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QUANTIZATION DIMENSION ESTIMATE FOR CONDENSATION SYSTEMS OF CONFORMAL MAPPINGS

Abstract

Let μ be the attracting measure of a condensation system consisting of a finite system of conformal mappings associated with a probability measure ν which is the image measure of an ergodic measure with bounded distortion. We have shown that for a given $r \in (0, +\infty)$ the lower and the upper quantization dimensions of order r of μ are bounded below by the quantization dimension $D_r(\nu)$ of ν and bounded above by a unique number $\kappa_r \in (0, +\infty)$ where κ_r has a relationship with the temperature function of the thermodynamic formalism that arises in multifractal analysis of μ.

1 Introduction

The term 'quantization' in this paper refers to the process of estimating a given probability by a discrete probability supported by a finite set. The quantization dimension D_r for $r \in (0, +\infty)$ of a probability measure is related to the asymptotic rate at which the expected distance (raised to the rth power) to the support of the quantized version of the probability goes to zero as the support is allowed to go to infinity. Such problem originated in the information theory and some engineering technology such as image compression and signal processing (see [4, 16]). Graf and Luschgy studied this problem systematically and gave a general mathematical treatment of it (see [3]). Given a Borel

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probability measure μ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the nth *quantization error* of order r of μ, is defined by

$$e_{n,r} := \inf\{(\int d(x,\alpha)^r d\mu(x))^{\frac{1}{r}} : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n\},$$

where $d(x,\alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. We note that if $\int \|x\|^r d\mu(x) < \infty$ then there is some set α for which the infimum is achieved (see [3]). The upper and the lower quantization dimensions of order r of μ are defined by

$$\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}}; \ \underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}}.$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call the common value the quantization dimension of order r of μ and denote it by $D_r(\mu)$.

Let $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ be a finite system of contractive conformal mappings, and let $\langle p_j \rangle := (p_0, p_1, p_2, \cdots, p_N)$ be a probability vector and ν be a probability measure on $\mathbb{R}^d$ with the compact support E. In the parallel lines as in [9, 10], it can be shown that there exists a unique Borel probability measure μ with the compact support K_E such that

$$\mu = p_0\nu + \sum_{j=1}^{N} p_j \mu \circ \varphi_j^{-1} \text{ and } K_E = \bigcup_{j=1}^{N} \varphi_j(K_E) \cup E. \tag{1}$$

Here μ is called the *inhomogeneous self-conformal measure*, and K_E is called the *inhomogeneous self-conformal set* associated with the list $(\{\varphi_j\}, \langle p_j \rangle, \nu) := (\varphi_1, \cdots, \varphi_N, p_0, p_1, \cdots, p_N, \nu)$. Note that inhomogeneous self-conformal sets and measures are the extensions of inhomogeneous self-similar sets and measures. Following [1, 8], we also call $(\{\varphi_j\}, \langle p_j \rangle, \nu)$ a *condensation system*, μ the *attracting measure* and the set K_E the *attractor* of the condensation system $(\{\varphi_j\}, \langle p_j \rangle, \nu)$.

Let the iterated function system $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ satisfy the *open set condition* (OSC): there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{j=1}^{N} \varphi_j(U) \subset U$ and $\varphi_i(U) \bigcap \varphi_j(U) = \emptyset$ for $1 \leq i \neq j \leq N$. Furthermore, the system satisfies the *strong open set condition* (SOSC) if U can be chosen such that $U \cap E \neq \emptyset$, and the *strong separation condition* (SSC) if $\varphi_j(E)$ are pairwise disjoint for $j = 1, 2, \cdots, N$.

For a finite system of self-similar mappings $\{S_1, S_2, \cdots, S_N\}$ satisfying the strong separation condition, in [15], Zhu considered a condensation system $(\{S_j\}, \langle p_j \rangle, \nu)$ where $\langle p_j \rangle = (p_0, p_1, p_2, \cdots, p_N)$, and ν is a self-similar measure which is generated by a probability vector $(t_1, t_2, \cdots, t_N)$, where $t_j > 0$ for all $1 \leq j \leq N$. For such a system Zhu determined the upper and the lower quantization dimensions of the attracting measure μ.

In this paper, we have considered a condensation system $(\{\varphi_j\}, \langle p_j \rangle, \nu)$ where $\{\varphi_j : 1 \leq j \leq N\}$ is a finite system of conformal mappings satisfying the strong separation condition, $\langle p_j \rangle = (p_0, p_1, p_2, \cdots, p_N)$, and ν is the image measure of an ergodic measure with bounded distortion over the symbolic space $\{1, 2, \cdots, N\}^{\mathbb{N}}$ (for the definition see next section) which has the support the limit set generated by the conformal mappings $\varphi_1, \varphi_2, \cdots, \varphi_N$. Let E be the support of ν, and then E satisfies the uniqueness condition: $\cup_{j=1}^{N} \varphi_j(E) = E$ (see [1, 5]). Thus for the support K_E of the attracting measure μ, by the uniqueness of the compact set K_E, it follows that $K_E = E$. Under the strong separation condition we have proved that for a given $r \in (0, +\infty)$ there exists a unique $\kappa_r \in (0, +\infty)$ such that

