

Michael P. Cohen, Department of Mathematics, University of North Texas,  
Denton, TX 76203, U.S.A. email: michaelcohen@my.unt.edu

## THE DESCRIPTIVE COMPLEXITY OF SERIES REARRANGEMENTS

### Abstract

We consider the descriptive complexity of some subsets of the infinite permutation group  $S_\infty$  which arise naturally from the classical series rearrangement theorems of Riemann, Levy, and Steinitz. In particular, given some fixed conditionally convergent series of vectors in Euclidean space  $\mathbb{R}^d$ , we study the set of permutations which make the series diverge, as well as the set of permutations which make the series diverge properly. We show that both collections are  $\Sigma_3^0$ -complete in  $S_\infty$ , regardless of the particular choice of series.

### 1 Introduction

The goal of this paper is to establish the exact descriptive complexity of some interesting subsets of the Polish group  $S_\infty$  of permutations of  $\omega$ , endowed with the topology of pointwise convergence on  $\omega$ , considered as a discrete set. Our methods will involve a blending of the techniques of classical real analysis and geometry, with the descriptive set theoretic notion of continuous reducibility between Polish spaces.

First we recall Bernhard Riemann's celebrated rearrangement theorem of 1876 [6], now a staple of every graduate course in real analysis, which states the following remarkable fact (presented here as in [8]): given a conditionally convergent series of real numbers  $\sum_{k=0}^{\infty} a_k$ , and two extended real numbers  $\alpha, \beta \in [-\infty, \infty]$  with  $\alpha \leq \beta$ , it is possible to find an infinite permutation  $\pi \in S_\infty$

---

Mathematical Reviews subject classification: Primary: 40A05

Key words: descriptive complexity, series rearrangement

Received by the editors September 10, 2012

Communicated by: Krzysztof Ciesielski

for which  $\liminf_{n \rightarrow \infty} \sum_{k=0}^n a_{\pi(k)} = \alpha$  and  $\limsup_{n \rightarrow \infty} \sum_{k=0}^n a_{\pi(k)} = \beta$ . In other words, by varying one's choice of  $\alpha$  and  $\beta$ , it is possible to rearrange the terms of a conditionally convergent infinite series so that the partial sums converge to any particular real number, or diverge to plus or minus infinity, or even diverge properly.

Almost as famous as Riemann's original theorem is the following  $d$ -dimensional analogue:

**Levy-Steinitz Theorem.** *Let  $\sum_{k=0}^{\infty} v_k$  be a conditionally convergent series of vectors in  $\mathbb{R}^d$ . Then there exists a non-trivial affine subspace  $A(v_k) \subseteq \mathbb{R}^d$  (that is, a space of the form  $A(v_k) = w + M$  where  $w \in \mathbb{R}^d$  and  $M \subseteq \mathbb{R}^d$  is a linear subspace with  $\dim M \geq 1$ ) such that whenever  $a \in A(v_k)$ , there is  $\pi \in S_{\infty}$  with  $\sum_{k=0}^{\infty} v_{\pi(k)} = a$ .*

The statement above implies that the set of all possible sums of rearrangements of a conditionally convergent series of  $d$ -dimensional vectors is at least as rich as in the 1-dimensional case. An incomplete proof was first given by Levy in 1905 [5], and the complete proof was furnished by Steinitz in 1913 [9]. Steinitz's proof relied on a particular geometric constant for Euclidean spaces which is now referred to as the *Steinitz constant*. The proof is nontrivial, and an excellent concise version of it may be found in the paper [7] by P. Rosenthal. Our proofs will also rely heavily on the existence of a Steinitz constant.

The Levy-Steinitz theorem gives rise to a natural partition of  $S_{\infty}$ , into the set  $\mathcal{D}$  of all permutations  $\pi$  for which  $\sum_{k=0}^{\infty} v_{\pi(k)}$  diverges (either properly or to  $\infty$ , where  $\infty$  denotes the point at infinity in the one-point compactification of  $\mathbb{R}^d$ ), and the complement set  $S_{\infty} \setminus \mathcal{D}$  of permutations  $\pi$  for which  $\sum_{k=0}^{\infty} v_{\pi(k)}$  converges to some vector in  $\mathbb{R}^d$ . Both  $\mathcal{D}$  and its complement are interesting nontrivial sets. For instance, it is easy to observe, as we will do briefly in Section 4, that both  $\mathcal{D}$  and  $S_{\infty} \setminus \mathcal{D}$  are uncountable and dense in  $S_{\infty}$ , and also that  $\mathcal{D}$  is a comeager set.

We wish to examine these collections from the vantage point of descriptive set theory, or, loosely speaking, the study of the definable subsets of Polish spaces. *Definable* here may refer to Borel, analytic, projective, or any other class of "well-behaved" sets, which are typically closed under continuous preimages. Of course the Borel sets may be stratified by their relative

complexity into a *Borel hierarchy* indexed by the countable ordinals, whose exact definition we will recall for the reader in Section 2. It is an empirical phenomenon that a great bulk of those Borel sets which present themselves in the everyday study of mathematics will fall into the very bottom few levels of the Borel hierarchy. Thus there has been some industry for descriptive set theorists in finding “natural” examples of Borel sets which are “more complex” than usual. For some instances of such sets, the reader may consult the well-known references [2] and Sections 23, 27, 33, and 37 of [4], or the paper [1], which produces many examples in the field of ordinary differential equations.

