RESEARCH

**Real Analysis Exchange**
Vol. 38(2), 2012/2013, pp. 391–408

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# MULTI-FRACTAL ANALYSIS OF CONVOLUTION POWERS OF MEASURES

**Abstract**

We investigate the multi-fractal analysis of (large) convolution powers of probability measures on $\mathbb{R}$. If the measure $\mu$ satisfies $(N)$ $\operatorname{supp}\mu = [0, N]$ for some $N$, then under weak assumptions there is an isolated point in the multi-fractal spectrum of $\mu^n$ for sufficiently large $n$. A formula is found for the limiting behaviour (as $n \to \infty$) of the $L^q$-spectrum of $\mu^n$ and this is related to the limit of the energy dimension of $\mu^n$ when $q \geq 1$.

## 1 Introduction

For a probability measure $\mu$, by the local dimension of $\mu$ at $x$ in its support, we mean the value

$$\dim \mu(x) := \lim_{r \to 0^+} \frac{\log \mu(B(x,r))}{\log r},$$

provided this limit exists. For measures that are not 'uniform' it is of interest to determine which values arise as local dimensions, the so-called multi-fractal spectrum, and to calculate $f_\mu(\alpha) = \dim_H\{x : \dim \mu(x) = \alpha\}$, the dimension spectrum. These spectra have been calculated for many classes of measures,

Mathematical Reviews subject classification: Primary: 28A80; Secondary: 42A85
Key words: multi-fractal analysis, local dimension, $L^q$-spectrum, self-similar measure
Received by the editors October 2, 2012
Communicated by: Zoltán Buczolich
*This research was supported in part by NSERC.
†This research was supported in part by NSERC.
This paper is in final form and no version of it will be submitted for publication elsewhere.

including self-similar measures which satisfy the open set condition and $p$-Cantor measures on central Cantor sets which satisfy an analogous separation condition (c.f. [2], [4], [12], [16]). For these classes of measures the multi-fractal spectrum is a closed interval and the function $f_\mu(\alpha)$ is the Legendre transform of the *$L^q$-spectrum*, $\tau_\mu(q)$, defined as

$$\tau_\mu(q) := \liminf_{r\to 0^+} \frac{\log(S_{r,\mu}(q))}{\log r} \text{ with } S_{r,\mu}(q) := \sup \sum \mu(B(x_i,r))^q,$$

where $\{B(x_i,r)\}_i$ is a countable family of disjoint balls centred at $x_i \in \operatorname{supp}\mu$ (we call this a *centred $r$ -packing of supp$\mu$*) and the supremum is taken over all such families. This is known as the multi-fractal formalism.

In surprising contrast, Hu and Lau in [13] discovered that the multi-fractal spectrum of the three-fold convolution of the classical Cantor measure (a self-similar measure not satisfying the OSC) is not an interval, but the union of an interval and an isolated point, the local dimension at 0. Further examples of self-similar measures exhibiting this phenomena were given by Shmerkin in [18], while in [8] it was shown that for quite general Cantor measures $\mu$, defined on Cantor sets with ratios of dissection bounded away from zero, $\dim \mu^n(0)$ is isolated in the multi-fractal spectrum provided $n$ is sufficiently large. An important ingredient in the proof was the fact that such Cantor sets $C$ have the property that $(N)C = [0,N]$ for sufficiently large $N$. In the first theorem of this paper we will show that this is the salient feature by proving that $\dim \mu^n(0)$ is always an isolated point in the multi-fractal spectrum of $\mu^n$ for sufficiently large $n$, provided that $\mu$ is a continuous, probability measure on $[0,1]$ with $(N)\operatorname{supp}\mu = [0,N]$ for some $N$, $\dim \mu(0) > 0$ and $\sup \dim \mu(x) < \infty$.

For self-similar measures which do not satisfy the OSC determining the $L^q$-spectrum is quite difficult. This has been done in [7], [14] and [18] for the three-fold convolution of the Cantor measure and small numbers of convolutions of certain other examples of self-similar measures.

Here we investigate what happens when the number of convolution powers is very large. We show that if $\mu$ is any compactly supported, probability measure on $\mathbb{R}$ and $q \geq 0$, then $\lim_{n\to\infty} \tau_{\mu^n}(q)$ exists. When $q \geq 1$, this limit is equal to $K(q-1)$ where $K$ is the limit of the energy dimensions of $\mu^n$. All values of $K \in [0,1]$ are possible and $K = 1$ in many important examples, including the Cantor measure and the self-similar measures studied in [18]. If $q < 0$, the results are not as complete. However we do show that if $\mu$ is a probability measure whose support is $[0,1]$, then for $q \leq -1$, the limit of $\tau_{\mu^n}(q) - nq \dim \mu(0)$ exists, and equals 0 for the standard Cantor measure and other similar examples.

## 2 Isolated points in the multi-fractal spectrum of convolutions of measures

By a measure we mean a compactly supported, finite, positive measure on $\mathbb{R}$. Given a measure $\mu$, we write $\mu^n$ to denote the $n$ 'th convolution power of $\mu$. As the functions of interest to us are unchanged under rescaling of the measure, there is no loss in assuming the measures are probability measures. In some cases we assume the measure is continuous, meaning non-atomic.

**Definition 1.** The upper local dimension of a measure $\mu$ at $x \in \operatorname{supp}\mu$ is defined as

$$\overline{\dim}\mu(x) = \limsup_{r \to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

The lower local dimension is defined similarly. If the upper and lower local dimensions are equal we write $\dim \mu(x)$ and call this the local dimension of $\mu$ at $x$.

We begin by proving that many measures have the property that sufficiently large convolutions of the measure has isolated points in the set of its upper local dimensions. By the notation $(N)\operatorname{supp}\mu$ we mean the $N$-fold sum of $\operatorname{supp}\mu$.