$$D_r(\nu) \leq \underline{D}_r(\mu) \leq \overline{D}_r(\mu) \leq \kappa_r,$$

where $\underline{D}_r(\mu)$ and $\overline{D}_r(\mu)$ are respectively the lower and the upper quantization dimensions of order r of μ, and $D_r(\nu)$ is the quantization dimension of the associated measure ν. Moreover, we have shown that κ_r has a relationship with the temperature function $\beta(q)$ of the thermodynamic formalism that arises in multifractal analysis of μ, i.e., κ_r satisfies: $\kappa_r = \frac{\beta(q_r)}{1-q_r}$ where $\beta(q_r) = rq_r$.

2 Basic definitions, lemmas and propositions

In this paper, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space equipped with a metric d compatible with the Euclidean topology. Let us write

$$V_{n,r}(\mu) = \inf\{\int d(x, \alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \ \mathrm{card}(\alpha) \leq n\},$$
$$u_{n,r}(\mu) = \inf\{\int d(x, \alpha \cup U^c)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \ \mathrm{card}(\alpha) \leq n\},$$

where U is a set which comes from the open set condition and U^c denotes the complement of U. We see that

$$u_{n,r}^{1/r} \leq V_{n,r}^{1/r} = e_{n,r}.$$

We call sets $\alpha_n \subset \mathbb{R}^d$, for which the above infimums are achieved, n-optimal sets for $e_{n,r}, V_{n,r}$ or $u_{n,r}$ respectively. If $n = 1$, we simply write e_r, V_r or u_r for $e_{n,r}, V_{n,r}$ or $u_{n,r}$ respectively. As stated above, Graf and Luschgy have shown that n-optimal sets exist when $\int \|x\|^r d\mu(x) < \infty$.

Let $V \subset \mathbb{R}^d$ be an open set. A $\mathcal{C}^1$-map $\varphi : V \to \mathbb{R}^d$ is conformal if the differential $\varphi'(x) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|\varphi'(x)y| = |\varphi'(x)| \cdot |y| \neq 0$ for all $x \in V$ and $y \in \mathbb{R}^d$, $y \neq 0$, where $|\varphi'(x)|$ is the norm of the derivative at $x \in \mathbb{R}^d$. Furthermore, $\varphi : V \to \mathbb{R}^d$ is contracting if there exists $0 < \gamma < 1$ such that $|\varphi(x) - \varphi(y)| \leq \gamma \cdot |x - y|$ for all $x, y \in V$. We say that $\{\varphi_i : X \to X\}_{i=1}^N$ is a *conformal iterated function system* (conformal IFS) on a compact set $X \subset \mathbb{R}^d$ if each φ_i extends to an injective contracting conformal map $\varphi_i : V \to V$ on an open set $V \supset X$. Let $\{\varphi_i\}_{i=1}^N$ be a conformal IFS on a compact set $X \subset \mathbb{R}^d$ for which there exists $0 < s < 1$ such that

$$d(\varphi_i(x), \varphi_i(y)) \leq s d(x, y) \tag{2}$$

for all $1 \leq i \leq N$ and all $x, y \in X$.

Let $I = \{1, 2, \cdots, N\}$. Let Ω_0 denote the set consisting only the empty word $\emptyset$. We define

$$\Omega_n := \prod_{k=1}^n I, \quad \Omega_* := \bigcup_{k=0}^\infty \Omega_k, \quad \Omega := \prod_{k=1}^\infty I.$$

For any $\sigma \in \Omega_*$, if $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \Omega_k$ we call k to be the length of σ and is denoted by $|\sigma|$, i.e., $|\sigma| = k$ ($k \geq 1$); the length of the empty word is zero. Note that if $\sigma \in \Omega$ then the length of σ is infinity, i.e., $|\sigma| = \infty$. For any $\sigma \in \Omega_* \cup \Omega$ with $|\sigma| \geq n \geq 1$, we write $\sigma|_n$ to represent the initial segment of σ of length n, i.e., $\sigma|_n := \sigma_1\sigma_2\cdots\sigma_n$, and $\sigma|_0 = \emptyset$. If $\sigma, \tau \in \Omega_*$ and $|\sigma| \leq |\tau|$, $\sigma = \tau|_{|\sigma|}$, then we call σ a predecessor of τ and denote this by $\sigma \prec \tau$; if $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, we say σ and τ are incomparable. For any two words $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ and $\tau = \tau_1\tau_2\cdots\tau_p$ in Ω_*, by $\sigma * \tau := \sigma\tau$ we mean the concatenation of the two words σ and τ, i.e., $\sigma\tau = \sigma_1\cdots\sigma_k\tau_1\cdots\tau_p$. For $\sigma = \sigma_1\sigma_2\cdots\sigma_{|\sigma|} \in \Omega_*$ let us write,

$$\sigma^- = \begin{cases} \emptyset & \text{if } |\sigma| = 1 \\ \sigma_1\sigma_2\cdots\sigma_{|\sigma|-1} & \text{if } |\sigma| > 1, \end{cases}$$

$$\varphi_\sigma = \begin{cases} \mathrm{Id}_{\mathbb{R}^d} & \text{if } \sigma = \emptyset \\ \varphi_{\sigma_1} \circ \varphi_{\sigma_2} \circ \cdots \circ \varphi_{\sigma_{|\sigma|}} & \text{if } |\sigma| \geq 1. \end{cases}$$

