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ATOMS AND SINGULAR INTEGRALS ON COMPLEX DOMAINS

Abstract

We study spaces of homogeneous type, and especially the theory of atoms, on the boundary of a domain in $\mathbb{C}^n$. We are particularly interested in atoms for small p, which must satisfy a higher-order moment condition. We have an axiomatic presentation of these ideas which avoids a lot of the usual nasty calculations. Examples show that this new theory is consistent with existing particular instances of atoms.

1 Introduction

There is considerable interest in developing the harmonic analysis of domains in $\mathbb{C}^n$. Pioneering work in this direction was done in [12], [13], [23]. Further studies occur in [17], [18], [19], [20], for instance.

Of particular interest in this endeavor is the study of the boundary behavior of H^p functions, and the action of singular integrals on the boundary. The situation is subtle because of the Levi geometry of the boundary. If the domain in question is the unit ball B, then each boundary point has the same geometry. If instead the domain Ω is strongly pseudoconvex, then boundary points are generically biholomorphically distinct (see [10]), but they are comparable from a variety of different viewponts.

The next level of complexity is finite type domains. In this situation, the boundary geometry varies in a semicontinuous fashion that is quite subtle. Strongly pseudoconvex points are still generic, but the points of type greater than 2 exert a strong influence over the shapes of boundary balls and approach regions for Fatou-type theorems. See [8] for the full story of this geometry.

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The calculations on the boundary of a finite-type domain, even in $\mathbb{C}^2$, can be quite subtle and technical.

In this paper we present an axiomatic geometric means to approach these problems that avoids a lot of the difficult calculations but still yields useful results. Specific examples show that this new approach is consistent with classical results obtained on the types of domains described above. Part of the inspiration for this work comes from [16] and also from [15]. The work [14] is a good general reference for the ideas discussed here.

It is a pleasure to thank the referee for a very careful reading of this paper, and for contributing a number of useful insights.

2 Basic principles

Given a domain $\Omega \subseteq \mathbb{C}^n$, we wish to think of $\partial\Omega$ as a space of homogeneous type (see [6, p 66]). This of course is an important device in the harmonic analysis of a space.

DEFINITION 1. Let X be a set and μ a measure on X. We call (X, μ) a *space of homogeneous type* if there is a collection of balls $B(P, r) \subseteq X$ satisfying

(a) $0 < \mu(B(P, r)) < \infty$ for every $P \in X$ and $r > 0$;

(b) There is a $C_1 > 0$ so that $\mu(B(P, 2r)) \leq C_1 \cdot \mu(B(P, r))$ for every $P \in X$ and $r > 0$;

(c) There is a $C_2 > 0$ so that if $B(Q, s) \cap B(P, r) \neq \emptyset$ and $s \geq r$ then $B(Q, C_2 s) \supseteq B(P, r)$.

In many instances the balls come from a metric, in which case Axiom **(c)** is automatically satisfied because of the triangle inequality.

DEFINITION 2. Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. We write

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\} ,$$

where ρ is a real function with nonvanishing gradient on $\partial\Omega$.

We say that a point $P \in \partial\Omega$ is *Levi pseudoconvex* if, for each $w = (w_1, w_2, \ldots, w_n)$ satisfying

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P) w_j = 0 ,$$

we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(P) w_j \overline{w}_k \geq 0 .$$

See [14, §3.2] for more about Levi pseudoconvexity.

In what follows, we shall be working with a fixed, smoothly bounded, Levi pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$.

DEFINITION 3. Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. The *Szegő kernel* for Ω, denoted by $S(z, \zeta)$, is the reproducing kernel for the Hardy space $H^2(\Omega)$. Thus

$$f(z) = \int_{\partial\Omega} f(\zeta) S(z, \zeta)\, d\sigma(\zeta)$$

for $f \in H^2$ and $z \in \Omega$. Furthermore, integration against $S(z, \zeta)$ gives the projection from $L^2(\partial\Omega)$ to H^2. [Note here that $d\sigma$ is the area measure on the boundary.] [14, §1.5] treats this idea.

