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ESSENTIAL DIVERGENCE IN MEASURE OF MULTIPLE ORTHOGONAL FOURIER SERIES

Abstract

In the present paper we prove the following theorem:
Let $\{\varphi_{m,n}(x,y)\}_{m,n=1}^{\infty}$ be an arbitrary uniformly bounded double orthonormal system on $I^2 := [0,1]^2$ such that for some increasing sequence of positive integers $\{N_n\}_{n=1}^{\infty}$ the Lebesgue functions $L_{N_n,N_n}(x,y)$ of the system are bounded below a. e. by $\ln^{1+\epsilon} N_n$, where ϵ is a positive constant. Then there exists a function $g \in L(I^2)$ such that the double Fourier series of g with respect to the system $\{\varphi_{m,n}(x,y)\}_{m,n=1}^{\infty}$ essentially diverges in measure by squares on I^2. The condition is critical in the logarithmic scale in the class of all such systems

1 Introduction

The role of Lebesgue functions in divergence phenomena is crucial. The fundamental inequality of A. M. Olevskii on growth of Lebesgue functions on sets of positive measure for general uniformly bounded ONS (orthonormal systems) is well known [6]-[8].

Theorem 1. *(A. M. Olevskii). Let $\{\varphi_n(x)\}_{n=1}^{\infty}$ be an arbitrary ONS on $I := [0,1]$ that satisfy the condition:*

$$|\varphi_n(x)| \leq M, \; n = 1, 2, 3, \ldots, \; \textit{for a.e. } x \in I \tag{1}$$

and for some positive constant M.

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Then for each $n > 1$ the following inequality holds

$$\mu_1 \left\{ x \in I : \max_{1 \le m \le n} L_m(x) \ge C_0 \log_2 n \right\} \ge \gamma > 0, \tag{2}$$

where C_0 and γ are positive constants that depend only on M, μ_1 denotes the one-dimensional Lebesgue measure and

$$L_m(x) = \int_0^1 \left| \sum_{k=1}^m \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta, \; m = 1, 2, \ldots, \; x \in I,$$

denotes the m-th Lebesgue function of the system.

Now we continue with the following definitions

Definition 1. *Let (X, Σ, ν) be σ-finite measurable space, $E \in \Sigma$ and $\nu(E) > 0$. Let also a sequence of measurable real-valued functions $\{f_n(x)\}_{n=1}^\infty$ be defined and a.e. finite on E. Then we say that the sequence $\{f_n(x)\}_{n=1}^\infty$ is essentially divergent in measure on E if for every $E_1 \subset E$, $E_1 \in \Sigma$ and $\nu(E_1) > 0$, the sequence is divergent in measure (that is does not converge in measure to an a.e. finite and measurable function) on E_1.*

Definition 2. *Let $\{\varphi_n(x)\}_{n=1}^\infty$ be a complete orthonormal system on $I := [0, 1]$ such that $\varphi_1(x) = 1$ on I ; Each function $\varphi_n(x)$ is a bounded function on I; There exists an integer $N > 1$ such that for every positive integer n there exists a number $k(n)$ such that $\varphi_n(Nx) = \varphi_{k(n)}(x)$ and for any $1 \le n_1 < n_2$ we have $k(n_1) < k(n_2)$. Then we say that the system $\{\varphi_n(x)\}_{n=1}^\infty$ is a system of type T.*

Note that the trigonometric system (contracted on I) is a system of type T (with arbitrary integer $N \ge 2$). The Walsh system in Paley's numeration also is a system of type T with $N = 2^l$ where l is an arbitrary positive integer.

A. N. Kolmogorov ([10], p. 267) proved that all trigonometric Fourier series converge in measure on $[0, 2\pi]$. S. V. Konyagin [5] and the author of this paper [4] constructed a double trigonometric Fourier series that diverges in measure by squares on $[0, 2\pi]^2$.