Inequality (2) implies that for all $i \in I$,

$$\|\varphi_i'\| = \sup_{x \in X} |\varphi_i'(x)| = \sup_{x \in X} \lim_{y \to x} \frac{d(\varphi_i(y), \varphi_i(x))}{d(y, x)} \leq \sup_{x \in X} \lim_{y \to x} \frac{s d(x, y)}{d(x, y)} = s,$$

and hence $\|\varphi_\sigma'\| \leq s^n$ for every $\sigma \in \Omega_n$, $n \geq 1$. Since given $\sigma = \sigma_1\sigma_2\cdots \in \Omega$, the diameters of the compact sets $\varphi_{\sigma|_n}(X) = \varphi_{\sigma_1} \circ \varphi_{\sigma_2} \circ \cdots \circ \varphi_{\sigma_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=1}^{\infty} \varphi_{\sigma|_n}(X)$$

is a singleton and therefore, if we denote its element by $\pi(\sigma)$, this defines the coding map $\pi : \Omega \to X$. Then the limit set of the iterated function system is

$$E := \pi(\Omega) = \bigcup_{\sigma\in\Omega} \bigcap_{n=1}^{\infty} \varphi_{\sigma|_n}(X). \tag{3}$$

Such a set E is unique and satisfies the natural invariance equality $E = \bigcup_{j=1}^{N} \varphi_j(E)$ (see [1, 5]), and is called the *self-conformal set* corresponding to the conformal IFS $\{\varphi_1, \cdots, \varphi_N\}$. Let the iterated function system $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ satisfy the *open set condition* (OSC): there exists a bounded nonempty open set $U \subset X$ such that $\bigcup_{j=1}^{N} \varphi_j(U) \subset U$ and $\varphi_i(U) \bigcap \varphi_j(U) = \emptyset$ for $1 \le i \ne j \le N$. Furthermore, the system satisfies the *strong open set condition* (SOSC) if U can be chosen such that $U \cap E \ne \emptyset$, and the *strong separation condition* (SSC) if $\varphi_j(E)$ are pairwise disjoint for $j = 1, 2, \cdots, N$. Note that in the case of a conformal iterated function system using a finite number of mappings open set condition implies the strong open set condition (see [12]). If $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ satisfies the strong separation condition then - as is easily seen - it also satisfies the open set condition, and hence the strong open set condition.

Let $\hat{\nu}$ be a shift-invariant ergodic measure on Ω satisfying the *bounded distortion property*, i.e., there exists a constant $K \ge 1$ such that for any two words σ and τ in Ω_*,

$$K^{-1}\hat{\nu}([\sigma])\hat{\nu}([\tau]) \le \hat{\nu}([\sigma\tau]) \le K\hat{\nu}([\sigma])\hat{\nu}([\tau]). \tag{4}$$

For the conformal iterated function system the following two lemmas are known.

Lemma 1. *(see [11, Lemma 2.1]) There exists a constant $C \ge 1$ such that $|\varphi_\sigma'(y)| \le C|\varphi_\sigma'(x)|$ for all $x, y \in X$ and all $\sigma \in \Omega_*$.*

Lemma 2. *(see [11, Lemma 2.2]) There exists a constant $\tilde{C} \ge C$ such that*

$$\tilde{C}^{-1}\|\varphi_\sigma'\|d(x, y) \le d(\varphi_\sigma(x), \varphi_\sigma(y)) \le \tilde{C}\|\varphi_\sigma'\|d(x, y)$$

for all $x, y \in X$ and all $\sigma \in \Omega_*$.

The following lemma easily follows from Lemma 1.

Lemma 3. *Let $C \geq 1$ be the constant as defined in Lemma 1. Then for any two words $\sigma, \tau \in \Omega_*$,*

$$C^{-1}\|\varphi'_\sigma\|\|\varphi'_\tau\| \leq \|\varphi'_{\sigma\tau}\| \leq \|\varphi'_\sigma\|\|\varphi'_\tau\|.$$

Let us now state the following three propositions which can be proved in the similar lines as in [9, 10].