**Theorem 1.** *Suppose $\mu$ is a continuous, probability measure supported on $[0,1]$ with $\sup\{\overline{\dim}\mu(x) : x \in supp\mu\} < \infty$ and $\overline{\dim}\mu(0) > 0$. In addition, assume that $0, 1 \in supp\mu$ and $(N)supp\mu = [0,N]$ for some $N$.*

(i) *There exists a positive integer $N_0$ such that for all $n \geq N_0$, $\overline{\dim}\mu^n(0)$ is isolated in the set of upper local dimensions of $\mu^n$.*

(ii) *There exist $N_0 \in \mathbb{N}$ and $q_0 < 0$ such that if $q < q_0$, $n \geq N_0$ and $\delta > 0$, then*

$$\tau_{\mu^n}(q) = \liminf_{r \to 0^+} \frac{\log(S'_r(q))}{\log r} \text{ with } S'_r(q) = \sup \sum \mu^n(B(x_i,r))^q,$$

*where $\{B(x_i,r)\}_i$ is a centred $r$-packing of $[0,\delta] \cup [n-\delta, n]$.*

We begin with a lemma.

**Lemma 1.** *Suppose $\mu$, $\nu$ are measures with $supp\nu = [0,n]$ and $0, 1 \in supp\mu \subseteq [0,1]$.*

(i) *If $\overline{\dim}\nu(x) \leq \lambda < \infty$ for all $x \in [0,n]$, then $\overline{\dim}\nu * \mu(z) \leq \lambda$ for all $z \in (0, n+1)$.*

(ii) *If, in addition, $\mu$ is a continuous measure, the same conclusion holds under the weaker assumption that $\overline{\dim}\nu(x) \leq \lambda$ for all $x \in (0,n)$.*

Proof. (i) Fix $z \in (0, n+1)$ and let $I = [0,1] \cap [z-n, z]$. Note that if $x \in I$, then $z - x \in [0, n]$ and (at least) one of $0, 1 \in I$. Since $z \neq 0, n+1$, $I$ has non-empty interior. We have $\mu(I) = \delta > 0$ since both $0, 1$ belong to the support of $\mu$.

Fix $\varepsilon > 0$. Since $\overline{\dim}\nu(z-x) \leq \lambda$ for every $x \in I$, by continuity of measure there exists $A \subseteq I$ and $r_0$ such that $\mu(A) \geq \delta/2$ and

$$\frac{\log \nu(B(z-x, r))}{\log r} \leq \lambda + \varepsilon$$

for all $r \leq r_0$ and for all $x \in A$. Equivalently, $\nu(B(z-x, r)) \geq r^{\lambda+\varepsilon}$ for all $r \leq r_0$ and $x \in A$. Thus, for all $r \leq r_0$,

$$\begin{aligned}\nu * \mu(B(z,r)) &= \int \nu(B(z-x,r))d\mu(x) \geq \int_A \nu(B(z-x,r))d\mu(x) \\ &\geq \int_A r^{\lambda+\varepsilon} d\mu(x) \geq r^{\lambda+\varepsilon}\delta/2.\end{aligned}$$

This clearly implies $\overline{\dim}\nu * \mu(z) \leq \lambda + \varepsilon$ and as $\varepsilon > 0$ was arbitrary, the result follows.

(ii) The same conclusion holds for a continuous measure $\mu$ under the weaker assumption that $\overline{\dim}\nu(x) \leq \lambda$ for all $x \in (0, n)$ because

$$\int_A \nu(B(z-x,r))d\mu(x) = \int_{A \setminus \{z-n, z\}} \nu(B(z-x,r))d\mu(x).$$

□

Here is another useful general fact.

**Lemma 2.** *If $\nu$ is a measure supported on $[0,1]$, then $\overline{\dim}\nu^m(0) = m\overline{\dim}\nu(0)$.*

Proof. Since $x_1 + \cdots + x_m \in B(0,r)$ whenever $x_j \in B(0, r/m)$, it follows that $\nu^m(B(0,r)) \geq (\nu(B(0, r/m)))^m$. Conversely, since $\text{supp}\nu \subseteq [0,1]$, $x_1 + \cdots + x_m \in B(0,r)$ only if $x_j \in B(0,r)$ for all $j$. Thus $\nu^m(B(0,r)) \leq (\nu(B(0,r)))^m$ and so

$$\frac{m \log(\nu(B(0,r)))}{\log r} \leq \frac{\log \nu^m(B(0,r))}{\log r} \leq \frac{m \log \nu(B(0, r/m))}{\log r}.$$

Now take the $\limsup$ as $r \to 0^+$. □

Proof of Theorem. (i) It is shown in [8] that if $x_j \in \operatorname{supp}\mu$ and $x = \sum_{j=1}^n x_j$, then $\overline{\dim}\mu^n(x) \le \sum_{j=1}^n \overline{\dim}\mu(x_j)$. Consequently, if $\overline{\dim}\mu(x) \le \lambda$ for all $x \in \operatorname{supp}\mu$, then $\overline{\dim}\mu^N(x) \le N\lambda$ for all $x \in [0, N]$.

As $\mu$ is a positive measure, the hypothesis $(N)\operatorname{supp}\mu = [0, N]$ implies $\operatorname{supp}\mu^N = [0, N]$. An application of Lemma 1 shows that $\overline{\dim}\mu^{N+1}(x) \le N\lambda$ for all $x \in (0, N + 1)$.

Moreover, $\operatorname{supp}\mu^{N+1} = [0, N + 1]$. Because $\mu$ is a continuous measure, this is enough to again apply the lemma, and by repeated application we deduce that $\overline{\dim}\mu^{N+m}(x) \le N\lambda$ for all positive integers $m$ and for all $x \in (0, N + m)$.

Since Lemma 2 implies $\overline{\dim}\mu^n(0) \to \infty$ as $n \to \infty$, the first part of the theorem holds.

