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# AVOIDING RATIONAL DISTANCES

**Abstract**

We show that for any set of reals $X$ there is a $Y \subseteq X$ such $X$ and $Y$ have same Lebesgue outer measure and the distance between any two distinct points in $Y$ is irrational.

## 1 Introduction

Péter Komjáth has asked the following question in [1]: Let $X$ be a subset of Euclidean space $\mathbb{R}^n$. Is there always a $Y \subseteq X$ such that $X$ and $Y$ have same outer measure and the distance between any two distinct points of $Y$ is irrational? In [2] he showed that $\mathbb{R}^n$ can be colored by countably many colors such that the distance between any two points of the same color is irrational. It follows that one can always find a subset of positive outer measure that avoids rational distances. Under the assumption that there is no weakly inaccessible cardinal below the continuum, he also showed in [1] that in dimension one we can always find a subset $Y$ of full outer measure in $X$, avoiding rational distances. Moti Gitik and Saharon Shelah showed the following in [4], [5]: For any sequence $\langle A_n : n \in \omega \rangle$ of sets of reals, there is disjoint refinement of full outer measure; i.e., there is a sequence $\langle B_n : n \in \omega \rangle$ of pairwise disjoint sets such that $B_n \subseteq A_n$ and they have the same outer measure. It follows that one can omit integer distances in dimension one while preserving outer measure. Their proof employs one of their results about forcing with ideals that says that forcing with a sigma ideal cannot be isomorphic to a product of Cohen and Random forcings. Here we answer Komjáth's question positively in dimension one.

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## 2 A theorem of Gitik and Shelah

Suppose $A$ is a subset of $\mathbb{R}^n$. We say that $B \subseteq A$ is full in $A$ if the difference of env($A$) and env($B$) is null, where by env($X$) we denote a $G_\delta$ measurable envelope of $X$; i.e., env($X$) is a $G_\delta$ set containing $X$ such that the inner measure of env($X$)$\backslash X$ is zero. Note that if the outer measure of $A$ is finite, a subset $B \subseteq A$ is full in $A$ iff $A$ and $B$ have same outer measure.

Let $\mathcal{T}$ be a subtree of $\omega^{<\omega}$ such that every node in $\mathcal{T}$ has at least two children; i.e., for every $\sigma \in \mathcal{T}$, $|\{n \in \omega : \sigma n \in \mathcal{T}\}| \geq 2$.

**Definition 2.1.** *Call a family $\langle A_\sigma : \sigma \in \mathcal{T}\rangle$ of subsets of a set $A$, a full tree on $A$ if:*

- *$A = A_\phi$, and for every $\sigma \in \mathcal{T}$,*
- *$A_\sigma$ is a disjoint union of $A_{\sigma n}$'s where $\sigma n \in \mathcal{T}$ and*
- *$A_\sigma$ is full in $A$.*

The following application of Theorem 2.3 is implicit in [4]:

**Theorem 2.2.** *Let $A \subseteq \mathbb{R}^n$ and let $\langle A_\sigma : \sigma \in \mathcal{T}\rangle$ be a full tree on $A$. Then there is a $B \subseteq A$ full in $A$ such that for every $\sigma \in \mathcal{T}$, $A_\sigma \backslash B$ is full in $A_\sigma$.*

This theorem is a consequence of the following theorem in [5]:

**Theorem 2.3.** *Suppose $I$ is a sigma ideal over a set $X$. Then forcing with $I$ cannot be isomorphic to Cohen $\times$ Random.*

Let us explain how Theorem 2.2 follows from Theorem 2.3. It is clearly enough to show that there is a non null $X \subseteq A$ such that $A_\sigma \backslash X$ is full in $A_\sigma$ for every $\sigma \in \mathcal{T}$, for then the union $B$ of a maximal family $\{X_n : n \in \omega\}$ of such sets with pairwise disjoint envelopes will be as required. Suppose that this fails so that for every non null $X \subseteq A$, there is some $\sigma \in \mathcal{T}$ such that env($A_\sigma$) is strictly larger than env($A_\sigma \backslash X$). Consider the map that sends every positive outer measure subset $X \subseteq A$ to the supremum, in the complete Boolean algebra Cohen $\times$ Random, of all pairs $(\sigma, E)$ where $\sigma \in \mathcal{T}$ and $E$ is a positive measure Borel subset of env($A$) such that $E$ is disjoint with env($A_\sigma \backslash X$). This gives a dense embedding from $\mathcal{P}(A)$/Null to Cohen $\times$ Random contradicting the fact that they cannot be forcing isomorphic.