Later we proved [3] the following

Theorem 2. *(R. Getsadze) Let $\{\varphi_k(x)\}_{k=1}^\infty$ be an arbitrary uniformly bounded orthonormal system (ONS) on I. Then there exists an integrable function on I^2 whose Fourier series with respect to the product system $\{\varphi_k(x)\varphi_l(y)\}_{k,l=1}^\infty$ diverges in measure by squares on I^2.*

The following theorem was proved in [1], p. 27,

Theorem 3. *(Dyachenko, M. I.; Kazaryan, K. S.; Sifuéntes, P.)* *Let $\{\varphi_m(x)\}_{m=1}^{\infty}$ be a uniformly bounded ONS on I that is a system of type T. Suppose that there exists a function $g_0 \in L(I^2)$ such that the Fourier series of g_0 with respect to the product system $\{\varphi_m(x)\varphi_n(y)\}_{m,n=1}^{\infty}$ unboundedly diverges in measure by squares on I^2. Then there exists a function $f_0 \in L(I^2)$ such that the Fourier series of f_0 with respect to the product system $\{\varphi_m(x)\varphi_n(y)\}_{m,n=1}^{\infty}$ essentially diverges in measure by squares on I^2.*

Let μ_n denote the n-dimensional Lebesgue measure, where $n = 1, 2, \dots$. Let

$$L_{m,p}(x,y) = \int_0^1 \int_0^1 \left| \sum_{k=1}^{m} \sum_{l=1}^{p} \varphi_{k,l}(x,y)\varphi_{k,l}(\vartheta,\eta) \right| d\vartheta d\eta, \qquad (3)$$
$$m, p = 1, 2, \dots,\ (x,y) \in I^2,$$

denote the (m,p)-th Lebesgue function of the double ONS $\{\varphi_{i,j}(x,y)\}_{i,j=1}^{\infty}$ on I^2 and let

$$S_{m,p}(f,x,y) := \sum_{i=1}^{m} \sum_{j=1}^{p} a_{i,j}(f)\varphi_{i,j}(x,y) \qquad (4)$$

denote the (m,p)-th rectangular partial sum of the Fourier series of an integrable on I^2 function f with respect to the system.

In the present article we prove the following

Theorem 4. *Let $\{\varphi_{m,n}(x,y)\}_{m,n=1}^{\infty}$ be an arbitrary double ONS on I^2 that satisfies the condition:*

$$|\varphi_{m,n}(x,y)| \leq M,\ m,n = 1,2,3,\dots, \quad \textit{for a.e.}\,(x,y) \in I^2 \qquad (5)$$

and for some positive constant M.

Suppose also that there exists an increasing sequence of positive integers $\{N_n\}_{n=1}^{\infty}$ such that for each $n = 1, 2, \dots$ the following inequality holds

$$L_{N_n,N_n}(x,y) \geq \ln^{1+\epsilon} N_n, \quad \textit{for a.e. } (x,y) \in I^2 \qquad (6)$$

and for some positive constant ϵ. Then there exists a function $g \in L(I^2)$ such that the Fourier series of g with respect to the system $\{\varphi_{m,n}(x,y)\}_{m,n=1}^{\infty}$ essentially diverges in measure by squares on I^2.

It follows from the inequality of A. M. Olevskii (see (1), (2)) that all product systems $\{\psi_m(x)\psi_n(y)\}_{m,n=1}^{\infty}$, where $\{\psi_m(x)\}_{m=1}^{\infty}$ is an arbitrary rearrangement of the trigonometric system (contracted on I) or of the Walsh-Paley system, satisfy all conditions and, consequently, the conclusion of Theorem 4. Indeed, for such systems $\{\psi_m(x)\}_{m=1}^{\infty}$ "the Lebesgue functions" are "Lebesgue constants," that is for all positive integers m and a. e. $x \in I$ we have

$$\int_0^1 \left|\sum_{i=1}^{m} \psi_i(x)\psi_i(\vartheta)\right| d\vartheta = \int_0^1 \left|\sum_{i=1}^{m} \psi_i(\vartheta)\right| d\vartheta.$$

On the other hand according to the inequality of A. M. Olevskii (see (1), (2))

$$\max_{1\le m\le n} \int_0^1 \left|\sum_{i=1}^{m} \psi_i(x)\psi_i(\vartheta)\right| d\vartheta \ge C_1 \ln n$$

for any positive integer n on a set E_n, $\mu_1 E_n \ge \gamma > 0$, where γ and C_1 are positive constants.