(ii) Let $\alpha = \overline{\dim}\mu(0)$ and $\sup\{\overline{\dim}\mu(x) : x \in \mathit{supp}\mu\} = \lambda$. Choose $N_0 > N$ and $s$ such that $N_0\alpha > s > N\lambda$. Given any fixed $n \ge N_0$, let

$$A_j = \{x : \min\{\mu^n(B(x, r)), \mu^n(B(n - x, r))\} \ge r^s \text{ for all } r < 1/j\}\,.$$

The sets $A_j$ are nested and symmetric about $n/2$. As explained in the first part of the proof, $\overline{\dim}\mu^n(x) \le N\lambda < s$ for all $x \in (0, n)$, thus $\cup_j A_j \supseteq (0, n)$. Consequently, we can find sets $A_j \subseteq [0, n]$ whose Lebesgue measures are arbitrarily close to $n$. Standard arguments imply there is a choice of $j$ such that $A_j + A_j$ contains an interval $I \subseteq [0, 2n]$, centred at $n$, with measure as close to $2n$ as we desire. In particular, given any $\delta > 0$, there is such a choice $j$ with $(\delta, 2n - \delta) \subseteq I$.

This ensures that if $x \in (\delta, 2n - \delta)$, then there exist $y_1, y_2 \in A_j$ with $x = y_1 + y_2$. If $r/2 < 1/j$, then

$$\mu^{2n}(B(x, r)) \ge \mu^n(B(y_1, r/2))\mu^n(B(y_2, r/2)) \ge 2^{-2s}r^{2s}.$$

Let $\{B(x_i, r)\}$ be any $r$-packing with centres in $(\delta, 2n - \delta)$ and let $q < 0$. For small $r$, there are at most $2n/r + 1 \le 3n/r$ balls in any such $r$-packing, hence

$$\sum \left(\mu^{2n}(B(x_i, r))\right)^q \le \frac{3n}{r} 2^{-2sq}r^{2sq} \le Cr^{2sq-1}.$$

As $2n\alpha = \overline{\dim}\mu^{2n}(0)$, if $\varepsilon > 0$ is fixed and $r$ is sufficiently small,

$$\left(\mu^{2n}(B(0, r))\right)^q \ge C' r^{(\alpha-\varepsilon)2nq}.$$

If $\varepsilon > 0$ is chosen so that $n(\alpha - \varepsilon) > s$, then for sufficiently negative $q$, $r^{2sq-1} << r^{(\alpha-\varepsilon)2nq}$. This shows that the points in $(\delta, 2n-\delta)$ do not contribute to the $L^q$-spectrum of $\mu^{2n}$.

To handle odd convolution powers, observe that

$$\mu^{2n+1}(B(x, r)) \ge \mu^{2n}(B(x, r/2))\mu(B(0, r/2)) \ge Cr^{2s}r^{\alpha+\varepsilon}$$

and argue similarly. □

**Remark 1.** We remark that $q_0 \to 0$ as $N_0 \to \infty$.

**Example 1.** Let $0 < r_k < 1/2$. We call the set $C_{\{r_k\}}$ a central Cantor set with ratios of dissection $\{r_k\}$ if $C_{\{r_k\}} = \bigcap C_k$ where $C_0 = [0,1]$ and if $C_{k-1}$ is the union of $2^{k-1}$ closed intervals of length $R_{k-1} = r_1 \cdots r_{k-1}$, then $C_k$ is constructed by removing the middle open subintervals of length $(1-2r_k)R_{k-1}$ from each of those intervals. The classical middle-third Cantor set is $C_{\{1/3\}}$.

It is known that a central Cantor set $C_{\{r_k\}}$ satisfies $(N)C_{\{r_k\}} = [0,N]$ for some $N$ if $\inf r_k > 0$ [3]. If $\mu$ is a $p$-Cantor measure with $p \neq 1/2$, then $\overline{\dim}\mu(0) = (\ln p)/c$ where $c = \liminf(\frac{1}{k}\ln R_k)$. Moreover, $\overline{\dim}\mu(x) \leq \max(\ln p/c, \ln(1-p)/c)$ for all $x \in \operatorname{supp}\mu$. As $\mu$ is a continuous measure, the theorem implies that if $\inf r_k > 0$, then $\overline{\dim}\mu^n(0)$ is isolated in the set of upper local dimensions of $\mu^n$ for sufficiently large $n$.

One might ask if the weaker assumption, $\inf {R_k}^{1/k} > 0$ (which is equivalent to saying $C_{\{r_k\}}$ has positive Hausdorff dimension) would still guarantee the existence of an isolated point in the spectrum of $\mu^n$ for large enough $n$. This is not true. The central Cantor set $C$ with $r_n = 3^{-k}$ if $n = k^3$ and $1/3$ otherwise is such an example. The details of this are omitted. We remark that this Cantor set $C$ also has the property that $\dim_H(C+C) = 1$.

# 3 The $L^q$-spectrum for convolutions of measures

## 3.1 Positive $q$

In this subsection we study the asymptotic behaviour of the $L^q$-spectrum of $\mu^n$ for $q \geq 0$. Intuitively, one would expect the $L^q$ -spectrum to be controlled by the balls of large measure when $q \geq 0$. This leads us to introduce the following notation:

$$\kappa_\mu = \liminf_{r \to 0^+} \left( \frac{\log\left(\sup_x \mu(B(x,r))\right)}{\log r} \right).$$

As $\mu(B(x,r)) \leq 1$ for all probability measures $\mu$, it is obvious that $\kappa_\mu \geq 0$. Moreover, it is easy to see that $\kappa_\mu = 0$ if $\mu$ is not continuous. More generally, if $\operatorname{supp}\mu \subseteq [-N,N]$, then the balls $B(-N+jr, r)$, for $j = 0, 1, \ldots, [2N/r]+1 = J$, cover $\operatorname{supp}\mu$. At least one of these balls must have $\mu$-measure at least $1/J$ and this implies $\kappa_\mu \leq 1$.

**Theorem 2.** *Suppose $\mu$ is a compactly supported, probability measure and $q \geq 0$. Then*

$$L_\mu(q) := \lim_{n\to\infty} \tau_{\mu^n}(q)$$

*exists. If* $q \geq 1$*, then* $L_\mu(q) = K_\mu(q-1) \leq q-1$*, where the constant* $K_\mu = L_\mu(2)$ *is given by* $K_\mu = \lim_n \kappa_{\mu^{4n}}$.