Proposition 4. *Let J be a compact subset of $\mathbb{R}^d$. Then for the given conformal iterated function system $\{\varphi_1, \cdots, \varphi_N\}$ there exists a unique compact subset K_J of $\mathbb{R}^d$ such that*

$$K_J = \bigcup_{j=1}^N \varphi_j(K_J) \cup J. \tag{5}$$

Proposition 5. *Let $\langle p_j \rangle := (p_0, p_1, p_2, \cdots, p_N)$ be a probability vector and let ν be a probability measure on $\mathbb{R}^d$ with compact support. Then for a given conformal iterated function system $\{\varphi_1, \cdots, \varphi_N\}$, there exists a unique probability measure μ such that*

$$\mu = \sum_{j=1}^N p_j \mu \circ \varphi_j^{-1} + p_0 \nu. \tag{6}$$

Proposition 6. *Let μ be a unique probability measure satisfying* (6) *and let J be the support of ν. Then the support of μ is equal to the unique nonempty compact set K_J satisfying* (5).

The unique nonempty compact set K_J given by Proposition 4 is called the *inhomogeneous self-conformal* set or the *attractor*, and the unique measure μ given by Proposition 5 is called the *inhomogeneous self-conformal measure* or the *attracting measure* of the condensation system $(\{\varphi_j\}, \langle p_j \rangle, \nu)$. In the rest of the paper, we take ν as the image measure under the coding map π of the shift-invariant ergodic measure $\hat{\nu}$ given by (4), i.e., $\nu := \hat{\nu} \circ \pi^{-1}$. Then ν has the support the unique invariant set E which is given by (3), and which satisfies $E = \bigcup_{j=1}^N \varphi_j(E)$. Then for the support K_E of the attracting measure μ, by the uniqueness of the compact set K_E, it follows that $K_E = E$. For this attracting measure μ we will determine the bounds of the lower and the upper quantization dimensions. For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \Omega_*$, $n \geq 1$, set

$$E_\sigma = \varphi_\sigma(E), \text{ and write } E_\sigma = E \text{ if } \sigma \text{ is the empty word } \emptyset,$$

$$p_\sigma = \begin{cases} 1 & \text{if } \sigma = \emptyset \\ p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} & \text{if } n \geq 1. \end{cases}$$

The set E_σ for $\sigma \in \Omega_*$ is called a cylinder set. If $\varphi_1, \varphi_2, \cdots, \varphi_N$ satisfy the strong separation condition, then it is easy to see that for any two incomparable words σ and τ in Ω_* we have $\mu(E_\sigma \cap E_\tau) = 0$.

Let us now prove the following lemma.

Lemma 7. *Let $K \geq 1$ be the constant arising in the bounded distortion property. Then the probability measure ν satisfies the following inequality: For any $n \geq 1$,*

$$K^{-1} \sum_{\sigma \in \Omega_n} \hat{\nu}[\sigma] \nu \circ \varphi_\sigma^{-1} \leq \nu \leq K \sum_{\sigma \in \Omega_n} \hat{\nu}[\sigma] \nu \circ \varphi_\sigma^{-1}.$$

PROOF. Since $E = \bigcup_{j=1}^N \varphi_j(E) = \bigcup_{\sigma \in \Omega_n} \varphi_\sigma(E)$ for all $n \geq 1$, and the Borel σ-algebra on E is generated by all sets of the form $\varphi_\sigma(E)$, to prove the inequality it is enough to prove that for any $\tau \in \Omega_*$ with $|\tau| \geq n$,

$$K^{-1} \sum_{\sigma \in \Omega_n} \hat{\nu}[\sigma] \nu \circ \varphi_\sigma^{-1}(E_\tau) \leq \nu(E_\tau) \leq K \sum_{\sigma \in \Omega_n} \hat{\nu}[\sigma] \nu \circ \varphi_\sigma^{-1}(E_\tau).$$

Let $\tau \in \Omega_*$ with $|\tau| \geq n$. Then $\tau = xy$ for some $x \in \Omega_n$ and $y \in \Omega_{|\tau|-n}$, and so

$$\begin{aligned} K^{-1} \sum_{\sigma \in \Omega_n} \hat{\nu}[\sigma] \nu \circ \varphi_\sigma^{-1}(E_\tau) =& K^{-1} \hat{\nu}[x] \nu(\varphi_y(E)) \\ =& K^{-1} \hat{\nu}[x] \hat{\nu}[y] \leq \hat{\nu}[xy] = \nu(E_\tau), \end{aligned}$$

and similarly, $\nu(E_\tau) \leq K \sum_{\sigma \in \Omega_n} \hat{\nu}[\sigma] \nu \circ \varphi_\sigma^{-1}(E_\tau)$. Thus the lemma is obtained. $\square$

We have

$$\mu = p_0 \nu + \sum_{j=1}^N p_j \mu \circ \varphi_j^{-1}.$$

Now substituting for μ in the right-hand side of the above equation, we have

$$\begin{aligned} \mu &= p_0 \nu + \sum_{j_1=1}^N p_{j_1} \Big(p_0 \nu + \sum_{j_2=1}^N p_{j_2} \mu \circ \varphi_{j_2}^{-1} \Big) \circ \varphi_{j_1}^{-1} \\ &= p_0 \nu + \sum_{j_1=1} p_0 p_{j_1} \nu \circ \varphi_{j_1}^{-1} + \sum_{j_1, j_2=1} p_{j_1 j_2} \mu \circ \varphi_{j_1 j_2}^{-1}. \end{aligned}$$