Consequently, for each positive integer n there is a positive integer m_n, independent of x, $1 \le m_n \le n$, such that

$$\int_0^1 \left|\sum_{i=1}^{m_n} \psi_i(\vartheta)\right| d\vartheta \ge C_1 \ln n.$$

Now it is clear that for all positive integers n and a.e. $(x, y) \in I^2$

$$\int_0^1 \int_0^1 \left|\sum_{i=1}^{m_n}\sum_{j=1}^{m_n} \psi_i(x)\psi_i(\vartheta)\psi_j(y)\psi_j(\eta)\right| d\vartheta d\eta \ge C_1^2 \ln^2 n \ge C_1^2 \ln^2 m_n.$$

Remark. The order of the lower bound in Theorem 4 is exact in the logarithmic scale for the class of all uniformly bounded double ONS $\{\varphi_{m,n}(x,y)\}_{m,n=1}^{\infty}$. Indeed, let $\{t_m(x)\}_{m=1}^{\infty}$ be the trigonometric system (contracted on I).

Let σ be a measurable mapping (see [9], p. 94) from I onto I^2 such that

1.) If $x \ne y$ then $\sigma(x) \ne \sigma(y)$,

2.) if $E \subset I$ is a measurable set, then $\sigma(E) \subset I^2$ is also measurable and $\mu_1(E) = \mu_2\sigma(E)$.

Let Z_+ be the set of all positive integers. Let also σ_0 be a one-to-one mapping from Z_+^2 onto Z_+ defined by

$$\sigma_0(m,j) := \begin{cases} (j-1)^2 + m, & if 1 \le m \le j \\ m^2 + 1 - j, & if 1 \le j \le m-1. \end{cases}$$

It is not difficult to see that

$$\{\sigma_0(m,j) : m, j = 1, 2 \dots, n\} = \{1, 2, 3, \dots, n^2\}$$

for any $n \in Z_+$.

We introduce an ONS $\{T_n(x,y)\}_{n=1}^{\infty}$ on I^2 by

$$T_n(x,y) := t_n(\sigma^{-1}(x,y))$$

for every positive integer n.

Finally, we introduce a double ONS $\{\Phi_{m,n}(x,y)\}_{m,n=1}^{\infty}$ on I^2 by

$$\Phi_{m,n}(x,y) := T_{\sigma_0(m,n)}(x,y)$$

for every ordered couple of positive integers m and n. It is obvious that for every positive integer n the following two sets are equal

$$\{T_1(x,y), T_2(x,y), \dots, T_{n^2}(x,y)\} = \{\Phi_{m,p}(x,y) : m,p = 1,2,\dots,n\}.$$

It is clear that the system $\{\Phi_{m,n}(x,y)\}_{m,n=1}^{\infty}$ has the following properties:

1.) The sequence of square Lebesgue functions of the system has logarithmic growth a.e.

and

2.) all Fourier series with respect to this system converge in measure by squares on I^2.

It is not difficult to see from the proofs of Theorems 1-3 in [2] that the following statement is true (see (3)-(6))

Lemma 1. *Let $\{\varphi_{m,n}(x,y)\}_{m,n=1}^{\infty}$ be an arbitrary ONS on I^2 that satisfies the conditions (5) and (6). Let $E \subset I^2$ be an arbitrary Lebesgue measurable set, $\mu_2 E > 0$. Then there exist an increasing sequence of positive integers $\{m_p = m_p(E)\}_{p=1}^{\infty}$ and a function $h = h(E) \in L(I^2)$, $\| h \|_{L(I^2)} \leq 1$, such that for each $p = 1,2,\dots$ we have*

$$\mu_2\{(x,y) \in E : | S_{m_p,m_p}(h,x,y) | \geq (\ln\ln m_p)^{\frac{\epsilon}{5}}\} \geq \frac{1}{36}\mu_2 E. \tag{7}$$

By a dyadic interval in I we shall mean an interval of the form

$$\Delta_n^{(k)} := [k2^{-n}, (k+1)2^{-n}), (k = 0,1,\dots,2^n-1, n = 0,1,2,\dots). \tag{8}$$