## 3 The main result

**Theorem 3.1.** *Let $X \subset \mathbb{R}$ be a set of positive outer measure. Then there is a $Y \subseteq X$ such that $Y$ is full in $X$ and the distance of any pair of distinct points in $Y$ is irrational.*

Proof of Theorem 3.1: Let $X_0$ be a set of representatives from the partition on $X$ induced by the equivalence relation $x \sim y$ iff $x - y$ is rational. Let $\langle r_n : n \geq 0 \rangle$ be a list of all rationals with $r_0 = 0$. For each $n \geq 0$, let $f_n : X_0 \to \mathbb{R}$ be defined by $f_n(x) = x + r_n$, if $x + r_n \in X$, otherwise $f_n(x) = x$, also put $X_n = \mathrm{range}(f_n)$. For $m, n \geq 0$, let $F_n^m = f_n \circ f_m^{-1} : X_m \to X_n$. Note that $f_n = F_n^0$. Also note that for every $m, n \geq 0$, $x \in X_m$, $F_n^m(x) = x + r$, for some $r \in \{0, r_n, -r_m, r_n - r_m\}$. This will allow us to use Lemma 3.3 below with $k \leq 4$.

We will inductively define a sequence $\langle K_n : n \geq 0 \rangle$ of pairwise disjoint subsets of $X_0$ such that for each $n \geq 0$, $f_n[K_n]$ is full in $X_n$. Theorem 3.1 will immediately follow by setting $Y = \bigcup\{f_n[K_n] : n \in \omega\}$. We need a definition for our next lemma.

**Definition 3.2.** *Let $Y \subseteq \mathbb{R}$ and $F : Y \to \mathbb{R}$. We say that $F$ is fullness preserving if whenever $W$ is a full subset of $Y$, $F[W]$ is a full subset of $F[Y]$.*

Observe that if $F : Y \to \mathbb{R}$ is fullness preserving, then for any $W \subseteq Y$ full in $Y$, $F \restriction W$ is also fullness preserving.

**Lemma 3.3.** *Suppose $F : Y \to \mathbb{R}$ acts by translating $k$ many pieces of $Y$; i.e., there are a partition $\{T_1, T_2, \ldots, T_k\}$ of $Y$ and reals $s_1, s_2, \ldots, s_k$ such that for every $x \in T_i$, $F(x) = x + s_i$. Then, there is another partition of size $k$, $\{Y_i : 1 \leq i \leq k\}$ of $Y$, such that for every $i \leq k$,*

- *$Y_i$ is full in $Y$ and*
- *$F \restriction Y_i$ is fullness preserving.*

Proof of Lemma 3.3: We will make several uses of the following result of Lusin: Any set of reals $X$ can be partitioned into two full subsets ([6]). Use induction on $k$. If $k = 1$, $Y_1 = Y$ works. So assume $k = l + 1$. Let $Z = \bigcup\{T_i : 1 \leq i \leq l\}$. Let $\{Z_i : 1 \leq i \leq l\}$ be a partition of $Z$ such that each $Z_i$ is full in $Z$ and $F \restriction Z_i$ is fullness preserving. Let $E_1 = \mathrm{env}(Z)$, $E_2 = \mathrm{env}(T_k)$ and $D = E_1 \bigcap E_2$. Let $W_1$, $W_2$ be a partition of $Z_1 \bigcap (E_1 \backslash D)$ into two full subsets. Let $\{V_j : 1 \leq j \leq k\}$ be a partition of $T_k \bigcap (E_2 \backslash D)$ into $k$ full subsets. Set $Y_1 = W_1 \bigcup (Z_1 \bigcap D) \bigcup V_1$. For $2 \leq i \leq l$, put $Y_i = Z_i \bigcup V_i$ and let $Y_k = W_2 \bigcup (D \bigcap T_k) \bigcup V_k$. Then $\{Y_i : 1 \leq i \leq k\}$ is a partition of $Y$ with the required properties. This finishes the proof of Lemma 3.3.