Again substituting for μ in the above expression successively for any $n \geq 1$,

$$\mu = p_0\nu + \sum_{j_1=1} p_0 p_{j_1}\nu \circ \varphi_{j_1}^{-1} + \cdots + \sum_{j_1,\cdots,j_{n-1}=1} p_0 p_{j_1\cdots j_{n-1}}\nu \circ \varphi_{j_1\cdots j_{n-1}}^{-1} + \sum_{j_1,\cdots,j_n=1} p_{j_1\cdots j_n}\mu \circ \varphi_{j_1\cdots j_n}^{-1}. \tag{7}$$

Using (7) and Lemma 7, for any $n \geq 1$, we have

$$\mu \leq p_0 K\Big(\sum_{j_1,\cdots,j_n=1} \hat{\nu}[j_1\cdots j_n]\nu \circ \varphi_{j_1\cdots j_n}^{-1} + \sum_{j_1,\cdots,j_n=1} p_{j_1}\hat{\nu}[j_2\cdots j_n]\nu \circ \varphi_{j_1\cdots j_n}^{-1} + \cdots + \sum_{j_1,\cdots,j_n=1} p_{j_1\cdots j_{n-1}}\hat{\nu}[j_n]\nu \circ \varphi_{j_1\cdots j_n}^{-1}\Big) + \sum_{j_1,\cdots,j_n=1} p_{j_1\cdots j_n}\mu \circ \varphi_{j_1\cdots j_n}^{-1}$$

and so,

$$\mu \leq \sum_{j_1,\cdots,j_n=1} \Big[p_0 K\Big(\hat{\nu}[j_1\cdots j_n] + p_{j_1}\hat{\nu}[j_2\cdots j_n] + \cdots + p_{j_1\cdots j_{n-1}}\hat{\nu}[j_n]\Big)\nu \circ \varphi_{j_1\cdots j_n}^{-1} + p_{j_1\cdots j_n}\mu \circ \varphi_{j_1\cdots j_n}^{-1}\Big]. \tag{8}$$

Similarly, for any $n \geq 1$, we have

$$\mu \geq \sum_{j_1,\cdots,j_n=1} \Big[p_0 K^{-1}\Big(\hat{\nu}[j_1\cdots j_n] + p_{j_1}\hat{\nu}[j_2\cdots j_n] + \cdots + p_{j_1\cdots j_{n-1}}\hat{\nu}[j_n]\Big)\nu \circ \varphi_{j_1\cdots j_n}^{-1} + p_{j_1\cdots j_n}\mu \circ \varphi_{j_1\cdots j_n}^{-1}\Big]. \tag{9}$$

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \Omega_*$, $n \geq 1$. Then by (8),

$$\begin{aligned}\mu(E_\sigma) &\leq p_0 K\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big) + p_{\sigma_1\cdots\sigma_n} \\ &\leq K\Big[p_0\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big) + p_{\sigma_1\cdots\sigma_n}\Big] \\ &\leq K\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n] + p_{\sigma_1\cdots\sigma_n}\Big),\end{aligned} \tag{10}$$

and by (9),

$$\begin{aligned}\mu(E_\sigma) &\geq p_0 K^{-1}\Big(\hat{\nu}[\sigma_1\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big) \\ &\qquad + p_{\sigma_1\cdots\sigma_n} \\ &\geq K^{-1}\Big[p_0\Big(\hat{\nu}[\sigma_1\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big) \\ &\qquad + p_{\sigma_1\cdots\sigma_n}\Big] \\ &\geq K^{-1}p_0\Big(\hat{\nu}[\sigma_1\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n] \\ &\qquad + p_{\sigma_1\cdots\sigma_n}\Big).\end{aligned} \tag{11}$$

Moreover, the inequality (11) implies that for all $\sigma \in \Omega_*$,

$$\mu(E_\sigma) \geq p_0 K^{-1}\hat{\nu}[\sigma_1\cdots\sigma_{|\sigma|}] = p_0 K^{-1}\hat{\nu}[\sigma] = p_0 K^{-1}\nu(E_\sigma). \tag{12}$$

Lemma 8. *For any two words $\sigma, \tau \in \Omega_*$, the probability measure μ satisfies*

$$\mu(E_{\sigma\tau}) \leq K^3 p_0^{-1}\mu(E_\sigma)\mu(E_\tau).$$

PROOF. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and $\tau = \tau_1\tau_2\cdots\tau_p$ where $n, p \geq 1$. Then by (8), (11) and (12),

$$\begin{aligned}\mu(E_{\sigma\tau}) &\leq p_0 K\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big)\nu(E_\tau) \\ &\qquad + p_{\sigma_1\cdots\sigma_n}\mu(E_\tau) \\ &\leq K^2\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big)\mu(E_\tau) \\ &\qquad + p_{\sigma_1\cdots\sigma_n}\mu(E_\tau) \\ &\leq K^2\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n] \\ &\qquad + p_{\sigma_1\cdots\sigma_n}\Big)\mu(E_\tau) \\ &\leq K^3 p_0^{-1}\mu(E_\sigma)\mu(E_\tau).\end{aligned}$$

