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MULTIFRACTAL ANALYSIS OF SOME MULTIPLE ERGODIC AVERAGES FOR THE SYSTEMS WITH NON-CONSTANT LYAPUNOV EXPONENTS

Abstract

We study certain multiple ergodic averages of an iterated functions system generated by two contractions on the unit interval. By using the dynamical coding $\{0,1\}^{\mathbb{N}}$ of the attractor, we compute the Hausdorff dimension of the set of points with a given frequency of the pattern 11 in positions $k, 2k$.

1 Introduction and statement of results

Initiated by the papers of Kifer [10] and Fan, Liao, and Ma [3], the study of the multiple ergodic average from the point view of multifractal analysis have attracted much attention. Major achievements have been made by Fan, Kenyon, Peres, Schmeling, Seuret, Solomyak, Wu et al. ([8, 5, 9, 12, 11, 6, 7, 13]). For a short history, we refer the readers to the paper of Peres and Solomyak [11].

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Considered the symbolic space $\Sigma = \{0,1\}^{\mathbb{N}}$ with the metric $d(x,y) = 2^{-\min\{n:\, x_n \neq y_n\}}$. In [3], the authors proposed to calculate the Hausdorff dimension spectrum of level sets of multiple ergodic averages. Among others, they asked the Hausdorff dimension of

$$A_\alpha := \left\{ (\omega_k)_1^\infty \in \Sigma : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k \omega_{2k} = \alpha \right\} \qquad (\alpha \in [0,1]). \tag{1.1}$$

As a first step to solve the question, they also suggested to study a subset of A_0:

$$A := \left\{ (\omega_k)_1^\infty \in \Sigma : \omega_k \omega_{2k} = 0 \quad \text{for all } k \geq 1 \right\}. \tag{1.2}$$

The Hausdorff dimension of A was later given by Kenyon, Peres, and Solomyak [9].

Theorem 1.1 (Kenyon-Peres-Solomyak)**.** *We have*

$$\dim_H A = -\log(1-p),$$

where $p \in [0,1]$ *is the unique solution of the equation*

$$p^2 = (1-p)^3.$$

Enlightened by the idea of [9], the question about A_α was finally answered by Peres and Solomyak [11], and independently by Fan, Schmeling, and Wu [6].

Theorem 1.2 (Peres-Solomyak, Fan-Schmeling-Wu)**.** *For any* $\alpha \in [0,1]$*, we have*

$$\dim_H A_\alpha = -\log(1-p) - \frac{\alpha}{2} \log \frac{q(1-p)}{p(1-q)},$$

where $(p,q) \in [0,1]^2$ *is the unique solution of the system*

$$\begin{cases} p^2(1-q) = (1-p)^3, \\ 2pq = \alpha(2+p-q). \end{cases}$$

We remark that a more general result on the Hausdorff dimension spectrum of level sets of multiple ergodic averages for a function depending only on one coordinate in Σ has been obtained in [6].

However, since the Lyapunov exponent is constant for the shift transformation on the symbolic space, what is obtained is in fact the entropy spectrum, i.e., the entropy (for Bowen's definition see [1]) of level sets of the multiple ergodic averages.

We also remark that the present work is then generalized by Fan, Liao, and Wu [4] by using nonlinear transfer equations introduced in [6].

Consider a piecewise linear map T on the unit interval with two branches. Let $I_0, I_1 \subset [0,1]$ be two closed intervals intersecting at most at one point. Let us also assume that $0 \in I_0$ and $1 \in I_1$. Suppose that on I_0, I_1, the map T is bijective and linear onto $[0,1]$ with slopes $e^{-\lambda_0} = 1/|I_0|$ and $e^{-\lambda_1} = 1/|I_1|$ $(\lambda_0, \lambda_1 > 0)$ correspondingly. Let

$$J_T := \cap_{n=1}^{\infty} T^{-n}[0,1].$$

Then (J_T, T) becomes a dynamical system. Similarly to [3, 11, 6], we would like to study the following sets

$$L := \left\{ x \in [0,1] : 1_{I_1}(T^k x) 1_{I_1}(T^{2k} x) = 0, \quad \text{for all } k \right\},$$

and

$$L_\alpha := \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{I_1}(T^k x) 1_{I_1}(T^{2k} x) = \alpha \right\} \quad (\alpha \in [0,1]).$$