DEFINITION 4. Let Ω be as above and $P \in \partial\Omega$. Let $r > 0$. Let $S(z, w)$ be the Szegő kernel for Ω. *We assume that S extends to be continuous on $(\overline{\Omega} \times \overline{\Omega}) \setminus \triangle$, where $\triangle$ is the boundary diagonal.* [Pseudolocality of the $\overline{\partial}_b$-Neumann problem is sufficient for this last to hold.]

For $P \in \partial\Omega$, define the ball

$$\beta(P, r) = \{\zeta \in \partial\Omega : |S(P, \zeta)| > 1/r\} \cup \{P\}\,.$$

[The adding on of $\{P\}$ here is somewhat redundant.]

If ζ^1, ζ^2 are points of $\partial\Omega$, then let us say the $\delta(\zeta^1, \zeta^2) < 2r$ if there is a ball $\beta(P, r)$, with $P \in \partial\Omega$, such that $\zeta^1 \in \beta(P, r)$ and $\zeta^2 \in \beta(P, r)$. We can define $\delta(\zeta^1, \zeta^2)$ by taking the infimum over r. In other words,

$$\delta(\zeta^1, \zeta^2) = \inf\{r > 0 : \zeta^1, \zeta^2 \in \beta(P, r) \text{ for some } P \in \partial\Omega\}\,.$$

If ζ^1, ζ^2 are points of $\partial\Omega$, then a *chain* from ζ^1 to ζ^2 is a sequence of points $p^0, p^1, \ldots, p^k$ such that $p^0 = \zeta^1$ and $p^k = \zeta^2$.

DEFINITION 5. If $\zeta^1, \zeta^2 \in \partial\Omega$, then we set

$$d(\zeta^1, \zeta^2) = \inf \left\{ \sum_{j=1}^{k} \delta(p^{j-1}, p^j) : \{p^j\}_{j=0}^k \text{ is a chain from } \zeta^1 \text{ to } \zeta^2 \right\}.$$

It is automatic that d is a metric. In particular, the triangle inequality follows just from definition chasing.

DEFINITION 6. If $P \in \partial\Omega$ and $r > 0$, then define

$$B(P, r) = \{\zeta \in \partial\Omega : d(P, \zeta) < r\}\,.$$

Equipping $\partial\Omega$ as above with surface measure (i.e., $(2n-1)$-dimensional Hausdorff measure—see [9, p 171]), we wish to claim that $\partial\Omega$ with the balls $B(P,r)$ forms a space of homogeneous type. Property **(a)** is obvious, as any open set in $\partial\Omega$ has positive, finite measure. Property **(c)** is also clear, because the balls $B(P,r)$ come from a metric (just use the triangle inequality). Verification of property **(b)** requires a bit of work.

We need to observe that $\partial\Omega$, equipped with the metric d, is a directionally limited metric space in the sense of Federer [9, p 146]. This means that there is an *a priori* constant $K>0$ so that, if $S(P,r)$ is a sphere in $\partial\Omega$ of radius r, then there can be at most K points in $S(P,r)$ that are spaced at least r apart. This assertion follows because the Cauchy estimates (see [11, p 87]) give us an *a priori* upper bound on the boundary growth of the Szegő kernel, hence a lower bound on the size of the balls $\beta(P,r)$.

