \cap (0, t).$$

The next theorem gives a constructive description of the $\boldsymbol{CSP}(0)$ - sets.

Theorem 42. *Let $E \subseteq \mathbb{R}^+$. Then E is a $\boldsymbol{CSP}(0)$ - set if and only if there are $q > 1$ and a strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}, x_n > 0$ for $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0 \tag{77}$$

and

$$E \sqsubseteq W(q) \tag{78}$$

where

$$W(q) := \bigcup_{n\in\mathbb{N}} (q^{-1}x_n, qx_n). \tag{79}$$

PROOF. The theorem is trivial if $0 \notin acE$. Let us consider the case when $0 \in acE$. Suppose that there are $q > 1$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive real numbers such that (77) and (78) hold. Let N_1 and N_2 be natural numbers such that

$$(q^{-1}x_{n+1}, qx_{n+1}) \cap (q^{-1}x_n, qx_n) = \varnothing \tag{80}$$

for $n \geq N_1$

$$E \cap (0,t) \subseteq W(q) \cap (0,t) \tag{81}$$

for $t \leq x_{N_2}$. Then we have

$$(qx_{n+1}, q^{-1}x_n) \subseteq ExtE$$

for $n \geq N_1 \vee N_2$ and write, in this case, (l_n, m_n) for the unique connected component of $ExtE$ satisfying the inclusion

$$(l_n, m_n) \supseteq (qx_{n+1}, q^{-1}x_n). \tag{82}$$

Let $(l_n, m_n) := (l_{N_1\vee N_2}, m_{N_1\vee N_2})$ for $n < N_1 \vee N_2$. We claim that $\tilde{L} = \{(l_n, m_n)\}_{n\in\mathbb{N}}$ is universal. Indeed, (82) implies that

$$\liminf_{n\to\infty} \frac{m_n}{l_n} \geq q^{-2} \liminf_{n\to\infty} \frac{x_n}{x_{n+1}} = \infty.$$

Thus

$$\lim_{n\to\infty} \frac{m_n}{l_n} = \infty,$$

so that $\tilde{L}$ belongs to $\tilde{I}^d_E$. Let $\tilde{A} = \{(a_j, b_j)\}_{j\in\mathbb{N}}$ be an arbitrary element of $\tilde{I}^d_E$. There is $N_3 \in \mathbb{N}$ such that

$$\frac{b_j}{a_j} > q^2 \tag{83}$$

and $b_j < (x_{N_1} \vee x_{N_2})$ for $j \geq N_3$. Let $j \geq N_3$. The interval (a_j, b_j) is a connected component of $ExtE$. Consequently, there is $n \geq (N_1 \vee N_2)$ such that either

$$(a_j, b_j) \supseteq (qx_{n+1}, q^{-1}x_n) \tag{84}$$

or

$$(a_j, b_j) \subseteq (q^{-1}x_n, qx_n). \tag{85}$$

Inclusion (85) implies

$$\frac{b_j}{a_j} \leq \frac{qx_n}{q^{-1}x_n} = q^2,$$

contrary to (83). Hence (84) holds. Since for every nonvoid interval $(s,t) \subseteq ExtE$ there is a unique connected component $(a,b) \supseteq (s,t)$, inclusions (82) and (84) imply the equality $(l_n, m_n) = (a_j, b_j)$. Hence $\tilde{L} \succeq \tilde{A}$ for every $\tilde{A} \in \tilde{I}_E^d$. Thus $\tilde{L}$ is an universal element of $(\tilde{I}_E^d, \preceq)$.

In accordance with Theorem 27 to prove that E is a $\boldsymbol{CSP}(0)$ - set it is sufficient to show

$$M(\tilde{L}) = \limsup_{n\to\infty} \frac{l_n}{m_{n+1}} < \infty. \tag{86}$$

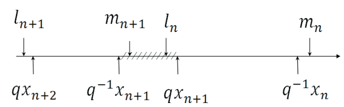


Fig. 3

Since, for all sufficiently large n, $(l_n, m_n) \supseteq (qx_{n+1}, q^{-1}x_n)$, $(l_{n+1}, m_{n+1}) \supseteq (qx_{n+2}, q^{-1}x_{n+1})$ and $l_{n+1} < m_{n+1} < l_n < m_n$, $qx_{n+2} < q^{-1}x_{n+1} < qx_{n+1} < q^{-1}x_n$ (see Fig. 3), we have

$$m_{n+1}, l_n \in [q^{-1}x_{n+1}, qx_{n+1}].$$

Consequently the inequality

$$\frac{l_n}{m_{n+1}} \leq \frac{qx_{n+1}}{q^{-1}x_{n+1}} = q^2$$

holds for all sufficiently large n. Inequality (86) follows.

Now assume that E is a $\boldsymbol{CSP}(0)$ - set. Let $\{(l_n, m_n)\}_{n\in\mathbb{N}} \in \tilde{I}_E^{sd}$ be universal. Without loss of generality, we may suppose that the sequence $\{l_n\}_{n\in\mathbb{N}}$ is strictly decreasing. Define $\{x_n\}_{n\in\mathbb{N}} := \{m_n\}_{n\in\mathbb{N}}$. Using the inequality $m_{n+1} \leq l_n$ we obtain, from the definition of $\tilde{I}_E^d$, that

$$\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \limsup_{n\to\infty} \frac{l_n}{m_n} = 0.$$

Thus

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0.$$

To complete the proof it is sufficient to show that there is $q > 1$ such that (78) holds. As in the proof of Lemma 30, one can easily note that

$$E \setminus \{0\} \subseteq [m_1, \infty) \cup \left(\bigcup_{n\in\mathbb{N}} [m_{n+1}, l_n] \right). \tag{87}$$

By formulas (20) and (21) we have

$$M(\tilde{L}) = \limsup_{n\to\infty} \frac{l_n}{m_{n+1}} < \infty.$$

Let $q \in (M(\tilde{L}), \infty)$. Then there is $N_4 \in \mathbb{N}$ such that $\frac{l_n}{m_{n+1}} < q$ for $n \geq N_4$. It is clear that $q > 1$. Consequently the inequalities $q^{-1}m_{n+1} < m_{n+1} \leq l_n < qm_{n+1}$ hold for $n \geq N_4$. These inequalities yield the inclusion $[m_{n+1}, l_n] \subseteq (q^{-1}m_{n+1}, qm_{n+1})$. The last inclusion and (87) imply

$$E \cap (0,t) \subseteq \left(\bigcup_{n\in\mathbb{N}} (q^{-1}m_n, qm_n) \right) \cap (0,t)$$

for every $t \in (0, m_{N_4+1})$. Relation (78) follows. □

In the case of the closed sets E we may modify Theorem 42 by the following way.

Theorem 43. *Let $E \subseteq \mathbb{R}^+$ be closed and let $0 \in acE$. Then E is a $\boldsymbol{CSP}(0)$ - set if and only if there are $q > 1$ and a strictly decreasing sequence of numbers $x_n > 0$, $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$ and*

$$W(1) \sqsubseteq E \sqsubseteq W(q)$$

where

$$W(a) = \left(\bigcup_{n\in\mathbb{N}} [x_n, ax_n] \right), \ a \in [1, \infty).$$

The last theorem shows that examples 8 and 9 give us, in a sense, "the extremal elements" among the closed $\boldsymbol{CSP}(0)$ - sets with accumulation point 0. The proof of Theorem 43 is similar to the proof of Theorem 42, so we omit it here.

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