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STRONGLY SEPARATELY CONTINUOUS AND SEPARATELY QUASICONTINUOUS FUNCTIONS $f: l^2 \rightarrow \mathbb{R}$

Abstract

In this paper we give a sufficient condition for the strongly separately continuous functions to be continuous on l^2 . Further we shall give notions of separately quasicontinuous function $f: l^2 \rightarrow \mathbb{R}$ and its properties. At the end we will expecting to determining sets $M \subset l^2$ for the class of separately continuous functions on l^2 .

1 Introduction

The notion of the strongly separately continuity $f: \mathbb{R}^m \rightarrow \mathbb{R}$, where $m \geq 1$ is introduced in paper [2]. It is proved, that the function f is continuous at the point $x^0 = (x_1^0, \dots, x_m^0)$ if and only if f is strongly separately continuous at x^0 .

Functions $f: l^2 \rightarrow \mathbb{R}$, where l^2 is the space of real sequences such that $\sum_{k=1}^{\infty} x_k^2 < +\infty$ with the metric $d(x, y) = \sqrt{\sum_{k=1}^{\infty} (x_k - y_k)^2}$ are investigated in paper [1].

It is known, that (l^2, d) is a separable and complete metric space. In [1], it is shown that there is a strongly separately continuous function f defined on l^2 , that is discontinuous everywhere on l^2 .

In the third part of this paper we shall deal with separately quasicontinuous functions on l^2 . We shall give a condition under which the separately quasicontinuity implies quasicontinuity on l^2 . Further it will be shown, that

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there is a strongly separately continuous function $f: l^2 \rightarrow \mathbb{R}$, that is not quasicontinuous and it belongs to the Baire class three.

The fourth part of the paper is devoted to the determining sets for the class of separately continuous functions on l^2 . A sufficient condition will be given for a set $M \subset l^2$ to be determining set for the class of separately continuous functions.

2 Strongly separately continuous functions $f: l^2 \rightarrow \mathbb{R}$

In paper [1] we find the definition of separately and strongly separately continuous function $f: l^2 \rightarrow \mathbb{R}$. If $x^0 \in l^2$ and $\delta > 0$, then $B(x^0, \delta)$ denotes the set $\{x \in l^2 : d(x, x^0) < \delta\}$.

Definition 1. A function $f: l^2 \rightarrow \mathbb{R}$ is said to be *separately continuous at the point* $x^0 = (x_j^0) \in l^2$ *with respect to the variable* x_k *under the assumption, that the function* $\varphi_k: \mathbb{R} \rightarrow \mathbb{R}$ *defined by* $\varphi_k(t) = f(x_1^0, \dots, x_{k-1}^0, t, x_{k+1}^0, \dots)$ *is continuous at* x_k . If f is separately continuous at x^0 with respect to x_k for all $k \in \mathbb{N}$, then f is said to be *separately continuous at* x^0 . If f is separately continuous at every point $x^0 \in l^2$, then f is said to be *separately continuous on* l^2 .

Definition 2. A function $f: l^2 \rightarrow \mathbb{R}$ is said to be *strongly separately continuous at the point* $x^0 = (x_j^0) \in l^2$ *with respect to the variable* x_k *under the assumption, that for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|f(x) - f(x')| < \varepsilon$ *holds for each* $x = (x_j) \in B(x^0, \delta)$ *and* $x' = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots)$. If f is strongly separately continuous at x^0 with respect to x_k for all $k \in \mathbb{N}$, then f is said to be *strongly separately continuous at* x^0 . The function f is said to be *strongly separately continuous on* l^2 *under the assumption, that it is strongly separately continuous at every point* $x^0 \in l^2$.

It is known that if $f: l^2 \rightarrow \mathbb{R}$ is continuous at $x^0 \in l^2$, then f is strongly separately continuous at x^0 . Also if $f: l^2 \rightarrow \mathbb{R}$ is strongly separately continuous at $x^0 \in l^2$, then f is separately continuous at x^0 .