The proof requires several preliminary results. We will frequently write $\tau_n$ or $\kappa_n$ in place of $\tau_{\mu^n}$ or $\kappa_{\mu^n}$ if $\mu$ is clear.

**Lemma 3.** *If* $q \geq 0$*, then* $-1 \leq q\kappa_\mu - 1 \leq \tau_\mu(q) \leq q\kappa_\mu \leq q$.

Proof. Suppose supp$\mu \subseteq [-N, N]$ and $\{B(x_i, r)\}$ is any centred $r$-packing. Then

$$\sup_i \mu(B(x_i,r))^q \leq \sum \mu(B(x_i,r))^q \leq \sup_x \mu(B(x,r))^q(2N/r+1).$$

Consequently, $q\kappa_\mu - 1 \leq \tau_\mu(q) \leq q\kappa_\mu$. The outer inequalities follow from the comments preceding the statement of the theorem. □

It is helpful to identify equivalent ways to define the $L^q$-spectrum when $q \geq 0$. Let

$$S^{(1)}_{r,\mu}(q) = \sup \sum_i \mu(B(x_i,r))^q \text{ and } S^{(2)}_{r,x,\mu}(q) = \sum_i \mu(B(x+ir,r))^q,$$

where in the first case the supremum is taken over all countable $r$-packings $\{B(x_i, r)\}_i$, but with $x_i$ not necessarily in supp$\mu$. One can readily check that following inequalities hold for any $x$: (note we suppress the dependence on $\mu$ in the notation)

$$S_r(q) \leq S^{(1)}_r(q) \leq 4S_{2r}(q) \text{ and}$$
$$S^{(2)}_{r,x}(q) \leq 2S^{(1)}_r(q) \leq 2^{q+1}S^{(2)}_{r,x}(q).$$

Consequently, the limiting behaviours of

$$\frac{\log(S_r(q))}{\log r}, \ \frac{\log(S^{(1)}_r(q))}{\log r} \text{ and } \frac{\log(S^{(2)}_{r,x}(q))}{\log r}$$

are the same and any of these functions can be used to calculate $\tau_\mu(q)$ for $q \geq 0$.

**Lemma 4.** *Suppose* $\mu$ *and* $\nu$ *are probability measures. If* $q \geq 1$*, then* $\tau_{\mu*\nu}(q) \geq \tau_\mu(q)$*. If* $0 < q < 1$*, then* $\tau_{\mu*\nu}(q) \leq \tau_\mu(q)$.

Proof. Fix $r>0$ and any $r$-packing $\{B(x_i,r)\}$. If $q\geq 1$, then an application of Holder's inequality shows that

$$\begin{aligned}\sum_i \mu*\nu(B(x_i,r))^q &= \sum_i \left(\int \mu(B(x_i-y,r))d\nu(y)\right)^q\\ &\leq \sum_i \left(\int \mu(B(x_i-y,r))^q d\nu(y)\right)\\ &\leq \int S^{(1)}_{r,\mu}(q)d\nu(y) = S^{(1)}_{r,\mu}(q).\end{aligned}$$

Hence $S^{(1)}_{r,\mu*\nu}(q)\leq S^{(1)}_{r,\mu}(q)$ and that proves the first inequality.

The second inequality is similar, but using $S^{(2)}_{r,x}$. The inequality is reversed because we apply Holder's inequality with exponent $1/q$ (and its dual index) to $\int \mu(B(x+ir,r))^q d\nu(y)$. □

**Lemma 5.** *If $\mu$ is a symmetric probability measure, then $\kappa_{\mu*\mu}=\underline{\dim}\mu*\mu(0)$.*

Proof. Fix $r>0$ and define

$$I_j=\begin{cases}[-r,r], & \text{if } j=0,\\ (2jr-r,2jr+r], & \text{if } j>0,\\ -I_{-j}, & \text{if } j<0.\end{cases}$$

For notational ease, put $z_j=\mu(I_j)$. As $\mu$ is symmetric, $z_j=z_{-j}$. We claim that for any $k$ and $x\in I_k$,

$$\begin{aligned}\mu*\mu(B(0,4r)) &\geq \sum z_j z_{-j}\\ &\geq \frac{1}{5}\Big(\sum z_{j+k}z_{-j}+z_{j+k-1}z_{-j}+z_{j+k+1}z_{-j}\\ &\qquad\qquad +z_{j+k+2}z_{-j}+z_{j+k-2}z_{-j}\Big)\\ &\geq \frac{1}{5}\mu*\mu(B(x,r)).\end{aligned}$$

The second inequality is simply Cauchy Schwartz together with the fact that $z_j=z_{-j}$. For the first, consider $\mu$ as the probability distribution of the random variable $X$. Then $\mu*\mu$ is the probability distribution of $X_1+X_2$, where $X_1,X_2$ are independent random variables with the same distribution as $X$. The collection $\{I_j\}$ is a partition of $\mathbb{R}$ into disjoint intervals of length $2r$. If $X_1\in I_j$ and $X_2\in I_{-j}$, then $X_1+X_2\in B(0,2r)\subseteq B(0,4r)$, and this happens with probability $z_j z_{-j}$. Summing gives the first inequality.

For the third inequality, suppose $x \in I_k$. Then $B(x,r) \subseteq I_k \cup I_{k-1} \cup I_{k+1}$. Moreover, if $X = X_1 + X_2$, $X_1 \in I_m$ and $X \in I_j$, then $X_2 \in I_{j-m-1} \cup I_{j-m} \cup I_{j-m+1}$. Thus if $X \in I_k \cup I_{k-1} \cup I_{k+1}$, then for each choice of $m$ with $X_1 \in I_m$ there are only five possible intervals to which $X_2$ can belong.