Let n and $0 \leq i,j \leq 2^n - 1$, be nonnegative integers. Set

$$\Delta_n^{(i,j)} := \Delta_n^{(i)} \times \Delta_n^{(j)}. \tag{9}$$

2 Proof of Theorem 4

Let S_n denote a finite one-dimensional sequence of all intervals $\Delta_k^{(i,j)}$ where $i, j, = 0, 1, 2, \ldots, 2^k - 1,\ k = 0, 1, 2, \ldots, n$. According to the following scheme

$$S_0, S_1, S_2, \ldots, S_n, \ldots,$$

we obtain a sequence of sets

$$E_1, E_2, \ldots, E_k, \ldots, \tag{10}$$

that has the following properties:

i.) For each positive integer k there exists a triple of non negative integers (n, i, j) where $0 \leq i, j \leq 2^n - 1$, such that

$$E_k = \Delta_n^{(i,j)} \tag{11}$$

and

ii.) for each triple of non negative integers (n, i, j), where $i, j = 0, 1, 2, \ldots, 2^n - 1$, there exists an increasing sequence of positive integers $\{r_p = r_p(n, i, j)\}_{p=1}^{\infty}$ such that

$$E_{r_p} = \Delta_n^{(i,j)} \tag{12}$$

for every $p = 1, 2, \ldots$.

Now for the sequence of sets in (10) we will construct by induction an increasing sequence of positive integers $\{l_j\}_{j=1}^{\infty}$, sequence of positive numbers $\{\delta_j\}_{j=2}^{\infty}$ and a sequence of functions $\{F_j(x, y)\}_{j=1}^{\infty}$ from $L^{\infty}(I^2)$ such that for all $j = 1, 2, \ldots$ we have

$$\| F_j \|_{L(I^2)} \leq 1, \tag{13}$$

$$\frac{1}{(\ln\ln l_{j+1})^{\frac{\epsilon}{10}}} \leq \frac{1}{2}\frac{1}{(\ln\ln l_j)^{\frac{\epsilon}{10}}}, \tag{14}$$

$$l_{j+1} > l_j > e^2, \tag{15}$$

$$\mu_2\{(x, y) \in E_j : | S_{l_j, l_j}(F_j, x, y) | \geq \frac{1}{2}(\ln\ln l_j)^{\frac{\epsilon}{5}}\} \geq \frac{1}{36}\mu_2 E_j, \tag{16}$$

$$l_j^2 M^2 \frac{2}{(\ln\ln l_{j+1})^{\frac{\epsilon}{10}}} < 1, \tag{17}$$

$$\mu_2\{(x,y)\in I^2 :| S_{l_{j+1},l_{j+1}}(\alpha_j,x,y)|\geq \delta_{j+1}\}\leq \frac{1}{108}\mu_2 E_{j+1}, \tag{18}$$

where

$$\alpha_j(x.y)=\sum_{i=1}^{j}\frac{1}{(\ln\ln l_i)^{\frac{\epsilon}{10}}}F_i(x,y) \tag{19}$$

and

$$\frac{1}{6}(\ln\ln l_{j+1})^{\frac{\epsilon}{10}} > \max(1,\delta_{j+1}). \tag{20}$$

The constructions of the integer l_1, the function F_1 and the number δ_2 are contained by the description of the general $k+1$-st step of the induction.

Let the numbers $\{l_j\}_{j=1}^k$, $\{\delta_j\}_{j=2}^k$ and functions $\{F_j(x,y)\}_{j=1}^k$ be already defined so that they satisfy (13)-(20).