For convenience, we will study the corresponding iterated function system and its natural coding. Let $\{f_0, f_1\}$ be the iterated function system on $[0,1]$ given by

$$f_0(x) = e^{-\lambda_0} x, \quad f_1(x) = e^{-\lambda_1} x + 1 - e^{-\lambda_1}, \qquad (\lambda_0, \lambda_1 > 0)$$

satisfying the open set condition, i.e., $e^{-\lambda_0} + e^{-\lambda_1} \leq 1$. It has the usual symbolic description by $\Sigma = \{0,1\}^{\mathbb{N}}$ with a natural projection

$$\pi(\omega) = \lim_{n\to\infty} f_{\omega_1} \circ f_{\omega_2} \circ \ldots \circ f_{\omega_n}(0).$$

Let us define in Σ the subsets A and A_α by (1.1), (1.2). Up to a countable set, the sets L, L_α can be written as

$$L = \pi(A), \quad L_\alpha = \pi(A_\alpha).$$

We remark that if $\lambda_0 = \lambda_1 = \lambda$, i.e., the Lyapunov exponent is constant, then

$$\dim_H L = \frac{\dim_H A}{\lambda/\log 2}, \quad \dim_H L_\alpha = \frac{\dim_H A_\alpha}{\lambda/\log 2}.$$

Furthermore, if $\lambda_0 = \lambda_1 = \log 2$, then $\pi(\Sigma) = [0,1]$, and the Hausdorff dimensions of L, L_α are the same as those of A, A_α. Our goal is to calculate the Hausdorff dimension of sets L and L_α for $\lambda_0 \neq \lambda_1$.

Our results are as follows:

Theorem 1.3. *We have*

$$\dim_H L = \dim_H L_0 = -\frac{\log(1-p)}{\lambda_0},$$

where $p \in [0,1]$ *is the unique solution of the equation*

$$p^{2\lambda_0} = (1-p)^{2\lambda_1+\lambda_0}.$$

For any $\alpha \in (0,1]$, *we have*

$$\dim_H L_\alpha = \frac{\alpha \log \frac{p(1-q)}{(1-p)q} - 2\log(1-p)}{2\lambda_0},$$

where $(p,q) \in [0,1]^2$ *is the unique solution of the system*

$$\begin{cases} \alpha(\lambda_1 - \lambda_0) \log \dfrac{p(1-q)}{(1-p)q} + \lambda_0 \log \dfrac{p^2(1-q)}{1-p} - 2\lambda_1 \log(1-p) = 0, \\ 2pq = \alpha(2+p-q). \end{cases}$$

The paper is strongly related to [11], we mostly repeat the calculations there in a more complicated situation. For the missing details, in particular for [11, Lemma 2] we refer the reader there. In the following two sections we calculate the lower bound: in Section 2 we introduce a family of measures and then we find the measure in this family that is supported on the set L_α and has maximal Hausdorff dimension, in Section 3 we find a formula for this dimension. In Section 4 we check that this formula is also the upper bound for the dimension of L_α.

2 Telescopic product measures

The same measures that were used to calculate the entropy spectrum (see [11]) will be useful for the Hausdorff spectrum as well.

Let us start from the multiplicative golden shift case. Given $p \in [0,1]$, let μ_p be a probability measure on S given by

– if k is odd then $\omega_k = 1$ with probability p,

– if k is even and $\omega_{k/2} = 0$ then $\omega_k = 1$ with probability p,

– if k is even and $\omega_{k/2} = 1$ then $\omega_k = 0$.

More precisely, let $(p_0, p_1) := (1-p, p)$ and let

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} := \begin{pmatrix} 1-p & p \\ 1 & 0 \end{pmatrix}.$$

Then the measure μ_p of a cylinder is given by

$$\mu_p([\omega_1 \cdots \omega_n]) = \prod_{k=1}^{\lceil n/2 \rceil} p_{\omega_{2k-1}} \cdot \prod_{k=1}^{\lfloor n/2 \rfloor} p_{\omega_k \omega_{2k}},$$

where $\lceil \cdot \rceil, \lfloor \cdot \rfloor$ denote the ceiling function and the integer part function correspondingly.

Let $\nu_p = \pi_* \mu_p$. The Hausdorff dimension of L will turn out to be the supremum of Hausdorff dimensions of ν_p.

Similarly, to deal with the spectrum of the sets L_α we will define a family of probabilistic measures of two parameters. Given $p, q \in [0,1]$ we define a measure $\mu_{p,q}$ on Σ as

– if k is odd then $\omega_k = 1$ with probability p,

– if k is even and $\omega_{k/2} = 0$ then $\omega_k = 1$ with probability p,

– if k is even and $\omega_{k/2} = 1$ then $\omega_k = 1$ with probability q.