From the claim we see that

$$\frac{1}{5}\sup_x \mu * \mu(B(x,r)) \leq \mu * \mu(B(0,4r)) \leq \sup_x \mu * \mu(B(x,4r)),$$

and that inequality establishes the lemma. □

**Corollary 1.** *If $\mu$ is a symmetric probability measure, then $\kappa_{\mu^4} = \tau_{\mu*\mu}(2)$.*

Proof. Let $\nu = \mu * \mu$. As $\nu$ is a symmetric probability measure, the previous lemma implies it is enough to show that $\tau_\nu(2) = \underline{\dim}\nu * \nu(0)$. It is convenient to use the integral formulation for the $L^q$-spectrum with $q \geq 0$ ([6]):

$$\tau_\nu(q) = \liminf_{r\to 0^+} \frac{\log\left(\int \nu(B(x,r))^q dx\right)}{\log r} - 1. \tag{3.1}$$

By symmetry and similar reasoning to the proof of the first inequality of the claim in the previous lemma,

$$\begin{aligned}\int \nu(B(x,r))^2 dx &= \int \nu(B(x,r))v(B(-x,r))dx \\ &= \sum_j \int_{-r}^{r} v(B(y+2jr,r)\nu(B(-y-2jr,r))dy \\ &\leq \int_{-r}^{r} \nu * \nu(B(0,2r)dy = 2r\nu * \nu(B(0,2r)).\end{aligned}$$

Thus

$$\frac{\log\left(\int \nu(B(x,r))^2 dx\right)}{\log r} - 1 \geq \frac{\log 2\nu * \nu(B(0,r))}{\log r}.$$

Passing to the lim inf as $r \to 0^+$ shows $\tau_\nu(2) \geq \underline{\dim}\nu * \nu(0)$.

On the other hand, if $x_1 \in B(y+jr, r/2)$, then $x_1 + x_2 \in B(0, r/2)$ only if $x_2 \in B(-y-jr, r)$. Hence $\sum_j v(B(y+jr,r)\nu(B(-y-jr,r)) \geq \nu*\nu(B(0,r/2))$. The usual arguments then give the required equality. □

Proof of Theorem. Lemmas 3 and 4 imply that for each $q \geq 0$, $\tau_{\mu^n}(q)$ is a bounded monotonic sequence, and hence converges pointwise. It only remains to verify that when $q \geq 1$, then $L_\mu(q) = K_\mu(q-1)$, for $K_\mu$ as specified.

First, we argue it is enough to prove this for $\mu$ symmetric. To see this, assume $\mu$ is an arbitrary, compactly supported, probability measure and put

$\mu'(E) = \mu(-E)$. Then $\nu = (\mu + \mu')/2$ is a symmetric, probability measure and $\nu^n = 2^{-n}\sum_{j=0}^{n}\binom{n}{j}\mu^j * \mu'^{n-j}$. Since $a^q + b^q \sim (a+b)^q$ when $q$ is positive, it is not hard to see that

$$\tau_{\mu_1+\mu_2}(q) = \min(\tau_{\mu_1}(q), \tau_{\mu_2}(q)) \text{ for } q \geq 0.$$

Lemma 4 implies that

$$\tau_{\mu^j * \mu'^{n-j}}(q) \geq \max(\tau_{\mu^j}(q), \tau_{\mu'^{n-j}}(q)) \geq \tau_{\mu^{n/2}}(q).$$

Since also $\nu^n \geq 2^{-n}\mu^n$, we have $\tau_{\mu^{n/2}} \leq \tau_{\nu^n} \leq \tau_{\mu^n}$. As $\tau_{\mu^n}(q)$ and $\tau_{\mu^{n/2}}(q)$ converge to the same limit, so does $\tau_{\nu^n}(q)$.

Hence we may assume $\mu$ is symmetric. It follows from Cor. 1 that

$$L_\mu(2) = \lim_n \tau_{2n}(2) = \lim_n \kappa_{4n} = K_\mu.$$

Lemma 3 implies $qK_\mu - 1 \leq L_\mu(q) \leq qK_\mu$.

Being a limit of concave functions, $L_\mu(q)$ is concave and hence has right and left-hand derivatives everywhere. Let $L'_\mu$ denote the right hand derivative. If there exists $q_0$ such that $L'_\mu(q_0) < K_\mu$, then by concavity $L'_\mu(q) < K_\mu$ for all $q > q_0$, and eventually $L_\mu(q) < K_\mu q - 1$. Thus $L'_\mu(q) \geq K_\mu$ for all $q \geq 0$.

As $\tau_\mu(1) = 0$ for all probability measures $\mu$, $L_\mu(2) - L_\mu(1) = K_\mu$. If $L'_\mu(1) > K_\mu$, then $L'_\mu(x) < K_\mu$ for some $x \in (1,2)$ and that's a contradiction. Hence $L'_\mu(1) = K_\mu$ and since $L'$ is decreasing, $L'_\mu(q) = K_\mu$ for all $q \geq 1$. One can similarly argue that the left-hand derivatives of $L_\mu$ are also identically $K_\mu$ at all $q \geq 1$, hence $L_\mu$ is the linear function $L_\mu(q) = K_\mu(q-1)$.

That $L_\mu(q) \leq q - 1$ now follows from the fact that $\kappa_\nu \leq 1$ for all $\nu$. □

Here is one class of examples where $L_\mu(q) = q - 1$ for $q \geq 1$. The multifractal analysis of these measures is studied in detail in [1] and [18].

**Proposition 1.** *Let $d \geq 3$ be an integer. Suppose $\mu$ is the self-similar probability measure associated with the IFS $\{x/d + i/d, p_i\}_{i \in A}$ where $A \subseteq \mathbb{N}$ is a finite set that is not a singleton, and the probabilities $p_i$ are strictly positive for $i \in A$. Then $L_\mu(q) = q - 1$ for all $q \geq 1$.*

Proof. There is no loss of generality in assuming $0 \in A$. Furthermore, replacing $\mu$ if necessary by the compressed measure that maps $E \longmapsto \mu(r^{-1}E)$ for $r = \gcd(A)$, there is no loss in assuming $\gcd(A) = 1$.

The measure $\mu^n$ is the self-similar measure arising from the IFS $\{x/d + i/d, p_i^{(n)}\}$, for suitable probabilities $p_i^{(n)}$. For each $n$, let $Z_n = \max_k \sum_j p_{k+jd}^{(n)}$.