According to the inequalities of Chebyshev and Bessel we obtain that for all positive numbers δ and for all positive integers n we have (see (4), (19))

$$\mu_2\{(x,y)\in I^2 :| S_{n,n}(\alpha_k,x,y)|\geq \delta\}\leq \frac{||\ \alpha_k\ ||^2_{L^2(I^2)}}{\delta^2}$$

and, consequently, one can choose a positive number δ_{k+1} such that for all positive integers n we have

$$\mu_2\{(x,y)\in I^2 :| S_{n,n}(\alpha_k,x,y)|\geq \delta_{k+1}\}\leq \frac{1}{108}\mu_2 E_{k+1}. \tag{21}$$

Now we apply Lemma 1 for the set $E:=E_{k+1}$ (see (10)) and obtain an increasing sequence of integers $\{n_p^{(k+1)}=n_p^{(k+1)}(E_{k+1})\}_{p=1}^\infty$ and a function $h_{k+1}:=h(E_{k+1})\in L(I^2)$, $||\ h_{k+1}\ ||_{L(I^2)}\leq 1$, such that for each $p=1,2,\ldots$ we have (see (7))

$$\mu_2\{(x,y)\in E_{k+1} :| S_{n_p^{(k+1)},n_p^{(k+1)}}(h_{k+1},x,y)|\geq (\ln\ln n_p^{(k+1)})^{\frac{\epsilon}{5}}\}\geq \frac{1}{36}\mu_2 E_{k+1}. \tag{22}$$

Now we choose the integer l_{k+1} as one of the numbers in the sequence $\{n_p^{(k+1)}\}_{p=1}^\infty$ large enough so that the following inequalities are satisfied

$$\frac{1}{(\ln\ln l_{k+1})^{\frac{\epsilon}{10}}}\leq \frac{1}{2}\frac{1}{(\ln\ln l_k)^{\frac{\epsilon}{10}}}, \tag{23}$$

$$l_{k+1} > l_k, \tag{24}$$

$$\mu_2\{(x,y)\in E_{k+1} :\mid S_{l_{k+1},l_{k+1}}(h_{k+1},x,y)\mid\geq (\ln\ln l_{k+1})^{\frac{\epsilon}{5}}\}\geq \frac{1}{36}\mu_2 E_{k+1}, \quad (25)$$

$$l_k^2 M^2 \frac{2}{(\ln\ln l_{k+1})^{\frac{\epsilon}{10}}} < 1, \quad (26)$$

and

$$\frac{1}{6}(\ln\ln l_{k+1})^{\frac{\epsilon}{10}} > \max(\delta_{k+1},1). \quad (27)$$

It is clear now (see (5), (25)) that we can find a function F_{k+1} such that $F_{k+1}\in L^\infty(I^2)$, $\| F_{k+1}\|_{L(I^2)}\leq 1$ and

$$\mu_2\{(x,y)\in E_{k+1} :\mid S_{l_{k+1},l_{k+1}}(F_{k+1},x,y)\mid\geq \frac{1}{2}(\ln\ln l_{k+1})^{\frac{\epsilon}{5}}\}\geq \frac{1}{36}\mu_2 E_{k+1}.$$

The construction of sequences $\{l_j\}_{j=1}^\infty$, $\{\delta_j\}_{j=1}^\infty$ $\{F_j(x,y)\}_{j=1}^\infty$ is now completed (see (13)-(27)).

Taking account of (18), (20) we obtain for all $k=1,2,\dots$

$$\mu_2\{(x,y)\in I^2 :\mid S_{l_{k+1},l_{k+1}}(\alpha_k,x,y)\mid\geq \frac{1}{6}(\ln\ln l_{k+1})^{\frac{\epsilon}{10}}\}\leq \frac{1}{108}\mu_2 E_{k+1}. \quad (28)$$

Introduce the following functions defined on I^2 by

$$g(x,y) := \sum_{i=1}^\infty \frac{1}{(\ln\ln l_i)^{\frac{\epsilon}{10}}} F_i(x,y) \quad (29)$$

and

$$\beta_k(x,y) := \sum_{i=k+1}^\infty \frac{1}{(\ln\ln l_i)^{\frac{\epsilon}{10}}} F_i(x,y). \quad (30)$$

It is obvious that then (see (23)) for any $k=1,2,\dots$

$$\int_0^1\int_0^1 \mid \beta_k(x,y)\mid dxdy \leq \sum_{i=k+1}^\infty \frac{1}{(\ln\ln l_i)^{\frac{\epsilon}{10}}} \leq \frac{2}{(\ln\ln l_{k+1})^{\frac{\epsilon}{10}}} \quad (31)$$

and

$$\int_0^1\int_0^1 \mid g(x,y)\mid dxdy \leq \sum_{i=1}^\infty \frac{1}{(\ln\ln l_i)^{\frac{\epsilon}{10}}} < \infty. \quad (32)$$