As a consequence, the Besicovitch covering theorem (see [9, p 147] and, in particular, see the *proof* of the Besicovitch covering theorem) are valid on $\partial\Omega$ with the metric and balls as indicated—see Section 3 (particularly Proposition 3.1) below.[1] So we know that, if $B(P,r)$ is a fixed ball in $\partial\Omega$, then there are at most K' pairwise disjoint balls $B(P_j,r)$ which touch $B(P,r)$. By the triangle inequality, all these balls are contained in $B(P,3r)$. Further, if x is a point of $B(P,3r)$ that is not contained in $B(P,r)$ nor in any $B(P_j,r)$ then $B(x,r)$ will intersect one of those balls. As a result,

$$B(P,3r)\subseteq \cup_{j=1}^{K'} B(P_j,2r)\,.$$

We see therefore that

$$\mu(B(P,3r)\le \sum_{j=1}^{K'}\mu(B(P_j,2r))\,.$$

This is a version of property **(b)**. Thus we have:

Modified Axioms for a Space of Homogeneous Type

(a) $0<\mu(B(P,r))<\infty$ for every $P\in X$ and $r>0$;

(b') Given a ball $B(P,r)$, we can find pairwise disjoint balls $B(P_j,r)$, $j=1,2,\ldots,K'$, such that

$$B(P,3r)\subseteq \cup_{j=1}^{K'} B(P_j,2r)\,;$$

[1] One of the charming features of the Besicovitch covering theorem is that there is no measure involved in either its statement or its proof.

(c) There is a $C_2 > 0$ so that if $B(Q,s) \cap B(P,r) \neq \emptyset$ and $s \geq r$ then $B(Q, C_2 s) \supseteq B(P,r)$.

While a bit more technical, these axioms will suffice for the results that we wish to prove below.

So we see that $\partial\Omega$ is a space of homogeneous type with the modified axioms.

3 A covering theorem

Because we have a directionally limited metric space, we can use standard arguments (again see [9]) to prove the following covering lemma of Besicovitch:

Proposition 1. *There is a number $M > 0$ with the following property. Let $B(P_1, r_1), B(P_2, r_2), \ldots, B(P_k, r_k)$ be balls in our metric space X with the property that no ball contains the center of any other (given a covering, it is always possible to extract a refinement with this property). Then our collection of balls can be partitioned into a union of at most M subfamilies so that each subfamily is pairwise disjoint.*

Now a standard argument shows that the Hardy-Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(t)|\, dt$$

is weak-type $(1,1)$ (see [STW, p. 184] for this concept). The operator $\mathcal{M}$ is trivially bounded on L^∞. By Marcinkiewicz interpolation (see [24, p 183]), we find that $\mathcal{M}$ is bounded on L^p for $1 < p \leq \infty$.

4 Singular integrals

Of course the Szegő kernel is created in the context of $H^2(\partial\Omega) \subseteq L^2(\partial\Omega)$. So it is automatic that the Szegő integral is bounded on L^2. There is some interest in seeing that the Szegő integral S maps L^p to L^p for $1 < p < \infty$ and also maps H^p to H^p for $0 < p \leq 1$. We shall first treat the case of the Hardy spaces. Ideally one would then like to apply an interpolation of operators theorem to the spaces H^p and L^2. Unfortunately such interpolation theorems do not exist in the generality that we are treating here. So we shall have to treat the L^p spaces separately.

We first need to define the real-variable Hardy spaces. This is done using the atomic theory (see [7, p 573]). In fact Coifman and Weiss only treat the atomic theory on a general space of homogeneous type for p less than or equal to 1 but close to 1. This is because atoms for that range of p require only a very simple mean-value property. Smaller p require a more sophisticated mean-value property, and it is not at all clear how to define such a property in a general context. The classical means of formulating the mean-value property is in terms of orthogonality to certain polynomials. But what is a polynomial on a general space of homogeneous type? One of our main purposes here is to present a new way to think about the mean value property that will work in considerable generality. These ideas are inspired by work in [16, p 165].

5 Atoms

First we treat the case $p = 1$. Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain for which the Szegő kernel extends to the boundary as described in Section 2 above. We use the space-of-homogeneous-type structure as also described in Section 2. We say that a measurable function a on X is a *1-atom* if

(a) a is supported in a ball $B(x, r)$;

(b) $|a(t)| \leq \dfrac{1}{\mu(B(x, r))}$ for all $t \in B(x, r)$;

(c) $\displaystyle\int a(x)\, d\mu(x) = 0$.