Let $X$ be a random variable with $P(X = i) = p_i$ and let $X_1, \ldots, X_n$ be independent random variables with the same distribution as $X$. Then

$$p_i^{(n)} = P(X_1 + \cdots + X_n = i)$$

and

$$\sum_j p_{k+jd}^{(n)} = P(X_1 + \cdots + X_n \equiv k \; mod(d).$$

Let $Y_n = X_1 + \cdots + X_n mod(d)$. Then $(Y_n)_{n=1}^{\infty}$ is an irreducible, aperiodic Markov chain and hence it approaches a steady distribution. As the random variables $Y_{n+1} - Y_n$ are independent of $Y_n$, this steady state is uniform. Thus for each $i = 0, \ldots, d-1$, $P(Y_n = i) \to 1/d$ and therefore $Z_n = P(Y_n = k) \to 1/d$ ([9, Thm. 11.7, 8])

Using the arguments of [1] it is not difficult to see that $\mu^n(B(x, d^{-k})) \leq c_n Z_n^k$ for an appropriate constant $c_n$. Thus $\tau_{\mu^n}(q) \geq -q \log Z_n / \log d - 1$ for $q \geq 0$ and therefore $L_\mu(q) \geq q - 1$. But always $L_\mu(q) \leq q - 1$ for all $q \geq 1$ and therefore we have equality. □

### 3.2 $L^q$-spectrum and energy dimension

In this subsection we will establish another formula for $K_\mu$ and investigate when it equals one. First, note that it is clear from the integral formulation of the $L^q$-spectrum (3.1) that for any measure $\mu$,

$$\tau_\mu(2) = \sup \left\{ a : \limsup_{t \to 0} \frac{1}{t^{1+a}} \int_{\mathbb{R}} (\mu(B(x,t)))^2 dx = 0 \right\}.$$

A Fourier transform formula can be deduced from this and Parseval's theorem.

**Proposition 2.** *If $\mu$ is any probability measure on supported on $[0,1]$, then for $0 \leq a < 1$,*

$$\begin{aligned}\limsup_{t \to 0^+} \frac{1}{t^{1+a}} \int_0^1 (\mu(B(x,t)))^2 dx &\sim \limsup_{N \to \infty} \frac{1}{N^{1-a}} \int_{|x| \leq N} |\widehat{\mu}(x)|^2 \, dx \\ &\sim \limsup_{N \to \infty} \frac{1}{N^{1-a}} \sum_{|n| \leq N} |\widehat{\mu}(n)|^2 ,\end{aligned}$$

*where in the first case $\mu$ is viewed as a measure on $\mathbb{R}$ and its Fourier transform has domain $\mathbb{R}$, and in the second case $\mu$ is viewed as a measure on the torus, $[0,1]$, where $0$ and $1$ are identified, and its Fourier transform is defined on $\mathbb{Z}$.*

Proof. The first relation is proven in [14] for the more general case of a measure on $\mathbb{R}^n$. Here we use a similar, but simpler, strategy to prove the second. (The same methods give a simpler proof of the first relation for $n = 1$.)

Since $\mu(B(x,t)) = \mu * 1_{(-t,t)}(x)$, where $1_{(-t,t)}$ denotes the characteristic function of the interval $(-t,t)$, Parseval's theorem implies

$$\begin{aligned}\frac{1}{t^{1+a}}\int_0^1 (\mu(B(x,t)))^2 dx &= \frac{1}{t^{1+a}}\sum_{n=-\infty}^{\infty} |\widehat{\mu}(n)|^2 \left|\widehat{1_{(-t,t)}}(n)\right|^2 \\ &= Ct^{1-a}\left(\sum_{n\neq 0}|\widehat{\mu}(n)|^2\left|\frac{\sin 2\pi nt}{nt}\right|^2 + |\widehat{\mu}(0)|^2\right) \\ &\geq \frac{C}{N^{1-a}}\sum_{|n|\leq N}|\widehat{\mu}(n)|^2\end{aligned}$$

when $N = [1/t]$ and the constant $C$ may change from one line to another. Taking the lim sup as $t \to 0^+$, equivalently $N \to \infty$, proves one inequality. For the other, let

$$A_j = \frac{1}{2^{j(1-a)}}\sum_{|n|=1}^{2^j}|\widehat{\mu}(n)|^2 .$$

If $2^{-(k+1)} \leq t < 2^{-k}$, then

$$\begin{aligned}t^{1-a}\sum_{n\neq 0}|\widehat{\mu}(n)|^2\left|\frac{\sin 2\pi nt}{nt}\right|^2 &\leq Ct^{1-a}\sum_{|n|=1}^{2^k}|\widehat{\mu}(n)|^2 + \frac{1}{t^{1+a}}\sum_{j=k}^{\infty}\sum_{|n|=2^j+1}^{2^{j+1}}\frac{|\widehat{\mu}(n)|^2}{|n|^2} \\ &\leq C\left(A_k + 2^{k(1+a)}\sum_{j=k}^{\infty}2^{-j(1+a)}A_{j+1}\right) \\ &\leq C\sup_{j\geq k}A_j.\end{aligned}$$

Again, take the lim sup. $\square$

Suppose $\mu$ is any compactly supported measure on $\mathbb{R}$. We denote by $[\mu]$ its quotient measure on $[0,1]$, the measure defined by

$$[\mu](E) = \sum_{n\in\mathbb{Z}}\mu(E+n) \text{ for } E \subseteq [0,1].$$

Since the sum is over a bounded number of integers $n$, $(\mu(B(x,r)))^q \sim ([\mu](B(x,r)))^q$ for any $q \geq 0$. Thus, $\tau_\mu(q) = \tau_{[\mu]}(q)$ for positive $q$. As $\widehat{[\mu^m]}(n) = \widehat{\mu}(n)^m$, this observation yields the following corollary.