A subset $S \subseteq l^2$ is said to be a *set of type* (P_1) *under the assumption, that the following holds: if* $x = (x_j) \in S$, $y = (y_j) \in l^2$ *and the set* $\{j \in \mathbb{N} : x_j \neq y_j\}$ *contains at most one element, then* $y = (y_j) \in S$. In [1] we find the following examples of such sets of type (P_1) .

Example 1. a) $S = \{x = (x_j) \in l^2 : \{j \in \mathbb{N} : x_j \text{ is a rational (irrational, algebraic, transcendent) number}\} \text{ is a finite set}\}$,

b) $S' = \{x = (x_j) \in l^2 : \sum_{j=1}^{\infty} x_k < +\infty\}$.

It is easy to see, that the sets S , $l^2 \setminus S$, S' and $l^2 \setminus S'$ are dense in l^2 .

Theorem 1 (Theorem 1.4 in [1]). *There exists a function $g: l^2 \rightarrow \mathbb{R}$ such that g is strongly separately continuous on l^2 and g is discontinuous at every point of l^2 .*

PROOF. Let $S \subseteq l^2$ be a set of type (P_1) such that S and $l^2 \setminus S$ are dense in l^2 . Let $c \in \mathbb{R}$, $c \neq 0$. Define a function $g: l^2 \rightarrow \mathbb{R}$ by $g(x) = c$ for all $x \in S$ and $g(x) = 0$ otherwise. If $x^0 \in l^2$, then for every neighbourhood U of x^0 we have $U \cap S \neq \emptyset$, $U \cap (l^2 \setminus S) \neq \emptyset$ and this means, that g is discontinuous at x^0 . On the other hand, let $k \in \mathbb{N}$ and $x^0 = (x_j^0)$, $x = (x_j)$, $x' = (x'_j)$ be arbitrary points of l^2 such that for all $j \neq k$, $x_j = x'_j$ and $x_k^0 = x'_k$. It is obvious that if $x \in S$ then also $x' \in S$ and if $x \notin S$ then also $x' \notin S$. Hence we always obtain $|g(x) - g(x')| = 0$ so that for each $x^0 \in l^2$ and each $k \in \mathbb{N}$ the function g is strongly separately continuous at x^0 with respect to x_k . \square

It follows from Theorem 1 that the function $g: l^2 \rightarrow \mathbb{R}$ defined in the proof of Theorem 1 does not belong to the first Baire class.

Now, we can formulate a sufficient condition for a strongly separately function to be continuous at the point $x^0 \in l^2$.

Theorem 2. *Let $f: l^2 \rightarrow \mathbb{R}$ be a strongly separately continuous at the point $x^0 = (x_j^0) \in l^2$. Let for each $\varepsilon > 0$ there exists $\delta > 0$ and $N \in \mathbb{N}$, such that for all $y = (y_j) \in l^2$ for which $\sqrt{\sum_{j=N+1}^{\infty} (x_j^0 - y_j)^2} < \delta$ implies*

$$|f(y_1, y_2, \dots) - f(y_1, \dots, y_N, x_{N+1}^0, x_{N+2}^0, \dots)| < \varepsilon.$$

Then f is continuous at $x^0 = (x_j^0) \in l^2$.

PROOF. Let $\varepsilon > 0$. According to the assumption there exists $\delta_0 > 0$ and $N \in \mathbb{N}$ such that for each $y = (y_j) \in l^2$ for which $\sqrt{\sum_{j=N+1}^{\infty} (x_j^0 - y_j)^2} < \delta_0$ we have

$$|f(y_1, y_2, \dots) - f(y_1, \dots, y_N, x_{N+1}^0, x_{N+2}^0, \dots)| < \frac{\varepsilon}{2}.$$