We see here in part **(c)** the very simple mean value property that is suitable for $p = 1$ and also for p less than 1 but very near[2] to 1. See [7] for a consideration of this point, and of the limitations on p.

We wish now to give a definition of atom for the full range of p, $0 < p \leq 1$. If a is to be an atom supported in $B(x, r)$, let $\mu(B(P, r)) = r^\alpha$. Of course the choice of α may depend on P. Then set

$$k = \left[\alpha\left(\frac{1}{p} - 1\right)\right] .$$

We demand as usual that $|a(t)| \leq 1/\mu(B(x, r))^{1/p}$. And the mean-value property now is

$$\int_{B(x,r)} a(t)\varphi(t)d\mu(t) \leq \|\varphi\|_{C^k}$$

[2]This idea is treated in considerable detail in [KRA3, §2.5, 2.6]. In $\mathbb{R}^N$, it is required that p be greater than $(N - 1)/N$. The issue is that, for this restricted range of p, an atom is only required to have mean value 0. For smaller p, some orthogonality to polynomials is needed. In fairly general settings, it is difficult to say what these polynomials should be, and what form the orthogonality should take.

for any $\varphi \in C_c^k(B(x,r))$.

An H^p_{Re} distribution is then any infinite linear combination of p atoms, with convergence in the distribution topology.

DEFINITION 7. Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Fix a $P \in \partial\Omega$. We say that Ω is *regular* if its Szegő kernel satisfies an estimate of the form

$$\|S(z,\,\cdot\,)\|_{C^k(B(P,r))} \le d(z,\,\cdot\,)^{-\alpha-k} r^\alpha \,,$$

for all positive integers k, as long as $d(z,P) \ge 3r$.

Given that the balls β are defined in terms of the Szegő kerneal, this condition is plausible. The reference [21] gives a specific instance (finite type domains in $\mathbb{C}^2$) where the condition may be verified concretely—see particularly Section 5 of that paper, Theorem 5.1, and the discussion at the bottom of page 133. Of course the regular property also holds on strongly pseudoconvex domains, as the results of [5] show. And it holds on the disc in $\mathbb{C}$.

Now our principal result is this:

Theorem 2. *Let $0 < p \le 1$. Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ which is regular. Then the Szegő integral maps $H^p(\partial\Omega)$ to $H^p(\partial\Omega)$.*

PROOF. Fix p and let a be a p-atom. Assume that a is supported on the ball $B(P,r)$. We take the σ-measure of $B(P,r)$ to be some positive number r^α.

We will of course make good use of the fact that the Szegő integral is bounded on $L^2(\partial\Omega)$.

We divide the calculation into two parts:

$$\begin{aligned}
& \int_{d(z,P)<3r} \left| \int_{\partial\Omega} a(\zeta) S(z,\zeta)\, d\sigma(\zeta) \right|^p \, d\sigma(z) \\
\le\ & \int_{d(z,P)<3r} \left| \int_{\partial\Omega} a(\zeta) S(z,\zeta)\, d\sigma(\zeta) \right|^2 \, d\sigma(z)^{p/2} \\
& \qquad \cdot \int_{d(z,P)<3r} 1^{2/(2-p)} \, d\sigma(z)^{(2-p)/2} \\
\le\ & \left\| \int a(\zeta) S(z,\zeta)\, d\sigma(\zeta) \right\|_{L^2}^p \cdot C \cdot (r^\alpha)^{(2-p)/2} \\
\le\ & C' \cdot \|a\|_{L^2}^p \cdot r^{\alpha(2-p)/2} \\
\le\ & C'' \cdot \left(r^{-\alpha/p} \cdot r^{\alpha/2} \right)^p \cdot r^{\alpha - \alpha p/2} \\
\le\ & C'' \,.
\end{aligned}$$

Our second calculation goes as follows. For $d(\zeta,P) > 3r$, we let $\psi \in C_c^\infty(\partial\Omega)$ be such that

(a) $\psi(\zeta) \equiv 1$ for $d(\zeta, P) \le r$;

(b) $\psi(\zeta) \equiv 0$ for $d(\zeta, P) > 3r/2$.