Our objective here will be to establish the exact descriptive complexity of our set  $\mathcal{D}$  and its complement. In classical terminology, we will show that  $\mathcal{D}$  is  $G_{\delta\sigma}$  but not  $F_{\sigma\delta}$  (and hence not  $F_\sigma$ ,  $G_\delta$ , open, nor closed). Using the more modern notation, we prove:

**Theorem 1.** Let  $\sum_{k=0}^{\infty} v_k$  be any conditionally convergent series of vectors in  $\mathbb{R}^d$ , and let  $\mathcal{D} \subseteq S_\infty$  be the set of all permutations  $\pi$  for which  $\sum_{k=0}^{\infty} v_{\pi(k)}$  diverges. Then  $\mathcal{D}$  is  $\Sigma_3^0$ -complete.

Of course, it follows immediately that  $S_\infty \setminus \mathcal{D}$  is  $\Pi_3^0$ -complete. Now, for  $\pi \in S_\infty$ , say that the rearrangement  $\sum_{k=0}^{\infty} v_{\pi(k)}$  *diverges properly* if the series diverges, but does not diverge to infinity. Our methods also give the following result:

**Theorem 2.** Let  $\sum_{k=0}^{\infty} v_k$  be any conditionally convergent series of vectors in  $\mathbb{R}^d$ , and let  $\mathcal{DP} \subseteq S_\infty$  be the set of all permutations  $\pi$  for which  $\sum_{k=0}^{\infty} v_{\pi(k)}$  diverges properly. Then  $\mathcal{DP}$  is  $\Sigma_3^0$ -complete.

It follows that the set  $S_\infty \setminus \mathcal{DP}$  of series rearrangements which either converge to a vector in  $\mathbb{R}^d$ , or which diverge to  $\infty$ , is also a  $\Pi_3^0$ -complete set in  $S_\infty$ . Notice that, remarkably, none of the above statements depend on the nature of the particular conditionally convergent series  $\sum_{k=0}^{\infty} v_k$  that we choose! Thus, for each conditionally convergent series, we exhibit a naturally defined subset of  $S_\infty$  which lies no lower on the Borel hierarchy than the third level.

## 2 Definitions and terminology

First we recall the definition of the Borel hierarchy. Given a Polish space  $X$ , we let  $\Sigma_1^0(X)$  be the family of all open subsets of  $X$ , and  $\Pi_1^0(X)$  the family of all closed subsets of  $X$ . We set  $\Delta_1^0(X) = \Sigma_1^0(X) \cap \Pi_1^0(X)$ , so  $\Delta_1^0(X)$  consists of the clopen sets in  $X$ . The rest of the levels of the hierarchy are defined recursively as follows: Suppose for some countable ordinal  $\beta$ , we have defined the classes  $\Sigma_\alpha^0(X)$  and  $\Pi_\alpha^0(X)$  for all  $\alpha < \beta$ . Then we set

$$\begin{aligned}\Sigma_\beta^0(X) &= \{\bigcup_{n \in \omega} A_n : A_n \in \Pi_{\alpha_n}^0 \text{ for some } \alpha_n < \beta\}, \\ \Pi_\beta^0(X) &= \{A^c : A \in \Sigma_\beta^0(X)\}, \text{ and} \\ \Delta_\beta^0(X) &= \Sigma_\beta^0(X) \cap \Pi_\beta^0(X).\end{aligned}$$

It is well known that  $\Delta_\beta^0(X) \subseteq \Sigma_\beta^0(X)$ ,  $\Pi_\beta^0(X) \subseteq \Delta_{\beta+1}^0$  for each  $\beta$ , and that the inclusions are all proper.

Let  $X, Y$  be Polish spaces and  $A \subseteq X$ ,  $B \subseteq Y$ . If there exists a continuous function  $f : X \rightarrow Y$  such that  $f^{-1}(B) = A$ , then we say that  $A$  is *Wadge reducible* or *continuously reducible* to  $B$ , and we write  $A \leq_W B$ . Intuitively, we think that  $A$  is “no more complex” than  $B$ .

Let  $\Gamma$  be any of the pointclasses  $\Sigma_\beta^0$ ,  $\Pi_\beta^0$ , or  $\Delta_\beta^0$ . A standard inductive argument through the hierarchy shows that  $\Gamma$  is closed under continuous preimages, i.e., whenever  $X$  and  $Y$  are Polish,  $A \subseteq X$ ,  $B \in \Gamma(Y)$ , and  $A$  is continuously reducible to  $B$ , then we have  $A \in \Gamma(X)$ .

The above comment provides a useful tool for determining the complexity of a set. We say that a subset  $B$  of a Polish space  $Y$  is  $\Gamma$ -hard if for every Polish space  $X$  and every  $A \in \Gamma(X)$  we have  $A \leq_W B$ . It follows from the above comments that if  $B$  is  $\Gamma$ -hard, then  $\Gamma$  is a lower bound for the descriptive complexity of  $B$ . If in addition we have  $B \in \Gamma(Y)$ , then we say that  $B$  is  $\Gamma$ -complete, and we have determined its exact complexity in the Borel hierarchy.