Since f is strongly separately continuous at $x^0 = (x_j^0) \in l^2$: for all $\varepsilon' > 0$ and for all $k \in \mathbb{N}$ there exists $\delta'_k > 0$ such that for all $x = (x_j) \in l^2$ for which $\sqrt{\sum_{j=1}^{\infty} (x_j - x_j^0)^2} < \delta'_k$ implies

$$|f(x_1, \dots) - f(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots)| < \varepsilon'$$

holds. Let $\varepsilon' = \frac{\varepsilon}{2N}$. Then for each $k \in \{1, 2, \dots, N\}$ there exists $\delta_k > 0$ such that if $x = (x_j) \in B(x^0, \delta_k)$ then

$$|f(x_1, \dots) - f(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots)| < \frac{\varepsilon}{2N}.$$

Let $\delta = \min\{\delta_0, \delta_1, \dots, \delta_k\}$ and $x = (x_j) \in B(x^0, \delta)$. Then also (x_1^0, x_2, \dots) , $(x_1^0, x_2^0, x_3, \dots)$, \dots , $(x_1^0, \dots, x_{N-1}^0, x_N, x_{N+1}, \dots) \in B(x^0, \delta)$ which is the subset of $B(x^0, \delta_k)$, $k = 0, 1, 2, \dots, N$ and

$$\sqrt{\sum_{j=N+1}^{\infty} (x_j - x'_j)^2} \leq \sqrt{\sum_{j=1}^{\infty} (x_j - x_j^0)^2} < \delta \leq \delta_0.$$

Then

$$\begin{aligned} & |f(x_1, x_2, \dots) - f(x_1^0, x_2^0, \dots)| \leq |f(x_1, x_2, \dots) - f(x_1^0, x_2, \dots)| \\ & + |f(x_1^0, x_2, x_3, \dots) - f(x_1^0, x_2^0, x_3, \dots)| \\ & + \dots + \\ & + |f(x_1^0, \dots, x_{N-1}^0, x_N, x_{N+1}, \dots) - f(x_1^0, \dots, x_{N-1}^0, x_N^0, x_{N+1}, \dots)| \\ & + |f(x_1^0, \dots, x_{N-1}^0, x_N^0, x_{N+1}, \dots) - f(x_1^0, \dots, x_N^0, x_{N+1}^0, x_{N+2}^0, \dots)| \\ & < N \cdot \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence f is continuous at the point $x^0 = (x_j^0) \in l^2$. \square

Corollary 3. *Let $f: l^2 \rightarrow \mathbb{R}$ be strongly separately continuous on l^2 . Let for each $\varepsilon > 0$ there exists $\delta > 0$ and $N \in \mathbb{N}$ such that for all $x, y \in l^2$: $\sqrt{\sum_{j=N+1}^{\infty} (x_j - y_j)^2} < \delta$ implies*

$$|f(y_1, y_2, \dots) - f(y_1, \dots, y_N, y_{N+1}, y_{N+2}, \dots)| < \varepsilon.$$

Then f is continuous on l^2 .

Let us recall the notion of quasicontinuity for a function defined on arbitrary metric space.

Definition 3. Let (X, d_1) , (Y, d_2) be metric spaces. A function $f: X \rightarrow Y$ is *quasicontinuous* at $x_0 \in X$ if for all $\varepsilon > 0$ and for all $\delta > 0$ there exists $B(x_1, \delta_1) \subseteq B(x_0, \delta)$ such that $f(B(x_1, \delta_1)) \subseteq B(f(x_0), \varepsilon)$.

It is known, that each continuous function on X is quasicontinuous on X . In general the converse is not true.

Further, the function $f: \mathbb{R}^2 \rightarrow \mathbb{R} : f(x, y) = 1, (x, y) \in A$ and $f(x, y) = 0, (x, y) \notin A$, where $A = \mathbb{Q}^2 \cup \{\{0\} \times \mathbb{R}\} \cup \{\mathbb{R} \times \{0\}\}$ is separately continuous at $(0, 0)$ and is not quasicontinuous at $(0, 0)$.