Thus the lemma is true. □

By Lemma 3, Lemma 8, and the standard theory of subadditive sequences, the following limit exists: For any two numbers $q, t \in \mathbb{R}$,

$$P(q,t) = \lim_{k\to\infty}\frac{1}{k}\log\sum_{\sigma\in\Omega_k}(\mu(E_\sigma))^q\|\varphi'_\sigma\|^t. \tag{13}$$

The above function $P(q,t)$ is called the *topological pressure* corresponding to the condensation system $(\{\varphi_j\}, \langle p_j\rangle, \nu)$. The following proposition states the well-known properties of the function $P(q,t)$ (see [2, 11]).

Proposition 9. *(i) $P(q,t) : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.*

(ii) $P(q,t)$ is strictly decreasing in each variable separately.

(iii) For fixed q we have $\lim_{t\to+\infty} P(q,t) = -\infty$ and $\lim_{t\to-\infty} P(q,t) = +\infty$.

(iv) $P(q,t)$ is convex: if $q_1, q_2, t_1, t_2 \in \mathbb{R}$, $a_1, a_2 \geq 0$, $a_1 + a_2 = 1$, then

$$P(a_1q_1 + a_2q_2, a_1t_1 + a_2t_2) \leq a_1P(q_1,t_1) + a_2P(q_2,t_2).$$

Now for fixed q, $P(q,t)$ is a continuous function of t. Its value ranges from $-\infty$ (when $t\to+\infty$) to $+\infty$ (when $t\to-\infty$). Therefore, by the intermediate value theorem there is a real number β such that $P(q,\beta) = 0$. The solution β is unique, since $P(q,\cdot)$ is strictly decreasing. This defines β implicitly as a function of q: for each q there is a unique $\beta = \beta(q)$ such that $P(q,\beta(q)) = 0$.

The following proposition gives the well-known properties of the function $\beta(q)$ (see [2, 11]).

Proposition 10. *Let $\beta = \beta(q)$ be defined by $P(q,\beta(q)) = 0$. Then*

(i) β is a continuous function of the real variable q.

(ii) β is strictly decreasing: if $q_1 < q_2$, then $\beta(q_1) > \beta(q_2)$.

(iii) $\lim_{q\to-\infty}\beta(q) = +\infty$ and $\lim_{q\to+\infty}\beta(q) = -\infty$.

(iv) β is convex: if $q_1, q_2, a_1, a_2 \in \mathbb{R}$ with $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$, then

$$\beta(a_1q_1 + a_2q_2) \leq a_1\beta(q_1) + a_2\beta(q_2).$$

The function $\beta(q)$ is sometimes denoted by $T(q)$ and called the temperature function. A more general discussion of this function can be found in [6], where our $\beta(q)$ function corresponds to $-\tau(q)$ in their notation.

Remark 11. *If $q = 0$, then $P(q,\beta(q)) = 0$ implies*

$$\lim_{k\to\infty}\frac{1}{k}\log\sum_{\sigma\in\Omega_k}\|\varphi'_\sigma\|^{\beta(0)} = 0,$$

and so $\beta(0)$ gives the Hausdorff dimension $\dim_H(E)$ of the attractor E (see [7]). Note that

$$P(1,0) = \lim_{k\to\infty}\frac{1}{k}\log\sum_{\sigma\in\Omega_k}\mu(E_\sigma) = \lim_{k\to\infty}\frac{1}{k}\log 1 = 0,$$

and hence $\beta(1) = 0$.

In the next section we state and prove the main result of the paper.

3 Main results

The following theorem gives the bounds of the lower and the upper quantization dimensions of the attracting measure μ in terms of the quantization dimension $D_r(\nu)$ of the associated measure ν and the temperature function $\beta(q)$.

Theorem 12. *Let μ be the attracting measure of the condensation system $(\{\varphi_j\}, \langle p_j \rangle, \nu)$, where $\{\varphi_1, \cdots, \varphi_N\}$ is a conformal iterated function system satisfying the strong separation condition, ν is the image measure of an ergodic measure with bounded distortion on the symbolic space $\{1, 2, \cdots, N\}^{\mathbb{N}}$. Let $\beta = \beta(q)$ be the temperature function of the thermodynamic formalism for the attracting measure μ. For each $r \in (0, +\infty)$ choose q_r such that $\beta(q_r) = rq_r$. Then the lower and the upper quantization dimensions $\underline{D}_r(\mu)$ and $\overline{D}_r(\mu)$ of order r of the attracting measure μ satisfy the following relation:*

$$D_r(\nu) \leq \underline{D}_r(\mu) \leq \overline{D}_r(\mu) \leq \frac{\beta(q_r)}{1 - q_r},$$

where $D_r(\nu)$ is the quantization dimension of order r of ν.

To prove the above theorem, we need to prove some lemmas and propositions. The following lemma plays a vital role.