The most common method for showing that a set  $B$  is  $\Gamma$ -hard is to find a set  $A$  which is already known to be  $\Gamma$ -complete, and prove that  $A \leq_W B$  by constructing an explicit continuous reduction. This will be the method of our proof in Section 4, and we will make use of the following subset  $\mathcal{C}$  of the Baire space  $\omega^\omega$ :

$$\mathcal{C} = \{x \in \omega^\omega : \lim_{n \rightarrow \infty} x(n) = \infty\}.$$

Exercise 23.2 of [4] asks the reader to show that  $\mathcal{C}$  is in fact  $\Pi_3^0$ -complete. It necessarily follows that the complement  $\omega^\omega \setminus \mathcal{C}$  is  $\Sigma_3^0$ -complete. Our proof

will continuously reduce this complement  $\omega^\omega \setminus \mathcal{C}$ , simultaneously, to both  $\mathcal{D}$  and  $\mathcal{DP}$ , and thus establish the  $\Sigma_3^0$ -hardness of the latter two sets.

We regard each nonnegative integer  $n$  as a von Neumann ordinal, i.e. we think of each  $n$  as the set  $\{0, \dots, n - 1\}$ . If a function  $\pi : n \rightarrow \omega$  is injective, then we call  $\pi$  a *finite partial permutation*. We use the notation  $\text{dom}(\pi)$  to refer to the map's domain  $n = \{0, \dots, n - 1\}$  and  $\text{ran}(\pi)$  to refer to its range  $\{\pi(0), \dots, \pi(n - 1)\}$ . If  $\sigma \in S_\infty$  or if  $\sigma$  is a finite partial permutation, then we say  $\sigma$  *extends*  $\pi$  if  $\sigma \upharpoonright \text{dom}(\pi) = \pi$ .

### 3 The Bounded Walk Lemma

In this section we will develop the main technical lemma on which our proof is built. An intuitive explanation for the Bounded Walk lemma is as follows: Consider a conditionally convergent series as an abstract infinite collection  $(v_k)_{k \in \omega}$  of vectors in  $\mathbb{R}^d$  from which we may build finite paths. Let  $\alpha$  and  $\beta$  be two points in  $\mathbb{R}^d$ . Suppose we have already chosen some finite subcollection

$\{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(J)}\}$ ,  $J \in \omega$ , of vectors from  $(v_k)_{k \in \omega}$  whose sum  $\sum_{k=0}^J v_{\pi(k)}$

(here visualized as a path of vectors laid end-to-end) is very close to  $\alpha$ . We wish to extend the path we have already built by choosing finitely many more vectors, from among those we have not already chosen, in such a way that the extended path will terminate very close to  $\beta$ , i.e. we wish to “walk from  $\alpha$  to  $\beta$ .”

We will show that if all the remaining vectors to choose from are sufficiently small (say less than  $\frac{1}{3C_d} \|\beta - \alpha\|$  where  $C_d$  is some constant to be determined later), then it is possible to build a finite path which (1) extends the path we have already walked; (2) uses up all except arbitrarily small remaining vectors; (3) takes us arbitrarily close to  $\beta$ ; and (4) does not wander arbitrarily far from the straight-line path connecting  $\alpha$  and  $\beta$ . In addition we may (5) use up any particular vector we wish. (Note that conditions (1) and (2) allow us to repeat this “bounded walk” process between as many points as we like, as often as we like.)

Now we aim to establish such a lemma. Before we do so, we first recall the following classical result as stated in [7], which is attributed to Steinitz, and which asserts the existence of a very useful “bounded rearrangement constant”  $C_d$  in Euclidean space, now referred to as the Steinitz constant:

**Lemma 3** (Polygonal Confinement Theorem). *Let  $d \geq 1$  be any integer. Then there exists a constant  $C_d$  which satisfies the following statement: Whenever  $v_0, v_1, \dots, v_m$  are vectors in  $\mathbb{R}^d$  which sum to 0 and satisfy  $\|v_i\| \leq 1$  for each*

$i \leq m$ , then there is a finite permutation  $P \in S_m$  with the property that

$$\left\| v_0 + \sum_{i=1}^j v_{P(i)} \right\| \leq C_d$$

for every  $j$ .

The Polygonal Confinement Theorem is the basis for the remaining lemmas in this section.

**Lemma 4.** *Let  $\alpha, v_1, \dots, v_m$  be vectors in  $\mathbb{R}^d$  which sum to  $\beta \in \mathbb{R}^d$ , and let  $C_d$  be as in the Polygonal Confinement Theorem. Further suppose we have  $\|v_i\| \leq \frac{1}{C_d} \|\beta - \alpha\|$  for each  $i \leq m$ . Then there is a finite permutation  $P \in S_m$  with the property that*

$$\left\| \sum_{i=1}^j v_{P(i)} \right\| \leq 2\|\beta - \alpha\|$$

for every  $j$ .

PROOF. Without loss of generality we may assume  $\alpha = 0$ , for if not, replace  $\alpha$  with 0 and  $\beta$  with  $\beta - \alpha$ . We may also without loss of generality take  $\|\beta\| = C_d$ , for if not, replace  $\beta$  with  $\beta \cdot \frac{C_d}{\|\beta\|}$  and  $v_i$  with  $v_i \cdot \frac{C_d}{\|\beta\|}$ . In this case we have  $\|v_i\| \leq 1$  for each  $i$ .