Theorem 4. *There exists a strongly separately continuous function on l^2 belonging to the third Baire class which is not quasicontinuous at any point in l^2 .*

We shall use the following Lemma:

Lemma 5. *Let $D = \{x = (x_k) \in l^2 : \sum_{k=1}^{\infty} x_k = +\infty\}$. The set D is a dense set in l^2 .*

PROOF. Let $x = (x_k) \in l^2$ be arbitrary and $\delta > 0$. We show, that there exists $\alpha = (\alpha_k) \in D$, for which

$$d(x, \alpha) = \sqrt{\sum_{k=1}^{\infty} (\alpha_k - x_k)^2} < \delta.$$

We can find $m \in \mathbb{N}$ such that $\sum_{k=m+1}^{\infty} x_k^2 < \frac{\delta^2}{4}$ and $\sum_{k=m+1}^{\infty} \frac{1}{k^2} < \frac{\delta^2}{4}$. Put $y_k = x_k$ for $k = 1, 2, \dots, m$ and $y_k = 0$ for $k = m+1, m+2, \dots$. Then $y = (y_k) \in l^2$ and

$$d(x, y) = \sqrt{\sum_{k=1}^{\infty} (x_k - y_k)^2} = \sqrt{\sum_{k=m+1}^{\infty} x_k^2} < \frac{\delta}{2}$$

holds. Further, we put $\alpha_k = y_k$ for $k = 1, 2, \dots, m$, $\alpha_k = \frac{1}{k}$ for $k = m+1, m+2, \dots$. Then $\alpha = (\alpha_k) \in D$ and

$$d(y, \alpha) = \sqrt{\sum_{k=1}^{\infty} (\alpha_k - y_k)^2} = \sqrt{\sum_{k=m+1}^{\infty} \frac{1}{k^2}} < \frac{\delta}{2}.$$

Then $d(\alpha, x) \leq d(\alpha, y) + d(y, x) < \delta$. Hence D is a dense set in l^2 . \square

PROOF OF THEOREM 4. Denote $H = \{x = (x_j) \in l^2 : \sum_{j=1}^{\infty} x_j \text{ is convergent}\}$. Define the function $h: l^2 \rightarrow \mathbb{R}$ in the following way: $h(x) = \sum_{j=1}^{\infty} x_j$ for $x \in H$ and $h(x) = 0$ for $x \in l^2 \setminus H$.

First, we show that h is strongly separately continuous on l^2 . Let $x^0 = (x_j^0) \in l^2$, $k \in \mathbb{N}$. We show that h is strongly separately continuous at x^0 with respect to the variable x_k . Let $\varepsilon > 0$. If $x = (x_j) \in B(x^0, \varepsilon)$, then also $x' = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots) \in B(x^0, \varepsilon)$. If $x \in H$ and $h(x) = \sum_{j=1}^{\infty} x_j$, then

$$|h(x) - h(x')| = |x_k - x_k^0| \leq d(x, x^0) < \varepsilon.$$

If $x \in l^2 \setminus H$, then $h(x) = h(x') = 0$ and we have $|h(x) - h(x')| = 0$. Hence h is strongly separately continuous at x^0 with respect to a variable x_k .

Now we show indirectly that h is quasicontinuous at no point $x^0 = (x_j^0) \in l^2$. Let h be quasicontinuous at $x^0 = (x_j^0) \in l^2$, $\varepsilon > 0$, $\delta > 0$. Then according to Definition 3, there is a ball $B(y, \delta_1) \subseteq B(x^0, \delta)$ such that $h(B(y, \delta_1)) \subset B(h(x^0), \varepsilon)$. Let $\alpha = (\alpha_j) \in D$ such that $\alpha \in B(y, \frac{\delta_1}{2})$. We can find $m \in \mathbb{N}$ such that $\sum_{j=m+1}^{\infty} \alpha_j^2 < \frac{\delta_1^2}{4}$ and $\sum_{j=1}^m \alpha_j > h(x^0) + \varepsilon$. Put $b_j = \alpha_j$, $j = 1, 2, \dots, m$ and $b_j = 0$, $j = m+1, m+2, \dots$. Then $\sum_{j=1}^{\infty} b_j$ converges and $b = (b_j) \in H$. According to the definition of the function h , $h(b) = \sum_{j=1}^{\infty} b_j = \sum_{j=1}^m \alpha_j$. Hence $h(b) > h(x^0) + \varepsilon$. On the other hand