Similarly, if we let $(p_0, p_1) := (1-p, p)$ and let

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} := \begin{pmatrix} 1-p & p \\ 1-q & q \end{pmatrix},$$

then we have

$$\mu_{p,q}([\omega_1 \cdots \omega_n]) = \prod_{k=1}^{\lceil n/2 \rceil} p_{\omega_{2k-1}} \cdot \prod_{k=1}^{\lfloor n/2 \rfloor} p_{\omega_k \omega_{2k}}.$$

Once again, let $\nu_{p,q} = \pi_* \mu_{p,q}$. Please note that this notation is a little bit different from that in [11]. Note also that $\mu_p = \mu_{p,0}$.

Lemma 2.1. *We have*

$$\mu_{p,q}(S_\alpha) = 1$$

for

$$\alpha = \frac{2pq}{2+p-q}.$$

PROOF. This lemma is proven in [11, Lemma 3]. However, we will need this proof as a starting point for the proof of Lemma 2.2.

Denote

$$x_n(\omega) = \frac{2}{n} \sum_{k=n/2+1}^{n} \omega_k.$$

For a $\mu_{p,q}$-typical ω the Law of Large Numbers implies

$$x_{2n}(\omega) = \frac{1}{2}p + \frac{x_n(\omega)}{2}q + \frac{1 - x_n(\omega)}{2}p + o(1).$$

Hence, as $k \to \infty$,

$$x_{2^k n}(\omega) \to \frac{2p}{2+p-q}.$$

By [11, Lemma 5], this implies that $\mu_{p,q}$-almost surely

$$\lim_{n\to\infty} x_n(\omega) = \frac{2p}{2+p-q}. \tag{2.1}$$

Then, for $\mu_{p,q}$-a.e. ω,

$$\frac{2}{n} \sum_{k=n/2+1}^{n} \omega_k \omega_{2k} = x_n(\omega)(q + o(1)) \to \frac{2pq}{2+p-q}.$$

Thus the assertion follows. □

Let us denote

$$H(p) = -p\log p - (1-p)\log(1-p)$$

with the convention $H(0) = H(1) = 0$.

Lemma 2.2. *We have*

$$\dim_H \nu_p = \frac{2H(p)}{2p\lambda_1 + (2-p)\lambda_0},$$

and

$$\dim_H \nu_{p,q} = \frac{(2-q)H(p) + pH(q)}{2p\lambda_1 + (2-p-q)\lambda_0}.$$

PROOF. As $\nu_p = \nu_{p,0}$, it is enough to prove the second part of the assertion. For $\omega \in \Sigma$ denote

$$C_n(\omega) = \{\tau \in \Sigma; \tau_k = \omega_k \ \forall k \leq n\}.$$

Let

$$h_n(\omega) := \log \mu_{p,q}(C_{2n}(\omega)) - \log \mu_{p,q}(C_n(\omega))$$

and

$$\lambda_n(\omega) := \log \operatorname{diam} \pi(C_{2n}(\omega)) - \log \operatorname{diam} \pi(C_n(\omega)).$$

We are going to prove that $h_n(\omega)/\lambda_n(\omega)$ converges to some limit. As $\lambda_n(\omega)/n$ is bounded from below and from above (by λ_1 and λ_2), [11, Lemma 5] will then imply that $\log \nu_{p,q}(\pi(C_n(\omega)))/\log \operatorname{diam}\pi(C_n(\omega))$ converges to the same limit.

By the Law of Large Numbers, for $\mu_{p,q}$-typical ω and for big enough n we have

$$\frac{2}{n} h_n(\omega) = (2 - x_n(\omega))(p \log p + (1-p) \log p) \\ + x_n(\omega)(q \log q + (1-q) \log(1-q)) + o(1)$$

and

$$\frac{2}{n} \lambda_n(\omega) = (2 - x_n(\omega))(-p\lambda_1 - (1-p)\lambda_0) + x_n(\omega)(-q\lambda_1 - (1-q)\lambda_0) + o(1).$$

Thus, by (2.1)

$$\frac{h_n(\omega)}{\lambda_n(\omega)} \to \frac{(2-q)H(p) + pH(q)}{2p\lambda_1 + (2-p-q)\lambda_0} \qquad \mu_{p,q} - a.e.$$