Now we have:

$$\begin{aligned}
&\int_{d(z,P)>3r} \left| \int_{\partial\Omega} a(\zeta) S(z,\zeta)\, d\sigma(\zeta) \right|^p d\sigma(z) \\
&\quad = \int_{d(z,P)>3r} \left| \int_{d(\zeta,P)\le 3r/2} a(\zeta)|^p |\psi(\zeta) S(z,\zeta)\, d\sigma(\zeta) \right|^p d\sigma(z).
\end{aligned}$$

Observe that, for fixed z with $d(z, P) > 3r$, the function

$$\zeta \longmapsto \psi(\zeta) S(z, \zeta)$$

lies in $C_c^\infty(P, 2r)$. So we may use the vanishing moment condition on the atom a to estimate our integral as follows:

$$\begin{aligned}
&\int_{d(z,P)\ge 3r} \left\{ \|\psi(\,\cdot\,) S(z,\,\cdot\,)\|_{C^k(\overline{B}(P,r))} \cdot r^{k-\alpha/p} \right\}^p d\sigma((z) \\
&\le \int_{d(z,P)\ge 3r} \left\{ d(z,\,\cdot\,)^{-\alpha-k} \cdot r^{\alpha-\alpha/p+k} \right\}^p d\sigma(\zeta) \\
&\le C \cdot r^{-\alpha p - kp + \alpha} \cdot r^{\alpha p - \alpha + kp}.
\end{aligned}$$

Here we have used the regular property of the domain. More specifically, if the derivative falls on the cutoff function then the estimate holds by inspection; if instead the derivative falls on the kernel then the estimate is precisely the regular property. □

In the next section we treat L^p estimates for the Szegő integral when $1 < p < \infty$. We shall use the Lions-Peetre K-functional,[3] which is an instance of the real interpolation method (see, for instance, [3, pp 38, 52]).

6 L^p estimates

We begin by proving an interpolation result that is of independent interest.

Proposition 3. *Let T be a linear operator that is bounded on H^1_{Re} and bounded on L^2. Then T is bounded on L^p for $1 < p < 2$.*

[3] This is a rather technical device. The idea is that one can decompose a function in an intermediate space L^p into a summand which is in H^1_{Re} and a summand which is in L^2. The construction in the next section illustrates the idea.

Proof. Fix a p, $1 < p < 2$. Then we shall establish the following two assertions.

(a) To show that, if $f \in L^p(\partial\Omega)$ and $K \geq 1$, then there exist $f_1 \in H^1_{\mathrm{Re}}$ and $f_2 \in L^2(\partial\Omega)$ with $f = f_1 + f_2$ and $\|f_1\|_{H^1} \leq K^{1-p}$ and $\|f_2\|_{L^2} \leq K^{2-p}$;

(b) To show that, if $f_1 \in H^1_{\mathrm{Re}}$ with $\|f_1\|_{H^1} \leq K^{1-p}$ and $f_2 \in L^2$ with $\|f_2\|_{L^2} \leq K^{2-p}$ then $f = f_1 + f_2 \in L^p$.

Proposition 6.1 then follows from standard techniques (see [3]). We now proceed to establish these facts **(a)** and **(b)**.