$$d(\alpha, b) = \sqrt{\sum_{j=1}^{\infty} (\alpha_j - b_j)^2} = \sqrt{\sum_{j=m+1}^{\infty} \alpha_j^2} < \frac{\delta_1}{2}$$

and we have $d(y, b) \leq d(y, \alpha) + d(\alpha, b) < \delta_1$. Hence $b \in B(y, \delta_1)$ and therefore $h(b) < h(x^0) + \varepsilon$, it is a contradiction. Therefore h is not quasicontinuous at $x^0 = (x_j^0) \in l^2$.

Further we show, that h belongs to the third Baire class. It is sufficient to show that the sets

$$M^t = \{x = (x_j) \in l^2 : h(x) > t\} \text{ and } M_t = \{x = (x_j) \in l^2 : h(x) < t\}$$

are of the type $F_{\sigma\delta\sigma}$ in l^2 for each $t \in \mathbb{R}$.

First we show that the set $H = \{x = (x_j) \in l^2 : \sum_{j=1}^{\infty} x_j \text{ is convergent}\}$ is of type $F_{\sigma\delta}$ in l^2 . It obviously holds: $x = (x_j) \in H$ if and only if for all $j \geq 1$ there exists $p \geq 1$ such that for all $m, n \geq p$, $|s_m(x) - s_n(x)| \leq \frac{1}{j}$, where $s_k(x) = \sum_{j=1}^k x_j$, $k = 1, 2, \dots$. It follows that

$$H = \bigcap_{j=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{m, n=p}^{\infty} \left\{ x = (x_j) \in l^2 : |s_m(x) - s_n(x)| \leq \frac{1}{j} \right\}.$$

It is easy to see, that for a fixed $k \geq 1$ the function s_k is continuous on l^2 . Hence H is an $F_{\sigma\delta}$ set in l^2 . Let $t \leq 0$. Then the set

$$M^t = H \cap \bigcup_{j=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} \left\{ x = (x_j) \in l^2 : s_n(x) \leq t - \frac{1}{j} \right\}$$

is of the type $F_{\sigma\delta}$ in l^2 .

Let $t > 0$. Then

$$M^t = (l^2 \setminus H) \cap \bigcup_{j=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} \left\{ x = (x_j) \in l^2 : s_n(x) \leq t - \frac{1}{j} \right\}.$$

The set $l^2 \setminus H$ is of the type $G_{\delta\sigma}$ in l^2 and from this it follows that M^t is of the type $G_{\delta\sigma}$ too. We can see, that for arbitrary $t \in \mathbb{R}$, M^t is the set of the type $F_{\sigma\delta\sigma}$.

We have proved, that $h: l^2 \rightarrow \mathbb{R}$ is strongly separately continuous on l^2 and it belongs to the third Baire class. \square

3 Separately quasicontinuous functions on l^2

It follows from Definition 3 that the notion of quasicontinuity is a weaker form of continuity. It is possible to formulate the notion of separately quasicontinuity for $f: l^2 \rightarrow \mathbb{R}$.

Definition 4. A function $f: l^2 \rightarrow \mathbb{R}$ is said to be *separately quasicontinuous at the point* $x^0 = (x_j^0) \in l^2$ *with respect to the variable* x_k *under the assumption, that the function* $\varphi_k: \mathbb{R} \rightarrow \mathbb{R}$ *defined by* $\varphi_k(t) = f(x_1^0, x_2^0, \dots, x_{k-1}^0, t, x_{k+1}^0, \dots)$ *is quasicontinuous at* x_k^0 . *If* f *is separately quasicontinuous at* x^0 *with respect to* x_k , *for all* $k \in \mathbb{N}$, *then* f *is said to be* *separately quasicontinuous at* $x^0 = (x_j^0) \in l^2$. *If* f *is separately quasicontinuous at every point* $x^0 \in l^2$ *then* f *is said to be* *separately quasicontinuous on* l^2 .