Now let  $s$  be an integer sufficiently large so that  $\frac{\|\beta\|}{s} \leq 1$ , and set  $v_{m+1} = v_{m+2} = \dots = v_{m+s} = -\beta/s$ . Then  $\alpha, v_1, \dots, v_m, v_{m+1}, \dots, v_{m+s}$  are a collection of vectors which satisfy the hypotheses of the Polygonal Confinement Theorem, and hence there exists a permutation  $P' \in S_{m+s}$  for which

$$\left\| \alpha + \sum_{i=1}^{j'} v_{P'(i)} \right\| = \left\| \sum_{i=1}^{j'} v_{P'(i)} \right\| \leq C_d$$

for every  $j' \leq m + s$ . Let  $P \in S_m$  be the unique permutation which arranges  $1, \dots, m$  in the same order as  $P'$ .

Now let  $j \leq m$  be arbitrary. Let  $j' \geq j$  be the least integer for which  $\{P(1), \dots, P(j)\} \subseteq \{P'(1), \dots, P'(j')\}$ . Note that since  $P$  and  $P'$  arrange  $1, \dots, j$  in the same order, then for any  $i \leq j'$ , we must have either  $(P')^{-1}(i) \in \{1, \dots, j\}$  or  $(P')^{-1}(i) \in \{m+1, \dots, m+s\}$ . Let  $I = \{i \leq j' : P^{-1}(i) \in \{m+1, \dots, m+s\}\}$ . Then we have:

$$\begin{aligned} \left\| \sum_{i=1}^j v_{P(i)} \right\| &= \left\| \sum_{i=1}^{j'} v_{P'(i)} - \sum_{i \in I} v_i \right\| \\ &\leq \left\| \sum_{i=1}^{j'} v_{P'(j)} \right\| + \sum_{i \in I} \|v_i\| \\ &\leq C_d + \sum_{i \in I} \frac{\|\beta\|}{s} \\ &\leq \|\beta\| + s \cdot \frac{\|\beta\|}{s} \\ &= 2\|\beta\| \end{aligned}$$

as required. □

**Lemma 5.** *Let  $\sigma$  be any finite partial permutation, and let  $\sum_{k=0}^{\infty} v_k$  be a series of vectors. If  $\pi \in S_{\infty}$  is any permutation for which  $\sum_{k=0}^{\infty} v_{\pi(k)}$  converges, then there exists another permutation  $\pi' \in S_{\infty}$  for which*

1.  $\pi'$  extends  $\sigma$ , and
2.  $\sum_{k=0}^{\infty} v_{\pi'(k)} = \sum_{k=0}^{\infty} v_{\pi(k)}$ .

PROOF. This may be accomplished by simply finding a finitely supported permutation  $\tau \in S_{\infty}$  for which  $\tau \circ \pi \upharpoonright \text{dom}(\sigma) = \sigma$ , and setting  $\pi' = \tau \circ \pi$ . □

For the remainder of the paper, given a conditionally convergent series  $\sum_{k=0}^{\infty} v_k$  in  $\mathbb{R}^d$ , let  $A(v_k) \subseteq \mathbb{R}^d$  denote the affine subspace promised in the statement of the Levy-Steinitz Theorem.

**Lemma 6** (Bounded Walk Lemma). *Let  $\sum_{k=0}^{\infty} v_k$  be a conditionally convergent series of vectors in  $\mathbb{R}^d$ . Let  $\epsilon > 0$  and  $n \in \omega$  be arbitrary. Let  $\alpha \in \mathbb{R}^d$ ,*

$\beta \in A(v_k)$ , and let  $C_d$  be as in the Polygonal Confinement Theorem. Suppose  $\pi$  is a finite partial permutation with  $\text{dom}(\pi) = J + 1 \in \omega$ , for which

$$\left\| \alpha - \sum_{k=0}^J v_{\pi(k)} \right\| < \frac{1}{3} \|\beta - \alpha\|, \text{ and such that } \|v_k\| \leq \frac{1}{3C_d} \|\beta - \alpha\| \text{ whenever } k \notin \text{ran}(\pi).$$

Then there exists a finite partial permutation  $\sigma$  with  $\text{dom}(\sigma) = I + 1 \in \omega$  which satisfies the following properties:

1.  $\sigma$  extends  $\pi$ ;
2.  $\|v_k\| \leq \frac{1}{C_d} \cdot \epsilon$  whenever  $k \notin \text{ran}(\sigma)$ ;
3.  $\left\| \beta - \sum_{k=0}^I v_{\sigma(k)} \right\| < \epsilon$ ;
4.  $\left\| \sum_{k=J+1}^i v_{\sigma(k)} \right\| \leq 6 \|\beta - \alpha\|$  whenever  $J + 1 \leq i \leq I$ ; and
5.  $n \in \text{ran}(\sigma)$ .

PROOF. Since  $\beta \in A(v_k)$ , there is  $\tau \in S_\infty$  for which  $\sum_{k=0}^\infty v_{\tau(k)} = \beta$ . By applying Lemma 5, we may assume without loss of generality that  $\tau$  extends  $\pi$ . Choose  $I \in \omega$  to be so large that  $\tau^{-1}(n) \leq I$ ,  $\left\| \beta - \sum_{k=0}^I v_{\tau(k)} \right\| < \min(\epsilon, \frac{1}{3} \|\beta - \alpha\|)$ , and that  $\|v_{\tau(k)}\| \leq \frac{1}{C_d} \cdot \epsilon$  for all  $k > I$ .