Lemma 13. *Let $0 < r < +\infty$ be fixed. Then there exists exactly one number $\kappa_r \in (0, +\infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} (\mu(E_\sigma) \|\varphi'_\sigma\|^r)^{\frac{\kappa_r}{r+\kappa_r}} = 0.$$

Proof. By (13), we have

$$P(t, rt) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} (\mu(E_\sigma) \|\varphi'_\sigma\|^r)^t.$$

If $t = 0$, then $P(0, 0) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} 1 = \lim_{k\to\infty} \frac{1}{k} \log N^k = \log N > 0$; and if $t = 1$,

$$\begin{aligned} P(1, r1) &= \lim_{n\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \mu(E_\sigma) \|\varphi'_\sigma\|^r \leq \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \mu(E_\sigma) s^{kr} \\ &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \mu(E_\sigma) + r \log s \\ &= r \log s < 0. \end{aligned}$$

Since $P(t, rt)$ is continuous and strictly decreasing, the unique $t \in \mathbb{R}$ for which $P(t, rt) = 0$ must lie between 0 and 1. Then $\kappa_r = \frac{rt}{1-t}$ satisfies the conclusion of the lemma. □

Lemma 14. *For any two words $\sigma, \tau \in \Xi_*$, we have $\mu(E_{\sigma\tau}) \geq p_0 K^{-2}\mu(E_\sigma)\nu(E_\tau)$.*

Proof. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and $\tau = \tau_1\tau_2\cdots\tau_p$ where $n, p \geq 1$. Then by (9), (10) and (12),

$$\begin{aligned}
&\mu(E_{\sigma\tau})\\
&\geq p_0K^{-1}\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big)\nu(E_\tau)\\
&\qquad\qquad\qquad + p_{\sigma_1\cdots\sigma_n}\mu(E_\tau)\\
&\geq p_0K^{-1}\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\Big)\nu(E_\tau)\\
&\qquad\qquad\qquad + p_{\sigma_1\cdots\sigma_n}p_0K^{-1}\nu(E_\tau)\\
&= p_0K^{-1}\Big(\hat{\nu}[\sigma_1\sigma_2\cdots\sigma_n] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_n] + \cdots + p_{\sigma_1\cdots\sigma_{n-1}}\hat{\nu}[\sigma_n]\\
&\qquad\qquad\qquad + p_{\sigma_1\cdots\sigma_n}\Big)\nu(E_\tau)\\
&\geq p_0K^{-2}\mu(E_\sigma)\nu(E_\tau),
\end{aligned}$$

which yields the lemma. □

From the above lemma the following corollary can be deduced.

Corollary 15. *Let $0 < r < +\infty$ be fixed. Then there exists a constant $0 < A < 1$ such that for all $\sigma \in \Omega_*$, we have $\mu(E_\sigma)\|\varphi'_\sigma\|^r \geq A\mu(E_{\sigma^-})\|\varphi'_{\sigma^-}\|^r$.*

Proof. Let $L = \min\{\nu(E_1)\|\varphi'_1\|^r, \cdots, \nu(E_N)\|\varphi'_N\|^r\}$. Write $A = p_0K^{-2}LC^{-r}$, and then $0 < A < 1$. Thus, using Lemma 3 and Lemma 14, we have

$$\begin{aligned}
\mu(E_\sigma)\|\varphi'_\sigma\|^r &\geq p_0K^{-2}C^{-r}\mu(E_{\sigma^-})\nu(E_{\sigma_{|\sigma|}})\|\varphi'_{\sigma^-}\|^r\|\varphi'_{\sigma_{|\sigma|}}\|^r\\
&\geq p_0K^{-2}LC^{-r}\mu(E_{\sigma^-})\|\varphi'_{\sigma^-}\|^r,
\end{aligned}$$

which yields the corollary. □

We call $\Gamma \subset \Omega_*$ a *finite maximal antichain* if Γ is a finite set of words in Ω_* such that every sequence in Ω is an extension of some word in Γ, but no word of Γ is an extension of another word in Γ. Of course, this requires that the index set $\{1, 2, \cdots, N\}$ is finite. We will make this assumption in the remainder of this paper. By $|\Gamma|$ we denote the cardinality of Γ. Note that from the definition of Γ it follows that finite maximal antichain does not contain the empty word $\emptyset$ as all words are extension of $\emptyset$.

Remark 16. *Let $0 < r < +\infty$ be fixed and κ_r be as in Lemma 13. We assume that there exists a constant $C_r > 1$ such that for any finite maximal antichain Γ,*

$$\sum_{\sigma\in\Gamma} \Big(\mu(E_\sigma)\|\varphi'_\sigma\|^r\Big)^{\frac{\kappa_r}{r+\kappa_r}} \leq C_r.$$

This assumption we need to prove Proposition 20.