**Corollary 2.** *If $\mu$ is any probability measure on $[0,1]$, then*

$$\tau_{\mu^m}(2) = \sup\left\{ a : \limsup_{N\to\infty} \frac{1}{N^{1-a}} \sum_{|n|\le N} |\widehat{\mu}(n)|^{2m} = 0 \right\}.$$

The energy dimension, defined as

$$\dim_e \mu = \sup\left\{ a : \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{d\mu(x)d\mu(y)}{|x-y|^a} < \infty \right\},$$

is another way to quantify the singularity of a measure. The energy dimension is always a lower bound for the Hausdorff dimension of the measure and has been shown to be equal to

$$\sup\left\{ a : \sum_{n\neq 0} \frac{|\widehat{\mu}(n)|^2}{|n|^{1-a}} < \infty \right\} \text{ or } \sup\left\{ a : \int_{\mathbb{R}} \frac{|\widehat{\mu}(x)|^2}{|x|^{1-a}} dx < \infty \right\},$$

depending on whether $\mu$ is viewed as a measure on the torus or on $\mathbb{R}$ ([4], [10]). Using these formulas and the previous proposition, it is straightforward to verify the following.

**Corollary 3.** *Suppose $\mu$ is any compactly supported, probability measure. Then $\tau_\mu(2) = \dim_e \mu$. Moreover, $L_\mu(q) = q-1$ for all $q \ge 1$ (equivalently, $L_\mu(2) = 1$) if and only if $\lim_n \dim_e \mu^n = 1$ (where the convolution can be understood either on $\mathbb{R}$ or $[0,1]$.)*

**Corollary 4.** *Suppose $\mu$ is a probability measure on the torus. If $L_\mu(2) = 1$, then any Borel subgroup of the torus on which $\mu$ is concentrated has Hausdorff dimension one.*

Proof. If $\mu$ is concentrated on $E$, then $\dim_H E \ge \dim_e \mu$. Since $\mu^n$ is concentrated on $E^n \subseteq Grp(E)$, it follows that $\dim_H Grp(E) \ge \dim_e \mu^n \to L_\mu(2) = 1$. $\square$

**Example 2.** A measure $\mu$ defined on $[0,1]$ is said to be $L^p$-improving if for some $p > 2$, the operator $T_\mu : L^2[0,1] \to L^p[0,1]$, given by $T_\mu(f) = f * \mu$, is bounded. The classical Cantor measure is an example of an $L^p$-improving measure [15]. It is known that if $\mu$ is an $L^p$-improving measure, then $\dim_e[\mu^n] \to 1$ [11]. Thus every $L^p$-improving measure has $L_\mu(q) = q-1$.

In [17] a construction is given of a random Cantor measure, supported on a set of Hausdorff dimension $s$ for any given $0 < s < 1$, with the property that $\widehat{\mu} \in l^p$ for some $p < \infty$. Such a measure is known to be $L^p$-improving.

We conjecture that the self-similar measure associated with the IFS $\{x/d+i/d, p_i\}_{i=0}^m$ with integer $d$ and $p_i > 0$ for at least two indices $i$ (see Prop. 1) is $L^p$-improving.

**Example 3.** A measure $\mu$ is said to belong to $Lip(\alpha)$ if there is a constant $C$ such that $\mu(x, x+h] \leq Ch^\alpha$ for all $h > 0$ and all $x$. It is known that if $\mu \in Lip(\alpha)$, then $\dim_e \mu \geq \alpha$ [11]. Thus, if for every $\alpha < 1$ there is some $n$ such that $[\mu^n] \in Lip(\alpha)$, then $L_\mu(q) = q - 1$.

To conclude this subsection, we prove that for each $0 < a < 1$ there is a probability measure $\mu$ such that $\tau_{\mu^n}(2) = a$ for all $n$. Consequently, $L_\mu(q) = a(q-1)$. Our method is constructive.

**Example 4.** Fix $0 < a < 1$ and put $s = (1-a)/a$. Inductively define positive integers $n_j$ and $d_j$ such that $d_j >> 2^j$, $n_j = d_j^s$ and $d_{j+1} >> n_j d_j$. Let $K_N$ denote the $N$'th Fejer kernel and put $F_j(x) = K_{n_j}(d_j x)$. Let $\mu = \sum_j F_j(x)/j^2$. Note that $\mu \geq 0$ and as $\|F_j\|_1 = 1$, $\mu$ is an absolutely continuous, finite measure supported on $[0,1]$.

Fix positive integer $m$. For each $0 \leq b < 1$, let

$$A_N^b = \frac{1}{N^{1-b}} \sum_{0<n|\leq N} |\widehat{\mu}(n)|^{2m} .$$

We will use the criterion $\tau_{\mu^m}(2) = \sup\left\{b : \limsup_{N\to\infty} A_N^b = 0\right\}$.

As $\mathrm{supp}\widehat{F_j} = \{0, \pm d_j, ..., \pm n_j d_j\}$, we have $\mathrm{supp}\widehat{F_j} \cap \mathrm{supp}\widehat{F_k} = \{0\}$ for $j \neq k$. Thus $\widehat{\mu}(n) \neq 0$ for $n \neq 0$ if and only if there exists some integer $j$ such that $\widehat{\mu}(n) = \widehat{F_j}(n)/j^2$. Since $0 \leq \widehat{F_j}(n) \leq 1$ and $\widehat{F_j}(n) \geq 1/2$ for at least $n_j$ integers $n$, $\sum_{n\neq 0} \widehat{F_j}(n)^{2m} \sim n_j$ . If $N = n_k d_k$, then

$$A_N^b = \frac{1}{(n_k d_k)^{1-b}} \sum_{j=1}^{k} \frac{1}{j^{4m}} \sum_{n\neq 0} \left|\widehat{F_j}(n)\right|^{2m} \geq C \frac{1}{(n_k d_k)^{1-b}} \frac{n_k}{k^{4m}} .$$

Suppose $b > a$. Since $n_k^{b-a} \geq k^{4m}$ for large $k$ and $n_k^a = d_k^{1-a}$, it is easily seen that $A_N^b \to \infty$ as $N \to \infty$. Thus $\tau_{\mu^m}(2) \leq b$ for all $b > a$ and hence $\tau_{\mu^m}(2) \leq a$.