Set  $\alpha_1 = \sum_{k=0}^J v_{\tau(k)} = \sum_{k=0}^J v_{\pi(k)}$  and set  $\beta_1 = \sum_{k=0}^I v_{\tau(k)}$ . Now notice that the images  $\tau(J + 1), \dots, \tau(I)$  do not lie in the range of  $\pi$ , since  $\tau$  is a bijection extending  $\pi$  and  $\text{dom}(\pi) = \{0, \dots, J\}$ . It follows that for each  $i \in \{J + 1, \dots, I\}$  we have

$$\begin{aligned}
 \frac{1}{C_d} \|\beta_1 - \alpha_1\| &= \frac{1}{C_d} \|\beta - \alpha - (\beta - \beta_1) + (\alpha - \alpha_1)\| \\
 &\geq \frac{1}{C_d} \left| \|\beta - \alpha\| - \|\beta - \beta_1\| - \|\alpha - \alpha_1\| \right| \\
 &\geq \frac{1}{C_d} \left| \|\beta - \alpha\| - \frac{1}{3} \|\beta - \alpha\| - \frac{1}{3} \|\beta - \alpha\| \right| \\
 &= \frac{1}{3C_d} \|\beta - \alpha\| \\
 &\geq \|v_i\|.
 \end{aligned}$$

Hence we may apply Lemma 4 to find a bijection  $P : \{\tau(J+1), \dots, \tau(I)\} \rightarrow \{\tau(J+1), \dots, \tau(I)\}$  which satisfies

$$\left\| \sum_{k=J+1}^i v_{P(\tau(k))} \right\| \leq 2\|\beta_1 - \alpha_1\|$$

whenever  $J+1 \leq i \leq I$ . If we define  $\sigma : I+1 \rightarrow \omega$  by  $\sigma(k) = \tau(k)$  for  $k \leq J$  and  $\sigma(k) = P(\tau(k))$  for  $J < k \leq I$ , then  $\sigma$  is a finite partial permutation with domain  $I+1$  which clearly satisfies (1) above.

Note that if  $k \notin \text{ran}(\sigma)$ , then  $k \notin \{\sigma(0), \dots, \sigma(J), \sigma(J+1), \dots, \sigma(I)\} = \{\tau(0), \dots, \tau(J), P(\tau(J+1)), \dots, P(\tau(I))\} = \{\tau(0), \dots, \tau(I)\}$ . So  $\tau^{-1}(k) > I$ , and hence by our choice of  $I$ , we have  $\|v_k\| = \|v_{\tau(\tau^{-1}(k))}\| \leq \frac{1}{C_d} \cdot \epsilon$ . Thus (2) is also satisfied.

Since  $P$  is a bijection, we have

$$\begin{aligned}
 \left\| \beta - \sum_{k=0}^I v_{\sigma(k)} \right\| &= \left\| \beta - \left( \sum_{k=0}^J v_{\tau(k)} + \sum_{k=J+1}^I v_{P(\tau(k))} \right) \right\| \\
 &= \left\| \beta - \sum_{k=0}^I v_{\tau(k)} \right\| < \epsilon,
 \end{aligned}$$

so (3) is satisfied.

Lastly note that  $\|\beta_1 - \alpha_1\| \leq \|\beta - \beta_1\| + \|\beta - \alpha\| + \|\alpha - \alpha_1\| \leq 3\|\beta - \alpha\|$ . Hence, we have

$$\begin{aligned} \left\| \sum_{k=J+1}^i v_{\sigma(k)} \right\| &= \left\| \sum_{k=J+1}^i v_{P(\tau(k))} \right\| \\ &\leq 2\|\beta_1 - \alpha_1\| \\ &\leq 6\|\beta - \alpha\| \end{aligned}$$

whenever  $J + 1 \leq i \leq I$ . So (4) is also satisfied. (5) holds simply because  $\tau^{-1}(n) \leq I = \text{dom}(\tau)$ , and hence  $n \in \text{ran}(\tau) = \text{ran}(\sigma)$ . So the lemma is proved.  $\square$

*Remark.* For simplicity, from now on when we apply the Bounded Walk lemma, we will say that we use it to *walk from  $\alpha$  to  $\beta$* , where  $\alpha$  and  $\beta$  are as in the statement of the theorem.

#### 4 Proof of Theorems 1 and 2

First let us make a few introductory observations. We will describe our sets of interest using logical notation, with the assumption in place that all quantified variables range over  $\omega$ . Notice that by Cauchy's criterion for convergence, the following equivalence holds:

$$\pi \in \mathcal{D} \leftrightarrow \exists m \forall n \exists i \exists j \left[ i, j \geq n \wedge \left\| \sum_{k=i}^j v_{\pi(k)} \right\| \geq \frac{1}{m} \right].$$

Since the latter predicate is an open condition in  $S_\infty$ , a count of quantifiers verifies that  $\mathcal{D}$  indeed lies in  $\Sigma_3^0(S_\infty)$ . Now fix any  $m \geq 1$ , and consider the set  $\mathcal{D}_m$  defined by the following rule:

$$\pi \in \mathcal{D}_m \leftrightarrow \forall n \exists i \exists j \left[ i, j \geq n \wedge \left\| \sum_{k=i}^j v_{\pi(k)} \right\| \geq \frac{1}{m} \right].$$

Then  $\mathcal{D}_m$  is a nonempty  $\Pi_2^0(G_\delta)$ -subset of  $\mathcal{D}$  which is invariant under multiplication by finitely supported permutations, and hence dense in  $S_\infty$ . This shows that  $\mathcal{D}$  is a comeager set. The complement  $S_\infty \setminus \mathcal{D}$  is also nonempty and invariant under finitely supported permutations, and hence dense as well.