**Claim 3.4.** *There exists $K_0 \subseteq X_0$, such that $K_0$ is full in $X_0$ and for every $n \geq 1$, $X_n \backslash f_n[K_0]$ is full in $X_n$.*

Proof of Claim 3.4: Using Lemma 3.3, construct a full tree $\langle Y_\sigma : \sigma \in 2^{<\omega} \rangle$ on $Y = X_0$ such that for each $n \geq 1$, $\sigma \in 2^n$, $f_n \restriction Y_\sigma$ is fullness preserving.

Now Theorem 2.2 will imply that there is some $K_0 \subseteq X_0$, full in $X_0$, such that for every $\sigma \in 2^{<\omega}$, $Y_\sigma \backslash X_0$ is full in $Y_\sigma$. Fix any $n \geq 1$. Since for each $\sigma \in 2^n$, $f_n \restriction Y_\sigma$ is fullness preserving, we get that $f_n[Y_\sigma \backslash X_0]$ is full in $f_n[Y_\sigma]$. It follows that $X_n \backslash f_n[K_0] = \bigcup \{f_n[Y_\sigma \backslash X_0] : \sigma \in 2^n\}$ is full in $\bigcup \{f_n[Y_\sigma] : \sigma \in 2^n\} = X_n$. This finishes the proof of Claim 3.4.

Next suppose that for some $N \geq 1$, we have a pairwise disjoint family $\{K_i : 0 \leq i < N\}$ of subsets of $X_0$ such that

- for each $0 \leq i < N$, $f_i[K_i]$ is full in $X_i$ and
- for each $j \geq N$, $f_j[X_0 \backslash \bigcup \{K_i : 1 \leq i < N\}]$ is full in $X_j$.

Following the arguments in the proof of Claim 3.4, we first construct, using Lemma 3.3, a full tree $\langle Y_\sigma : \sigma \in 4^{<\omega} \rangle$ on $Y = f_N[X_0 \backslash \bigcup \{K_i : 1 \leq i < N\}]$ such that for each $n \geq 1$, $\sigma \in 2^n$, $F^N_{N+n} \restriction Y_\sigma$ is fullness preserving. Using Theorem 2.2, we get some $K \subseteq Y$, full in $Y$, such that for every $\sigma \in 4^{<\omega}$, $Y_\sigma \backslash K$ is full in $Y_\sigma$. Putting $K_N = f_N^{-1}[K]$ it follows that

- for each $0 \leq i < N$, $K_i \cap K_N = \phi$,
- $f_N[K_N]$ is full in $X_N$ and
- for each $j \geq N + 1$, $f_j[X_0 \backslash \bigcup \{K_i : 1 \leq i \leq N\}]$ is full in $X_j$.

  This concludes the proof of Theorem 3.1.

## 4 Concluding remarks

One can easily see that the above arguments can be applied to avoid any countable set of distances by replacing the rationals with the additive subgroup generated by this countable set. One can also obtain a category analogue in the following sense: Let $X \subseteq \mathbb{R}$. Then there is a subset $Y \subseteq X$ such that $Y$ is everywhere non meager in $X$ and the distance between any two distinct points of $Y$ is irrational. Here we call a subset $Y \subseteq X$ everywhere non meager in $X$ if for every open set $U$, if $X \cap U$ is non meager then $Y \cap U$ is also non meager. The proof follows essentially the same lines except that one has to use a category analogue of Theorem 2.2 which depends on the following result of Gitik and Shelah ([3]): Suppose $I$ is a sigma ideal over a set $X$. Then forcing with $I$ cannot be isomorphic to Cohen forcing. We do not know the answer to Komjáth's question (and its category analogue) in higher dimensions.

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