The following Theorem (see [3]) is published for all functions $f: \mathbb{R}^m \rightarrow \mathbb{R}$, where $m \geq 1$.

Theorem 6. *If* $f: \mathbb{R}^m \rightarrow \mathbb{R}$ *is separately quasicontinuous on* \mathbb{R}^m , *then* f *is quasicontinuous on* \mathbb{R}^m .

The next example shows, that for the functions $f: l^2 \rightarrow \mathbb{R}$ the situation is different.

Example 2. Let

$$g(x_1, x_2, \dots) = \begin{cases} c, & x \in S, \\ 0, & x \in l^2 \setminus S, \end{cases}$$

where $c \in \mathbb{R}, c \neq 0$ and the set S is defined by Example 1a). We show, that g is separately quasicontinuous on l^2 but it is not quasicontinuous at any point of l^2 . It was shown in the proof of Theorem 1, that g is strongly separately continuous on l^2 . According to Proposition 1.2 from [1] it follows that g is separately continuous on l^2 and obviously every separately continuous function on l^2 is also separately quasicontinuous on l^2 .

It is sufficient to show, that g is not quasicontinuous at any point of l^2 . Each of the sets $S, l^2 \setminus S$ are dense in l^2 . Then every ball $B(x^0, \delta)$ ($x^0 = (x_j^0) \in l^2, \delta > 0$) contains some elements $y^{(1)} \in S$ and $y^{(2)} \in l^2 \setminus S$. If $\varepsilon = |c|$, then either $g(y^{(1)}) \notin B(g(x^0), \varepsilon)$ or $g(y^{(2)}) \notin B(g(x^0), \varepsilon)$. Hence g is not quasicontinuous at the point $x^0 = (x_j^0) \in l^2$.

Furthermore we give a sufficient condition for the separately quasicontinuous function on l^2 to be quasicontinuous.

Theorem 7. *Let $f: l^2 \rightarrow \mathbb{R}$ be a separately quasicontinuous on l^2 . Let for each $\varepsilon > 0$ there exists $\delta > 0$ and $N \in \mathbb{N}$ such that for all $x, y \in l^2$:*

$$\sqrt{\sum_{j=N+1}^{\infty} (x_j - y_j)^2} < \delta \text{ implies}$$

$$|f(y_1, y_2, \dots) - f(y_1, \dots, y_N, x_{N+1}, x_{N+2}, \dots)| < \varepsilon. \tag{1}$$

Then f is quasicontinuous on l^2 .

PROOF. Let $x^0 = (x_j^0) \in l^2, \varepsilon > 0$ and $\delta > 0$. According to (1) for $\frac{\varepsilon}{2} > 0$ there exists $\delta_1 > 0, \delta_1 \leq \frac{\delta}{2}$ and there exists $N \in \mathbb{N}$ such that for all $x, y \in l^2$ for which $\sqrt{\sum_{j=N+1}^{\infty} (x_j - y_j)^2} < \delta$ implies

$$|f(y_1, y_2, \dots) - f(y_1, \dots, y_N, x_{N+1}, x_{N+2}, \dots)| < \frac{\varepsilon}{2}. \tag{2}$$

Let $\varphi_0: \mathbb{R}^m \rightarrow \mathbb{R}$ be a function defined by

$$\varphi_0(x_1, \dots, x_N) = f(x_1, x_2, \dots, x_N, x_{N+1}^0, x_{N+2}^0, \dots).$$

Then φ_0 is separately quasicontinuous and according to the Theorem 6 is quasicontinuous on \mathbb{R}^m .