Now Let $E_0\subset I^2$ be an arbitrary Lebesgue measurable set, $\mu_2 E_0>0$. It is clear that there exist a triple of non negative integers (n_0,i_0,j_0), where

$0 \leq i_0, j_0, \leq 2^n - 1$, and an increasing sequence of positive integers $\{k_q\}_{q=1}^{\infty}$ such that (see (8), (9), (12))

$$\mu_2\{E_0 \cap \Delta_{n_0}^{(i_0,j_0)}\} \geq \frac{215}{216}\mu_2\Delta_{n_0}^{(i_0,j_0)} \tag{33}$$

and

$$E_{k_q} = \Delta_{n_0}^{(i_0,j_0)} \tag{34}$$

for all $q = 1, 2, \dots$.

From (19), (29), (30) we have for all $k = 2, 3, \dots$

$$g(x,y) = \alpha_{k-1}(x,y) + \frac{1}{(\ln\ln l_k)^{\frac{\epsilon}{10}}}F_k(x,y) + \beta_k(x,y)$$

It is obvious that (see (4)) for all $q = 1, 2, 3, \dots$

$$\mu_2\{(x,y) \in E_{k_q} :\mid S_{l_{k_q},l_{k_q}}(\frac{1}{(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}}F_{k_q}, x, y) \mid\geq \frac{1}{2}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\}$$

$$\leq \mu_2\{(x,y) \in E_{k_q} :\mid S_{l_{k_q},l_{k_q}}(\alpha_{k_q-1}, x, y) \mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\}$$

$$+\mu_2\{(x,y) \in E_{k_q} :\mid S_{l_{k_q},l_{k_q}}(\beta_{k_q}, x, y) \mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\}$$

$$+\mu_2\{(x,y) \in E_{k_q} :\mid S_{l_{k_q},l_{k_q}}(g, x, y) \mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\}.$$

Using (4), (17), (31) we obtain that for all $(x,y) \in I^2$ and any $q = 1, 2, \dots$

$$\mid S_{l_{k_q},l_{k_q}}(\beta_{k_q}, x, y) \mid\leq l_{k_q}^2 M^2 \frac{2}{(\ln\ln l_{k_q+1})^{\frac{\epsilon}{10}}} < 1.$$

Thus (see (20)),

$$\mu_2\{(x,y) \in E_{k_q} :\mid S_{l_{k_q},l_{k_q}}(\beta_{k_q}, x, y) \mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\} = 0.$$

According to (28) we have for any $q = 1, 2, \dots$

$$\mu_2\{(x,y) \in E_{k_q} :\mid S_{l_{k_q},l_{k_q}}(\alpha_{k_q-1}, x, y) \mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\} \leq \frac{1}{108}\mu_2 E_{k_q}.$$

Consequently, (see (16)) we conclude that for any $q = 1, 2, \dots$

$$\mu_2\{(x,y) \in E_{k_q} :\mid S_{l_{k_q},l_{k_q}}(g, x, y) \mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\} \geq \frac{1}{54}\mu_2 E_{k_q}.$$

That is (see (34)),

$$\mu_2\{(x,y)\in\Delta_{n_0}^{(i_0,j_0)} :\mid S_{l_{k_q},l_{k_q}}(g,x,y)\mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\}\geq \frac{1}{54}\mu_2\Delta_{n_0}^{(i_0,j_0)},$$

and, consequently, (see (33)), for any $q=1,2,\dots$

$$\mu_2\{(x,y)\in E_0\cap\Delta_{n_0}^{(i_0,j_0)} :\mid S_{l_{k_q},l_{k_q}}(g,x,y)\mid\geq \frac{1}{6}(\ln\ln l_{k_q})^{\frac{\epsilon}{10}}\}\geq \frac{3}{216}\mu_2\Delta_{n_0}^{(i_0,j_0)}.$$

Theorem 4 (see (32)) is proved.

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