Fix $1 < p < 2$ and let $f(\zeta) = \chi_{B(P,r)}$. Let B_a be that portion of the ball $B(P,r)$ with $\mathrm{Re}\,(\zeta_1 - P_1) < \epsilon$ and B_b be that portion of the ball $B(P,r)$ with $\mathrm{Re}\,(\zeta_1 - P_1) \geq \epsilon$. We choose ϵ so that these two sub-balls have the same σ-measure (although, in the end, this will not really be very important—it is just convenient). Then we define

$$f_1(\zeta) = K^{2-p+1}\chi_{B(P',K^{-2})} - K^{2-p+1}\chi_{B(P'',K^{-2})} .$$

Here the points P' and P' are points near P so that the two balls $B(P', K^{-2})$ and $B(P'', K^{-2})$ are disjoint. Also we define

$$f_2(\zeta) = f - f_1 .$$

Then it is easy to calculate that $\|f_1\|_{H^1} \approx K^{1-p}$ and $\|f_2\|_{L^p} \approx K^{2-p} + 1$. These are the sorts of estimates that we want for this particularly simple f. But a perfectly arbitrary $f \in L^p$ can be approximated in norm by a finite linear combination of such functions with disjoint supports. So the result follows for general $f \in L^p$.

For the converse direction, suppose that $f = f_1 + f_2$ with $f_1 \in H^1_{\mathrm{Re}}$, $f_2 \in L^2$, $\|f_1\|_{H^1} \leq K^{1-p}$, and $\|f_2\|_{L^2} \leq K^{2-p}$. Define

$$\varphi(\zeta) = K^{p-1}\chi_{B(P,K^{-2})} , \tag{$\star$}$$

where P is an arbitrarily chosen point in $\partial\Omega$. Then it is easy to calculate that

$$\begin{aligned} \int f(\zeta)\varphi(\zeta)\,d\sigma(\zeta) &= \int f_1(\zeta)\varphi(\zeta)\,d\sigma(\zeta) + \int f_2(\zeta)\varphi(\zeta)\,d\sigma(\zeta) \\ &\leq \|f_1\|_{H^1} \cdot \|\varphi\|_{BMO} + \|f_2\|_{L^2}\|\varphi\|_{L^2} \\ &\leq C \cdot K^{1-p} \cdot K^{p-1} + C \cdot K^{2-p} \cdot K^{p-2} \\ &\leq C . \end{aligned}$$

This is the sort of estimate we seek, but only for the special φ indicated in line ($\star$). But in point of fact any function in BMO or L^2 can be approximated in norm by a finite linear combination of such functions with disjoint support. This is a standard idea from measure theory, but see [25]. So that establishes the result for general $\varphi \in L^{p'}(\partial\Omega)$.

We have established the standard K-functional decomposition of Lions and Peetre (see [3]). That in turn gives the interpolation result that we seek. So the Szegő integral is bounded on L^p, $1 < p < 2$. By duality (as noted above), the Szegő integral is also bounded on L^p, $2 < p < \infty$. □

We shall take it (the matter is treated in the last section) that the Szegő integral maps H^1_{Re} to L^1 and L^2 to L^2. Now we prove the following:

Theorem 4. *The Szegő integral maps $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$ for $1 < p \leq 2$. It also maps H^1_{Re} to H^1_{Re}.*

PROOF. We know from the last section that the Szegő projection is bounded on H^1_{Re} and L^2. The result now follows from Proposition 6.1. □

Theorem 5. *The Szegő integral maps $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$ for $1 < p < \infty$. It also maps H^1_{Re} to H^1_{Re}.*

PROOF. The Szegő projection operator is self-adjoint. So a standard duality argument allows us to derive from Theorem 6.2 that the Szegő operator maps L^p to L^p for $2 < p < \infty$. □

One could also use duality to prove that the Szegő integral maps BMO to BMO, but we shall not treat that matter here.

7 Concluding remarks

This paper has presented an abstract and relatively soft way to look at the question of L^p boundedness of the Szegő integral on domains in $\mathbb{C}^n$. Of course there are some ambient hypotheses, so the results are not perfectly general.