Hence, for $\mu_{p,q}$-a.e. ω we have

$$\lim_{n\to\infty} \frac{\log \nu_{p,q}(\pi(C_n(\omega)))}{\log \operatorname{diam}\pi(C_n(\omega))} = \frac{(2-q)H(p) + pH(q)}{2p\lambda_1 + (2-p-q)\lambda_0}.$$

□

We will denote

$$\gamma_\alpha = \left\{ (p,q) \in [0,1]^2 : \alpha = \frac{2pq}{2+p-q} \right\}.$$

Lemma 2.3. *The maximal Hausdorff dimension among measures ν_p is achieved for p satisfying*

$$p^{2\lambda_0} = (1-p)^{2\lambda_1+\lambda_0}. \tag{2.2}$$

For $\alpha \in (0,1)$, the maximal Hausdorff dimension among measures $\{\nu_{p,q} : (p,q) \in \gamma_\alpha\}$ is achieved for (p,q) satisfying

$$\alpha(\lambda_1 - \lambda_0)\log\frac{p(1-q)}{(1-p)q} + \lambda_0 \log\frac{p^2(1-q)}{1-p} - 2\lambda_1\log(1-p) = 0. \tag{2.3}$$

Such (p,q) is unique in γ_α and is always in $(0,1)^2$.

PROOF. Let us start from the second part of assertion. We need to find the maximum of the function

$$D(p,q) = \frac{(2-q)H(p) + pH(q)}{2p\lambda_1 + (2-p-q)\lambda_0}$$

over the curve γ_α. For $\alpha > 0$ this curve's endpoints are $(1, 3\alpha/(2+\alpha))$ and $(\alpha/(2-\alpha), 1)$. Moreover, we have

$$\mathrm{d}\alpha = \frac{2}{(2+p-q)^2}(q(2-q)\mathrm{d}p + p(2+p)\mathrm{d}q).$$

Hence, we need to solve the equation

$$p(2+p)\frac{\partial D}{\partial p} - q(2-q)\frac{\partial D}{\partial q} = 0.$$

After expanding the left hand side and collecting the terms, it turns out that it is divisible by $p(2-q)$. We get

$$\begin{aligned}
&(2pq\lambda_1 + (4+2p-2q-2pq)\lambda_0)\cdot\log p \\
&+((-4-2p+2q-2pq)\lambda_1 + (-2-p+q+2pq)\lambda_0)\cdot\log(1-p) \\
&+(-2pq\lambda_1 + 2pq\lambda_0)\cdot\log q \\
&+(2pq\lambda_1 + (2+p-q-2pq)\lambda_0)\cdot\log(1-q) = 0.
\end{aligned} \tag{2.4}$$

It will be convenient to use $\beta = 2/\alpha$. As $(p,q) \in \gamma_\alpha$, we have

$$2 + p - q = \beta pq.$$

Substituting this into (2.4), we get

$$\begin{aligned}
&(2\lambda_1 + (2\beta-2)\lambda_0)\log p + ((-2\beta-2)\lambda_1 + (-\beta+2)\lambda_0)\log(1-p) \\
&+(-2\lambda_1 + 2\lambda_0)\log q + (2\lambda_1 + (\beta-2)\lambda_0)\log(1-q) = 0
\end{aligned} \tag{2.5}$$

and (2.3) follows.

To get the first part of assertion it is enough to remove all terms with q and substitute $\alpha = 0$ into (2.3).

What remains is the third part of the assertion. Denoting by $F(p,q)$ the left hand side of (2.5), we have

$$F(1, 3\alpha/(2+\alpha)) = \infty$$

and

$$F(\alpha/(2-\alpha), 1) = -\infty.$$

We will check that F restricted to γ_α is strictly monotone. We have

$$p(p+2)\frac{\partial F}{\partial p} - q(2-q)\frac{\partial F}{\partial q} = \lambda_0((2\beta-2)(p+2) - 2(2-q)) + \text{spt},$$

where spt stands for some positive terms (in particular, all the terms with λ_1 are positive). However, as

$$(2\beta-2)(p+2) - 2(2-q) = 2p + 2q + 2(\beta-2)(p+2) > 0,$$

the coefficient for λ_0 is also positive. Hence, F restricted to γ_α indeed has no extrema, so it must have only one zero. $\square$

Remark. When $\alpha = 0$, the curve γ_0 degenerates into two segments : $p = 0$ and $q = 0$. On the first segment, the dimension of $\dim_H \nu_{0,q}$ is zero. On the second segment, we have the assertion on $\nu_{p,0} = \nu_p$ in Lemma 2.3. When $\alpha = 1$, the curve γ_1 degenerates into one point $(1,1)$, and we have $\dim_H \nu_{1,1} = 0$.