Let $j_0: \mathbb{R}^m \rightarrow l^2$ be a map defined by the following way: $j_0(x_1, \dots, x_N) = (x_1, x_2, \dots, x_N, x_{N+1}^0, x_{N+2}^0, \dots)$. j_0 is a continuous map on \mathbb{R}^m and $\varphi_0 = f \circ j_0$. From the quasicontinuity of φ_0 it follows, that for $\frac{\varepsilon}{2} > 0$ and for each $\delta_2 >$

0, $\delta_2 \leq \delta_1$ there exists a ball $B((x'_1, \dots, x'_N), \delta_3) \subseteq B((x_1^0, \dots, x_N^0), \delta_2)$ such that $\varphi_0(B((x'_1, \dots, x'_N), \delta_3)) \subseteq B(\varphi_0(x_1^0, \dots, x_N^0), \frac{\varepsilon}{2}) = B(f(x_1^0, \dots, x_N^0, x_{N+1}^0, \dots), \frac{\varepsilon}{2})$. Obviously $\delta_3 \leq \delta_2$. Denote

$$\begin{aligned}\xi^1 &= (x_1^1, \dots, x_N^1), & x^1 &= j_0(\xi^1) = (x_1^1, \dots, x_N^1, x_{N+1}^0, \dots), \\ \xi^0 &= (x_1^0, \dots, x_N^0), & x^0 &= j_0(\xi^0) = (x_1^0, \dots, x_N^0, x_{N+1}^0, \dots).\end{aligned}$$

Denote d as a metric on l^2 and d' as a metric on \mathbb{R}^m . Obviously for all $\xi, \eta \in \mathbb{R}^m$: $d'(\xi, \eta) = d(j_0(\xi), j_0(\eta))$, $d'(\xi^1, \xi^0) = d(j_0(\xi^1), j_0(\xi^0)) = d(x^1, x^0) < \delta_2$ therefore $\xi^1 \in B(\xi^0, \delta_2)$. Let us consider a ball $B(x^1, \delta_3)$ in l^2 . If $x \in B(x^1, \delta_3)$ then $d(x, x^0) \leq d(x, x^1) + d(x^1, x^0) < \delta_3 + \delta_2 \leq 2\delta_2 < \delta$. Hence $B(x^1, \delta_3) \subseteq B(x^0, \delta)$.

Let $x \in B(x^1, \delta_3)$. Then $d(x, x^1) < \delta_3$ and it holds

$$d'((x_1, \dots, x_N), (x_1^1, \dots, x_N^1)) \leq d(x, x^1) < \delta_3.$$

Then $(x_1, \dots, x_N) \in B((x_1^1, \dots, x_N^1), \delta_3)$ that implies that $\varphi_0(x_1, \dots, x_N) = f(x_1, \dots, x_N, x_{N+1}^0, \dots) \in B(f(x^0), \frac{\varepsilon}{2})$ that means

$$|f(x_1, \dots, x_N, x_{N+1}^0, x_{N+2}^0, \dots) - f(x_1^0, \dots, x_N^0, x_{N+1}^0, \dots)| < \frac{\varepsilon}{2}. \quad (3)$$

From the condition (2) we have

$$|f(x_1, x_2, \dots) - f(x_1, \dots, x_N, x_{N+1}^0, x_{N+2}^0, \dots)| < \frac{\varepsilon}{2},$$

since $\sqrt{\sum_{j=N+1}^{\infty} (x_j - x_j^0)^2} \leq d(x, x^1) < \delta_3 \leq \delta_2 \leq \delta_1$. Then

$$\begin{aligned}|f(x_1, x_2, \dots) - f(x_1^0, x_2^0, \dots)| &\leq |f(x_1, x_2, \dots) - f(x_1, \dots, x_N, x_{N+1}^0, \dots)| \\ &\quad + |f(x_1, \dots, x_N, x_{N+1}^0, \dots) - f(x_1^0, x_2^0, \dots)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

For $x^0 = (x_j^0) \in l^2$, $\varepsilon > 0$, $\delta > 0$ there exists $B(x^1, \delta_3) \subseteq B(x^0, \delta)$ such that $f(B(x^1, \delta_3)) \subseteq B(f(x^0), \varepsilon) = (f(x^0) - \varepsilon, f(x^0) + \varepsilon)$ and it means that f is quasicontinuous at $x^0 = (x_j^0) \in l^2$. \square