One can similarly show that if $b < a$, then $A_N^b \leq C/d_k^{1-b-sb} \to 0$ as $N = n_k d_k \to \infty$. Of course, if $N \in (n_k d_k, d_{k+1})$, then $A_N^b = A_{n_k d_k, m}^b$. If, instead, $N \in [J d_{k+1}, (J+1) d_{k+1})$ for $1 \leq J < n_{k+1}$, then again one can check that

$$\begin{aligned} A_N^b &= \frac{1}{N^{1-b}} \left( \sum_{0<n|<d_{k+1}} |\widehat{\mu}(n)|^{2m} + \sum_{|n|=d_{k+1}}^{N} |\widehat{\mu}(n)|^{2m} \right) \\ &\leq C \left( \frac{n_k + n_{k+1}^b}{d_{k+1}^{1-b}} \right) \to 0. \end{aligned}$$

Thus $\limsup_{N\to\infty} A_N^b = 0$ for any $b < a$. Together these observations prove $\tau_{\mu^m}(2) = a$.

### 3.3 Negative $q$

Finally, we consider the case $q < 0$. Here we expect the balls of small measure to control the behaviour of the $L^q$-spectrum. This leads to introducing

$$\beta_\mu = \limsup_{r\to 0^+} \frac{\log\left(\inf_{x\in supp\mu} \mu(B(x,r))\right)}{\log r}.$$

Throughout the subsection, let $\alpha_\mu = \overline{\dim}\mu(0)$. Assuming $0 \in \mathrm{supp}\mu$, then $\beta_{\mu^n} \geq \overline{\dim}\mu^n(0) = n\alpha_\mu$. As $\sup \sum \mu(B(x_i,r))^q \geq \inf_{x\in supp\mu} \mu(B(x,r))^q$, it also follows that $\tau_\mu(q) \leq q\beta_\mu$.

In general, $\tau_{\mu^n}(q)$ does not converge for $q < 0$. Instead, we consider the asymptotic behaviour of

$$\begin{aligned} \delta_{\mu^n}(q) &:= \tau_{\mu^n}(q) - n\alpha_\mu q \\ \gamma_{\mu^n} &:= \beta_{\mu^n} - n\alpha_\mu. \end{aligned}$$

Again, we write $\beta_n$, $\delta_n$, $\gamma_n$ for $\beta_{\mu^n}$ etc., provided $\mu$ is clear.

**Proposition 3.** *Assume supp$\mu = [0,1]$ and that $\overline{\dim}\mu(0) \geq \overline{\dim}\mu(1)$. For $q \leq -1$,*

$$2q \lim_n \gamma_n \leq \lim_n \delta_n(q) \leq q \lim_n \gamma_n. \tag{3.2}$$

PROOF. Note that

$$\mu^{n+1}(B(x,r)) \geq \mu^n(B(x,r/2))\mu(B(0,r/2)) \text{ if } x \in [0,n],$$

while

$$\mu^{n+1}(B(x,r)) \geq \mu^n(B(x-1,r/2))\mu(B(1,r/2)) \text{ if } x \in [n,n+1].$$

Thus $\beta_{n+1} \leq \beta_n + \alpha_\mu$ and hence $\{\gamma_n\}$ is a decreasing sequence.

If we consider $\left(\mu^{n+1}(B(x,r))\right)^q$ instead, the inequalities reverse and therefore $\tau_{n+1}(q) \geq \tau_n(q) + q\alpha$, showing $\{\delta_n\}$ is increasing.

Since

$$\sup \sum \mu^n(B(x_i,r))^q \geq \mu^n(B(0,r))^q \geq \mu(B(0,r))^{qn},$$

we have $\tau_n(q) \leq qn\alpha$. Thus $\delta_n(q) \leq 0$ for all $q < 0$. Further, $\delta_n(q) \leq q\beta_n - n\alpha q = q\gamma_n$, so $\{\gamma_n\}$ is bounded above. Thus both sequences, $\{\delta_n\}$ and $\{\gamma_n\}$, converge and the right side of (3.2) holds.

By the definition of $\beta_n$, given any $\varepsilon > 0$ there exists $r_0$ such that for all $r \leq r_0$, $\inf_x \mu^n(B(x,r)) \geq r^{\beta_n+\varepsilon}$. Thus for any $x \in [0,2n]$,

$$\mu^{2n}(B(x,2r)) \geq \sum_{k=0}^{[x/r]} \mu^n(B(x-kr,r))\mu^n(B(kr,r)) \geq \frac{x}{r} r^{2(\beta_n+\varepsilon)}.$$

If $\{B(x_i,2r)\}$ is any centred $r$-packing of $\mathrm{supp}\mu^{2n}$, then if $q < -1$,

$$\begin{aligned} \sum_i \mu^{2n}(B(x_i,2r))^q &\leq \sum_i \left(\frac{x_i}{r}\right)^q r^{2q(\beta_n+\varepsilon)} \\ &\leq \sum_{k \leq 2n/r} \left(\frac{kr}{r}\right)^q r^{2q(\beta_n+\varepsilon)} \leq C_q r^{2q(\beta_n+\varepsilon)}, \end{aligned}$$

while $\sum_i \mu^{2n}(B(x_i,2r))^{-1} \leq C\,|\log r|\, r^{2q(\beta_n+\varepsilon)}$. Taking the supremum over all centred $r$-packings, then log-limits and letting $\varepsilon \to 0$ gives $\tau_{2n}(q) \geq 2q\beta_n$. Thus $\delta_{2n}(q) = \tau_{2n}(q) - 2n\alpha q \geq 2q\gamma_n$ and that establishes the left side of (3.2). □

**Corollary 5.** *If $\gamma_n \to 0$, then $\delta_n(q) \to 0$ for $q \leq -1$.*

**Corollary 6.** *If $\mu(B(0,r)) = \inf_{x \in supp\mu} \mu(B(x,r))$, then $\delta_n \to 0$.*

Proof. In this case $\mu^n(B(0,r)) = \inf_{x \in supp\mu^n} \mu^n(B(x,r))$, so $\gamma_n = 0$ for all $n$. □

**Example 5.** The three-fold convolution of the standard Cantor measure, compressed to be supported on $[0,1]$, is an example of such a measure.

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