Remark. The curves γ_α cover the whole $(0,1)^2$. However, not all pairs $(p,q) \in (0,1)^2$ are solutions of (2.5) for any λ_1, λ_0. Indeed, we can write (2.5) in the form

$$\frac{\lambda_1}{\lambda_0} a_1 + a_2 = 0$$

with

$$a_1 = \alpha \log p + (-2-\alpha)\log(1-p) - \alpha \log q + \alpha \log(1-q)$$

and

$$a_2 = (2-\alpha)\log p + (\alpha - 1)\log(1-p) + \alpha \log q + (1-\alpha)\log(1-q).$$

Both a_1 and a_2 converge to ∞ as $p \to 1$ and to $-\infty$ as $q \to 1$. They are also both strictly monotone on γ_α, which can be checked like in the third part of the proof of Lemma 2.3 (using $(2-\alpha)(p+2) > \alpha(2-q)$ in case of a_2), so they both have unique zeros. As the equation

$$ra_1 + a_2 = 0$$

can have positive solution only if a_1 and a_2 have different signs, only those $(p,q) \in \gamma_\alpha$ between zeros of a_1 and a_2, or equivalently satisfying

$$\alpha \log \frac{p(1-q)}{(1-p)q} > \max\left(2\log(1-p), \log\frac{p^2(1-q)}{1-p}\right),$$

are solutions of (2.5) for some choice of λ_1, λ_0.

Remark. The measures $\mu_{p,q}$ for $p = q$ are Bernoulli. Each γ_α intersects the diagonal $\{p = q\}$ in exactly one point $(\alpha^{1/2}, \alpha^{1/2})$ and at this point $a_1 > 0, a_2 < 0$. So, (2.5) has a Bernoulli measure as a solution for each $\alpha \in (0,1)$. It happens when

$$\lambda_0 \log p = \lambda_1 \log(1-p),$$

that is, when $\nu_{\alpha^{1/2},\alpha^{1/2}}$ is the Hausdorff measure (in dimension $\dim_H \pi(\Sigma)$) on $\pi(\Sigma)$.

3 Exact formulas

To be able to provide the upper bounds in the following section, we need to substitute the results of Lemma 2.3 to Lemma 2.2 and obtain simpler formulas for our lower bound. We start with the golden shift case. Given λ_1, λ_0 let p be given by (2.2).

Lemma 3.1. *We have*

$$\dim_H \nu_p = -\frac{\log(1-p)}{\lambda_0}.$$

PROOF. By Lemma 2.2,

$$\dim_H \nu_p = \frac{2H(p)}{2p\lambda_1 + (2-p)\lambda_0}.$$

Applying (2.2) it is easy to check that

$$(2p\lambda_1 + (2-p)\lambda_0)\log(1-p) = -2H(p)\lambda_0$$

and the assertion follows. □

The calculations for the multifractal case are a little bit more complicated. Given λ_1, λ_0, and α, let p, q be given by (2.3).

Lemma 3.2. *We have*

$$\dim_H \nu_{p,q} = \frac{\alpha \log \frac{p(1-q)}{(1-p)q} - 2\log(1-p)}{2\lambda_0}. \tag{3.1}$$

If $\lambda_1 \neq \lambda_0$ then we have another formula:

$$\dim_H \nu_{p,q} = \frac{\log \frac{p^2(1-q)}{(1-p)^3}}{2(\lambda_0 - \lambda_1)}. \tag{3.2}$$

PROOF. By Lemma 2.2,

$$\dim_H \nu_{p,q} = \frac{(2-q)H(p) + pH(q)}{2p\lambda_1 + (2-p-q)\lambda_0}.$$

Using (2.3) one can check that

$$(2p\lambda_1 + (2-p-q)\lambda_0)\left(\alpha \log \frac{p(1-q)}{(1-p)q} - 2\log(1-p)\right) = 2\lambda_0((2-q)H(p) + pH(q)).$$

This gives (3.1). Applying (2.3) once again we get

$$\dim_H \nu_{p,q} = \frac{\alpha \log \frac{p(1-q)}{(1-p)q} + \log \frac{1-p}{p^2(1-q)}}{2\lambda_1}. \tag{3.3}$$

Together with (3.1) this gives (3.2). $\square$

4 Upper bounds

The last part of the proof is the upper bound.