An example of a function $f: l^2 \rightarrow \mathbb{R}$ that fulfils the assumption of Theorem 7 is $f(x) = \sum_{n=1}^{\infty} \alpha_n \cdot g_n(x)$, where $g(t) = 1$ for $t \geq 0$, $g(t) = 0$ for $t < 0$ and $\alpha_n = 2^{-n}$, $n = 1, 2, \dots$. The function f is quasicontinuous on l^2 , but it is not continuous there (see [8]).

4 Determining sets for separately continuous functions

Let \mathcal{F} be a class of functions defined on the set X . The set $M \subset X$ is called determining set for a class \mathcal{F} if for any couple of function $f, g \in \mathcal{F}$ the equality $f(x) = g(x)$ for $x \in M$ implies $f(x) = g(x)$ for every $x \in X$. Let \mathcal{F}_0 be the class of all separately continuous functions on \mathbb{R}^m , $m \geq 2$. It is known, that $M \subset \mathbb{R}^m$ is a determining set for the class \mathcal{F}_0 if and only if M is dense in \mathbb{R}^m (see [6]).

It is shown in [1] that the analogous statement does not hold for the class \mathcal{F}_1 of all separately continuous functions on l^2 . In [1] the following statement is formulated:

Theorem 8. *There exists a strongly separately continuous function $h: l^2 \rightarrow \mathbb{R}$ and a residual (and, consequently, dense) set E in l^2 such that $h(x) = 0$ for all $x \in E$ and $h(y) \neq 0$ for some $y \in l^2 \setminus E$.*

PROOF. See [1] Theorem 3.1. □

The function h and the set H introduced in the proof of Theorem 4 has these properties.

Further in [1], there is a statement formulated that when $M \subset l^2$ is not a determining set for the class \mathcal{F}_1 . In the following a sufficient condition will be given under a subset of l^2 is the determining set for \mathcal{F}_1 .

First we define the property (P_2) : It is said that $M \subset l^2$ has the property (P_2) if for all $x = (x_k) \in l^2$ there exists $m(x) \in \mathbb{N}$ such that for all $\delta > 0$ there exists $y = (y_k) \in M : x_k = y_k$ for $k \neq m$ and $|x_m - y_m| < \delta$.

Theorem 9. *If $M \subset l^2$ has the property (P_2) , then M is a determining set for the class \mathcal{F}_1 of all separately continuous functions on l^2 .*

PROOF. It is sufficient to show that if $f \in \mathcal{F}_1$ and $f(x) = 0$ for all $x = (x_j) \in M$, then $f(x) = 0$ for all $x = (x_j) \in l^2$.

Let $x^0 = (x_j^0) \in l^2$. It follows from the property (P_2) , that there exists $m = m(x^0) \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exists $y^{(k)} = (y_j^{(k)}) \in M$ for which $y = (x_1^0, \dots, x_{m-1}^0, y_m^{(k)}, x_{m+1}^0, \dots)$ and $|y_m^{(k)} - x_m^0| < \frac{1}{k}$.

Since f is separately continuous at x^0 , the function $\varphi_m: \mathbb{R} \rightarrow \mathbb{R}: \varphi_m(t) = f(x_1^0, \dots, x_{m-1}^0, t, x_{m+1}^0, \dots)$ is continuous. Obviously, $\lim_{k \rightarrow \infty} y_m^{(k)} = x_m^0$ and we have $\lim_{k \rightarrow \infty} \varphi_m(y_m^{(k)}) = \varphi_m(x_m^0)$. Since for all $k \in \mathbb{N} : \varphi_m(y_m^{(k)}) = f(y^{(k)}) = 0$ we have $\varphi(x_m^0) = f(x^0) = 0$. It means, that for all $x \in l^2 : f(x) = 0$. □

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