Define a map  $\phi : 2^\omega \rightarrow S_\infty$  by the rule

$$\begin{aligned} [\phi(x)](2k) &= 2k \text{ and } [\phi(x)](2k+1) = 2k+1 \text{ if } x(k) = 0; \text{ and} \\ [\phi(x)](2k) &= 2k+1 \text{ and } [\phi(x)](2k+1) = 2k \text{ if } x(k) = 1; \end{aligned}$$

for  $x \in 2^\omega$  and  $k \in \omega$ . Let  $T = \text{ran}(\phi) \subseteq S_\infty$ . The map  $\phi$  is injective and hence  $T$  is uncountable. The permutations in  $T$  act only by transposing consecutive integers, and as a consequence it is simple to check that both  $\mathcal{D}$  and  $S_\infty \setminus \mathcal{D}$  are invariant under multiplication on the left by elements of  $T$ . Thus both sets  $\mathcal{D}$  and  $S_\infty \setminus \mathcal{D}$  are uncountable dense, i.e. in some sense they are “large” nontrivial sets in  $S_\infty$ , as promised in the introduction. The reader may consult [3] for similar observations about some other sets in  $S_\infty$  which are closely related to our  $\mathcal{D}$  and  $\mathcal{DP}$ .

Next define a set  $\mathcal{I} \subseteq S_\infty$  by the rule

$$\pi \in \mathcal{I} \leftrightarrow \exists m \forall n \exists i \left[ i \geq n \wedge \left\| \sum_{k=0}^i v_{\pi(k)} \right\| \leq m \right].$$

Then  $\mathcal{I}$  is a  $\Sigma_3^0$  set, which consists exactly of those permutations whose corresponding series rearrangements  $\sum_{k=0}^\infty v_{\pi(k)}$  do not diverge to infinity. Since  $\mathcal{DP} = \mathcal{D} \cap \mathcal{I}$ , so too we have  $\mathcal{DP} \in \Sigma_3^0(S_\infty)$ .

PROOF OF THEOREMS 1 AND 2. In light of our comments above, it suffices to show that  $\mathcal{D}$  and  $\mathcal{DP}$  are  $\Sigma_3^0$ -hard. Recall that the set  $\mathcal{C} = \{x \in \omega^\omega : \lim_{n \rightarrow \infty} x(n) = \infty\}$  is known to be  $\Pi_3^0$ -complete, and the complement  $\omega^\omega \setminus \mathcal{C}$  is  $\Sigma_3^0$ -complete. We will build a function  $f : \omega^\omega \rightarrow S_\infty$  that will be a continuous reduction from  $\omega^\omega \setminus \mathcal{C}$  to both  $\mathcal{D}$  and  $\mathcal{DP}$  simultaneously. That is, both of the following will hold:

$$\begin{aligned} x \in \omega^\omega \setminus \mathcal{C} &\leftrightarrow f(x) \in \mathcal{D}, \\ x \in \omega^\omega \setminus \mathcal{C} &\leftrightarrow f(x) \in \mathcal{DP}. \end{aligned}$$

Fix an arbitrary  $x \in \omega^\omega$ . Let  $\mathbf{v} = \sum_{k=0}^\infty v_k$ . We will recursively construct a sequence of integers  $(J_n)_{n \geq -1}$  and a sequence of finite partial permutations  $(\pi_n)_{n \geq -1}$ , each with domain  $\{0, \dots, J_n\}$ , which satisfy the following seven conditions whenever  $n \geq 0$ :

- (I)  $\pi_n$  extends  $\pi_{n-1}$ ;
- (II)  $n \in \{\pi_n(0), \dots, \pi_n(J_n)\}$ ;

(III) the definitions of  $\pi_n(J_{n-1} + 1), \dots, \pi_n(J_n)$  depend only on the values of  $x(n)$  and  $x(n + 1)$ ;

(IV)  $\left\| \mathbf{v} - \sum_{k=0}^{J_n} v_{\pi_n(k)} \right\| < \frac{1}{x(n+1)+1}$ ;

(V)  $\left\| \sum_{k=J_{n-1}+1}^j v_{\pi_n(k)} \right\| \leq 36 \cdot \frac{1}{x(n)+1}$  for every  $j \in \{J_{n-1} + 1, \dots, J_n\}$ ;

(VI) there exist  $i, j \in \{J_{n-1}, \dots, J_n\}$  for which  $\left\| \sum_{k=i}^j v_{\pi_n(k)} \right\| > \frac{1}{x(n)+1}$ ; and

(VII)  $\|v_k\| \leq \frac{1}{C_d} \cdot \frac{1}{x(n+1)+1}$  for all  $k \notin \text{ran}(\pi_n)$ .

After this construction is finished, we will let  $\pi$  be the unique permutation which extends all the  $\pi_n$ 's, and set  $f(x) = \pi$ . Conditions (I) and (II) will guarantee that  $\pi$  is indeed a permutation, while (III) will guarantee that the map  $f$  is continuous. Conditions (IV) and (V) will ensure that if  $x \in \mathcal{C}$ , then  $\sum_{k=0}^{\infty} v_{\pi(k)}$  will converge to  $\mathbf{v}$ , while condition (VI) will guarantee that if  $x \notin \mathcal{C}$ , then  $\sum_{k=0}^{\infty} v_{\pi(k)}$  will diverge properly. (Condition (VII) is just a technical requirement to facilitate our recursive definition.)