We hope to explore other, and more general, versions of these results in future work.

References

[1] S. R. Barker, *Two theorems on boundary values of analytic functions*, Proc. Amer. Math. Soc., **68** (1978), 48–54.

[2] Tomek Bartoszyński and Boaz Tsaban, *Hereditary Topological Diagonalizations and the Menger-Hurewicz Conjectures*, Proc. Amer. Math. Soc., to appear.

[3] J. Bergh and J. Löfstrom, *Interpolation Spaces*, Springer-Verlag, Berlin, 1976.

[4] A. S. Besicovitch, *On density of perfect sets*, J. Lond. Math. Soc., **31** (1956), 48–53.

[5] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et Szegö,* Astérisque, **34-35** (1976), 123–164.

[6] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, Springer Lecture Notes **242**, Springer Verlag, Berlin, 1971.

[7] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83(4)** (1977), 569–645.

[8] F. Di Biase and S. G. Krantz, *The Boundary Behavior of Holomorphic Functions*, Birkhäuser, Boston, 2013.

[9] H. Federer, *Geometric Measure Theory*, Springer Verlag, New York, 1969.

[10] R. E. Greene and S. G. Krantz, *Deformations of complex structures, estimates for the $\overline{\partial}-$ equation, and stability of the Bergman kernel*, Adv. Math., **43** (1982), 1–86.

[11] R. E. Greene, *Function Theory of One Complex Variables*, John Wiley and Sons, New York, 1997.

[12] A. Koranyi, *Harmonic functions on Hermitian hyperbolic space*, Trans. Amer. Math. Soc., **135** (1969), 507–516.

[13] A. Koranyi, *Boundary behavior of Poisson integrals on symmetric spaces*, Trans. Amer. Math. Soc., **140** (1969), 393–409.

[14] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[15] S. G. Krantz, *Invariant metrics and the boundary behavior of holomorphic functions on domains in* $\mathbb{C}^n$, J. Geom. Anal., **1** (1991), 71–98.

[16] S. G. Krantz, *Geometric Analysis and Function Spaces*, CBMS and the American Mathematical Society, Providence, 1993.

[17] S. G. Krantz and S.-Y. Li, *A Note on Hardy Spaces and Functions of Bounded Mean Oscillation on Domains in* $\mathbb{C}^n$*,* Michigan Math. J., **41** (1994), 51–72.

[18] S. G. Krantz and S.-Y. Li, *On decomposition theorems for Hardy spaces on domains in* $\mathbb{C}^n$ *and applications,* J. Fourier Anal. Appl., **2** (1995), 65–107.

[19] S. G. Krantz and S.-Y. Li, *Duality theorems for Hardy and Bergman spaces on convex domains of finite type in* $\mathbb{C}^n$*,* Ann. Inst. Fourier (Grenoble), **45** (1995), 1305–1327.

[20] S. G. Krantz and S.-Y. Li, *Area integral characterizations of functions in Hardy spaces on domains in* $\mathbb{C}^n$*,* Complex Var. Elliptic Equ., **32** (1997), 373–399.

[21] A. Nagel, E. M. Stein, J.-P. Rosay, and S. Wainger, *Estimates for the Bergman and Szegö kernels in* $\mathbb{C}^2$, Ann. of Math., **129** (1989), 113–149.

[22] A. Nagel, E. M. Stein, and S. Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math., bf 155 (1985), 103–147.

[23] E. M. Stein, *Boundary Behavior of Holomorphic Functions of Several Complex Variables*, Princeton University Press, Princeton, 1972.

[24] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.

[25] M. Taibleson and G. Weiss, *Spaces generated by blocks,* Probability Theory and Harmonic Analysis, Dekker, New York, 1986.

[26] A. Zygmund, *Trigonometric Series*, 2nd ed., **1** and **2**, Cambridge University Press, Cambridge, (1959) reprinted (1993).