Lemma 4.1. *We have*

$$\dim_H L \leq \sup_p \dim_H \nu_p,$$

and for all $\alpha \in [0,1]$,

$$\dim_H L_\alpha \leq \sup_{(p,q)\in\gamma_\alpha} \dim_H \nu_{p,q}.$$

PROOF. As $L \subset L_0$, it is enough to prove the second part of the assertion. Fix α and let $\omega \in S_\alpha$. Let p, q be as in (2.3). We denote for all $n \in \mathbb{N}$

$$X_1^n = \sharp\{k \in [1, n] : \omega_k = 1\}$$

and for all even $n \in \mathbb{N}$

$$X_{11}^n = \sharp\{k \in [1, n/2] : \omega_k = \omega_{2k} = 1\}.$$

We also denote

$$\tilde{h}_n = -\log \mu_{p,q}(C_n(\omega))$$

and

$$\tilde{l}_n = -\log \operatorname{diam} \pi(C_n(\omega)).$$

The following result was proven in [11], we give the proof for completeness.

Lemma 4.2. *For any even n we have*

$$-\tilde{h}_n = n \log(1-p) + X_1^{n/2} \log \frac{1-q}{1-p} + X_1^n \log \frac{p}{1-p} - X_{11}^n \log \frac{p(1-q)}{(1-p)q}.$$

PROOF. We will need additional notations. Let

$$\begin{aligned}
X_{0odd}^n &= \sharp\{k \in [1, n/2] : \omega_{2k-1} = 0\}, \\
X_{1odd}^n &= \sharp\{k \in [1, n/2] : \omega_{2k-1} = 1\}, \\
X_{00}^n &= \sharp\{k \in [1, n/2] : \omega_k = \omega_{2k} = 0\}, \\
X_{01}^n &= \sharp\{k \in [1, n/2] : \omega_k = 0, \omega_{2k} = 1\}, \\
X_{10}^n &= \sharp\{k \in [1, n/2] : \omega_k = 1, \omega_{2k} = 0\}.
\end{aligned}$$

We have

$$\begin{aligned}
-\tilde{h}_n = X_{0odd}^n &\log(1-p) + X_{1odd}^n \log p + X_{00}^n \log(1-p) \\
&+ X_{01}^n \log p + X_{10}^n \log(1-q) + X_{11}^n \log q.
\end{aligned}$$

Substituting

$$\begin{aligned}
X_{00}^n + X_{01}^n &= \frac{n}{2} - X_1^{n/2}, \\
X_{10}^n + X_{11}^n &= X_1^{n/2}, \\
X_{0odd}^n + X_{00}^n + X_{10}^n &= n - X_1^n, \\
X_{1odd}^n + X_{01}^n + X_{11}^n &= X_1^n,
\end{aligned}$$

we obtain

$$\begin{aligned}-\tilde{h}_n &= (X_1^n - X_{11}^n)\log p + (n - X_1^n - X_1^{n/2} + X_{11}^n)\log(1-p) \\ &\quad + X_{11}^n \log q + (X_1^{n/2} - X_{11}^n)\log(1-q)\end{aligned}$$

and the assertion follows. □

We also have

$$\tilde{l}_n = (\lambda_1 - \lambda_0)X_1^n + n\lambda_0.$$

Substituting (3.1) and (3.2) we get

$$\tilde{l}_n \dim_H \nu_{p,q} = -\frac{1}{2}X_1^n \log\frac{p^2(1-q)}{(1-p)^3} + \frac{n}{2}\left(\alpha \log\frac{p(1-q)}{(1-p)q} - 2\log(1-p)\right).$$

Hence,

$$\frac{1}{n}(\tilde{l}_n \dim_H \nu_{p,q} - \tilde{h}_n) = \left(\frac{\alpha}{2} - \frac{X_{11}^n}{n}\right)\log\frac{p(1-q)}{(1-p)q} + \frac{1}{2}\left(\frac{X_1^{n/2}}{n/2} - \frac{X_1^n}{n}\right)\log\frac{1-q}{1-p}.$$

As the first summand converges to 0 and the second telescopes,

$$\limsup_{n\to\infty} \frac{1}{n}(\tilde{l}_n \dim_H \nu_{p,q} - \tilde{h}_n) \geq 0.$$

Then we have

$$\liminf_{n\to\infty} \frac{\tilde{h}_n}{\tilde{l}_n} \leq \dim_H \nu_{p,q}.$$

Applying Frostman Lemma [2, Proposition 2.3], we are done. □

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