We will now proceed with our construction. Here for the sake of convenience our base case will be  $n = -1$ . Let  $J_{-1} \geq 0$  be so large that  $\left\| \mathbf{v} - \sum_{k=0}^{J_{-1}} v_k \right\| < \frac{1}{x(0)+1}$ , and that  $\|v_k\| \leq \frac{1}{C_d} \cdot \frac{1}{x(0)+1}$  for all  $k > J_{-1}$ . Let  $\pi_{-1} : J_0 + 1 \rightarrow J_0 + 1$  be the identity permutation. Note that  $\pi_{-1}$  and  $J_{-1}$  trivially satisfy (IV) and (VII) above; this will be enough to facilitate our induction.

Now we assume that  $J_i$  and  $\pi_i$  are defined for all  $i < n$ , and satisfy at least (IV) and (VII), and we proceed with the inductive step of defining  $J_n$  and  $\pi_n$ . As we go we will verify that  $J_n$  and  $\pi_n$  really do satisfy all of conditions (I)-(VII).

By the Levy-Steinitz theorem, there is a non-trivial affine subspace  $A(v_k) \subseteq \mathbb{R}^d$  of points  $\beta$  such that some rearrangement of  $\sum_{k=0}^{\infty} v_k$  converges to  $\beta$ . In

particular, there is at least a line of such points. So we may choose some  $\beta \in A(v_k)$  for which  $\|\beta - \mathbf{v}\| = 3 \cdot \frac{1}{x(n)+1}$ . Now we will define  $\pi_n$  and  $J_n$  by applying the Bounded Walk lemma twice: first, we will use the lemma to “walk out” to a point near  $\beta$ , and then we will use the lemma to “walk back in” to a point near  $\mathbf{v}$ .

To “walk out”: observe that by condition (VII) of our inductive hypothesis, we have  $\|v_k\| \leq \frac{1}{C_d} \cdot \frac{1}{x(n)+1} \leq \frac{1}{3C_d} \|\beta - \mathbf{v}\|$  for all  $k \notin \text{ran}(\pi_{n-1})$ , and by condition (IV) of our inductive hypothesis, we have

$$\left\| \mathbf{v} - \sum_{k=0}^{J_{n-1}} v_{\pi_{n-1}(k)} \right\| < \frac{1}{x(n)+1} = \frac{1}{3} \|\beta - \mathbf{v}\|.$$

So we may apply the Bounded Walk lemma to walk from  $\mathbf{v}$  to  $\beta$ , extending the finite partial permutation  $\pi_{n-1}$  and with  $\epsilon = \frac{1}{x(n)+1}$ . Thus we obtain an index  $I > J_{n-1}$  and a finite partial permutation  $\sigma : I+1 \rightarrow I+1$  which satisfies properties (1)–(5) of the lemma. In particular, condition (3) ensures that we have

$$\begin{aligned} \left\| \sum_{k=J_{n-1}+1}^I v_{\sigma(k)} \right\| &= \left\| \beta - \mathbf{v} - \beta + \sum_{k=0}^I v_{\sigma(k)} + \mathbf{v} - \sum_{k=0}^{J_{n-1}} v_{\sigma(k)} \right\| \\ &\geq \|\beta - \mathbf{v}\| - \left\| \beta - \sum_{k=0}^I v_{\sigma(k)} \right\| - \left\| \mathbf{v} - \sum_{k=0}^{J_{n-1}} v_{\sigma(k)} \right\| \\ &> 3 \cdot \frac{1}{x(n)+1} - \frac{1}{x(n)+1} - \frac{1}{x(n)+1} \\ &= \frac{1}{x(n)+1} \end{aligned}$$

while condition (4) guarantees that  $\left\| \sum_{k=J_{n-1}+1}^i v_{\sigma(k)} \right\| \leq 6 \|\beta - \mathbf{v}\| = 18 \cdot \frac{1}{x(n)+1}$  whenever  $J+1 \leq i \leq I$ .

Next we “walk back in.” By property (2) in our previous application of the Bounded Walk lemma, we have  $\|v_k\| \leq \frac{1}{C_d} \cdot \frac{1}{x(n)+1} = \frac{1}{3C_d} \|\mathbf{v} - \beta\|$  whenever

$k \notin \text{ran}(\sigma)$ , and by property (3) we have  $\left\| \beta - \sum_{k=0}^I v_{\sigma(k)} \right\| < \frac{1}{x(n)+1} = \frac{1}{3} \|\mathbf{v} - \beta\|$ .

So we may apply the Bounded Walk lemma to walk from  $\beta$  to  $\mathbf{v}$ , extending

the finite partial permutation  $\sigma$ , with  $\epsilon = \frac{1}{x(n+1)+1}$ . Then we obtain an index  $J_n > I$  and a finite partial permutation  $\pi_n : J_n + 1 \rightarrow J_n + 1$  which again satisfies properties (1)–(5). By the previous inequality, and applying condition (4) for  $\pi_n$ , we see that

$$\begin{aligned} \left\| \sum_{k=J_{n-1}+1}^i v_{\pi_n(k)} \right\| &\leq \left\| \sum_{k=J_{n-1}+1}^I v_{\pi_n(k)} \right\| + \left\| \sum_{k=I+1}^i v_{\pi_n(k)} \right\| \\ &\leq \left\| \sum_{k=J_{n-1}+1}^I v_{\sigma(k)} \right\| + 6 \|\beta - \mathbf{v}\| \\ &\leq 18 \cdot \frac{1}{x(n)+1} + 18 \cdot \frac{1}{x(n)+1} \\ &= 36 \cdot \frac{1}{x(n)+1} \end{aligned}$$

whenever  $I+1 \leq i \leq J_n$ . Thus we have shown that (V) holds for  $\pi_n$ . (I), (III), and (IV) obviously hold from our definition of  $\pi_n$ , and (II) holds if we utilize condition (5) in either of our two applications of the Bounded Walk lemma to ensure that  $n \in \text{ran}(\pi_n)$ . We have shown that (VI) holds if we take  $i = J_{n-1}$  and  $j = I$ , and (VII) follows from condition (2) in our second application of the Bounded Walk lemma. So our construction is complete and we may let  $\pi \in S_\infty$  be the unique permutation which extends all of the  $\pi_n$ 's.

Define the map  $f : \omega^\omega \rightarrow S_\infty$  by  $f(x) = \pi$ , where  $\pi$  is as we have constructed above. The function  $f$ , as a map between the Polish space  $\omega^\omega$  and its Polish subspace  $S_\infty$ , is continuous by condition (III). We claim that  $f$  is in fact the continuous reduction we desire.

To see this, suppose  $x \in \mathcal{C}$ , so  $\lim_{n \rightarrow \infty} x(n) = \infty$  and hence  $\lim_{n \rightarrow \infty} \frac{1}{x(n)+1} = 0$ . For any  $i \in \omega$ , let  $n_i$  be the greatest integer for which  $J_{n_i-1} < i \leq J_{n_i}$ . Then by (IV) and (V) we have

$$\begin{aligned} \left\| \mathbf{v} - \sum_{k=0}^i v_{\pi(k)} \right\| &\leq \left\| \mathbf{v} - \sum_{k=0}^{J_{n_i-1}} v_{\pi_{n_i}(k)} \right\| + \left\| \sum_{k=J_{n_i-1}+1}^i v_{\pi(k)} \right\| \\ &\leq \frac{1}{x(n_i)+1} + 36 \cdot \frac{1}{x(n_i)+1} \\ &= 37 \cdot \frac{1}{x(n_i)+1}. \end{aligned}$$

Now taking the limit as  $i \rightarrow \infty$  (and as  $n_i \rightarrow \infty$ ) we see that  $\sum_{k=0}^{\infty} v_{\pi(k)}$  converges to  $\mathbf{v}$ . Hence  $f(x) \in S_{\infty} \setminus \mathcal{D}$  and  $f(x) \in S_{\infty} \setminus \mathcal{DP}$ .

On the other hand, suppose  $x \in \omega^{\omega} \setminus \mathcal{C}$ . Then the sequence  $(x(n))$  is cofinally bounded, i.e. there is an  $M < \infty$  such that  $x(n) < M$  infinitely often. Hence  $\frac{1}{x(n)+1} > \frac{1}{M+1}$  infinitely often. It follows from (VI) that there are infinitely many blocks  $i, \dots, j$  of integers for which  $\left\| \sum_{k=i}^j v_{\pi(k)} \right\| > \frac{1}{x(n)+1} > \frac{1}{M+1}$ , and hence  $\sum_{k=0}^{\infty} v_{\pi(k)}$  diverges by the Cauchy criterion. In addition, we have already demonstrated that for any  $n_i$  depending on  $i$  as above, we have

$$\left\| \mathbf{v} - \sum_{k=0}^{n_i} v_{\pi(k)} \right\| \leq 37 \cdot \frac{1}{x(n_i)+1} \leq 37.$$

This implies that all partial sums of the rearranged series are bounded, and so the series must diverge properly. Thus in this case we have  $f(x) \in \mathcal{D}$  and  $f(x) \in \mathcal{DP}$ . So  $f$  is the reduction we seek, and  $\mathcal{D}$  and  $\mathcal{DP}$  are  $\Sigma_3^0$ -complete.  $\square$

## References

- [1] A. Andretta and A. Marcone, *Ordinary differential equations and descriptive set theory: uniqueness and globality of solutions of Cauchy problems in one dimension*, *Fund. Math.*, **153** (1997), 157–190.
- [2] H. Becker, *Descriptive set theoretic phenomena in analysis and topology*, in: *Set Theory of the Continuum*, H. Judah, W. Just, and H. Woodin (eds.), *Math. Sci. Res. Inst. Publ.*, **26**, Springer (1992), 1–25.
- [3] R. G. Bilyeu, R. R. Kallman, and P. W. Lewis, *Rearrangements and category*, *Pacific J. Math.*, **12(1)** (1986), 41–46.
- [4] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.
- [5] P. Levy, *Sur les séries semi-convergentes*, *Nouv. Ann. d. Math.*, **64** (1905), 506–511.
- [6] B. Riemann, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, *Gesammelte Mathematische Werke* (Leipzig 1876): 213–253.

- [7] P. Rosenthal, *The remarkable theorem of Levy and Steinitz*, Amer. Math. Monthly, **94(4)** (1987), 342–351.
- [8] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, Inc., 1976.
- [9] E. Steinitz, *Bedingt Konvergente Reihe und Konvexe Systeme*, J. Reine Angew. Math., **143** (1913), 128–175.