Lemma 17. *Let Γ be a finite maximal antichain. Then for all $\sigma \in \Omega_*$,*

$$\mu \leq \sum_{\sigma\in\Gamma} \Big[p_0 K\Big(\hat{\nu}[\sigma_1\cdots\sigma_{|\sigma|}] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_{|\sigma|}] + \cdots + p_{\sigma_1\cdots\sigma_{|\sigma|-1}}\hat{\nu}[\sigma_{|\sigma|}]\Big)\nu\circ\varphi_\sigma^{-1} + p_{\sigma_1\cdots\sigma_{|\sigma|}}\mu\circ\varphi_\sigma^{-1}\Big].$$

Proof. Let $m = \max\{|\sigma| : \sigma \in \Gamma\}$. Since $E = \bigcup_{j=1}^N \varphi_j(E) = \bigcup_{\sigma\in\Omega_n}\varphi_\sigma(E)$ for all $n \geq 1$, and the Borel σ-algebra on E is generated by all sets of the form $\varphi_\sigma(E)$, to prove the inequality it is enough to prove that for any $\tau \in \Omega_*$ with $|\tau| \geq m$,

$$\mu(E_\tau) \leq \sum_{\sigma_1\cdots\sigma_{|\sigma|}\in\Gamma} \Big[p_0 K\Big(\hat{\nu}[\sigma_1\cdots\sigma_{|\sigma|}] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_{|\sigma|}] + \cdots + p_{\sigma_1\cdots\sigma_{|\sigma|-1}}\hat{\nu}[\sigma_{|\sigma|}]\Big)\nu\circ\varphi^{-1}_{\sigma_1\cdots\sigma_{|\sigma|}} + p_{\sigma_1\cdots\sigma_{|\sigma|}}\mu\circ\varphi^{-1}_{\sigma_1\cdots\sigma_{|\sigma|}}\Big](E_\tau).$$

Let $\tau \in \Omega_*$ with $|\tau| \geq m$. As Γ is a finite maximal antichain, corresponding to τ there exist $x \in \Gamma$ and $y \in \Omega_*$ such that $\tau = xy$. Then by (8), we have

$$\begin{aligned}
&\sum_{\sigma_1\cdots\sigma_{|\sigma|}\in\Gamma} \Big[p_0 K\Big(\hat{\nu}[\sigma_1\cdots\sigma_{|\sigma|}] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_{|\sigma|}] + \cdots \\
&\qquad + p_{\sigma_1\cdots\sigma_{|\sigma|-1}}\hat{\nu}[\sigma_{|\sigma|}]\Big)\nu\circ\varphi^{-1}_{\sigma_1\cdots\sigma_{|\sigma|}} + p_{\sigma_1\cdots\sigma_{|\sigma|}}\mu\circ\varphi^{-1}_{\sigma_1\cdots\sigma_{|\sigma|}}\Big](E_\tau) \\
&= p_0 K\Big(\hat{\nu}[x_1\cdots x_{|x|}] + p_{x_1}\hat{\nu}[x_2\cdots x_{|x|}] + \cdots + p_{x_1\cdots x_{|x|-1}}\hat{\nu}[x_{|x|}]\Big)\nu(E_y) \\
&\qquad + p_{x_1\cdots x_{|x|}}\mu(E_y) \\
&\geq \mu(E_\tau),
\end{aligned}$$

and thus the lemma is true. □

Lemma 18. *Let $\Gamma \subset \Omega_*$ be a finite maximal antichain, $n \in \mathbb{N}$ with $n \geq |\Gamma|$, and $0 < r < +\infty$. Then for any sequence $\{n_\sigma : \sigma \in \Gamma\}$ of natural numbers satisfying $1 \leq n_\sigma$, $\sum_{\sigma \in \Gamma} n_\sigma \leq n$, we have*

$$V_{2n,r}(\mu) \leq \tilde{C}^r \sum_{\sigma \in \Gamma} \Big[p_0 K \|\varphi'_\sigma\|^r \Big(\hat{\nu}[\sigma_1 \cdots \sigma_{|\sigma|}] + p_{\sigma_1} \hat{\nu}[\sigma_2 \cdots \sigma_{|\sigma|}] + \cdots + p_{\sigma_1 \cdots \sigma_{|\sigma|-1}} \hat{\nu}[\sigma_{|\sigma|}] \Big) V_{n_\sigma, r}(\nu) + p_{\sigma_1 \cdots \sigma_{|\sigma|}} \|\varphi'_\sigma\|^r V_{n_\sigma, r}(\mu) \Big].$$

Proof. Suppose $n_\sigma \geq 1$ for each $\sigma \in \Gamma$, and $\sum_{\sigma \in \Gamma} n_\sigma \leq n$. For each $\sigma \in \Gamma$ let α_σ be an n_σ-optimal set for $V_{n_\sigma, r}(\nu)$ and β_σ be an n_σ-optimal set for $V_{n_\sigma, r}(\mu)$. Since $|\cup_{\sigma \in \Gamma} \varphi_\sigma(\alpha_\sigma \cup \beta_\sigma)| \leq 2n$, by the previous lemma, we have

$$\begin{aligned} V_{2n,r}(\mu) &\leq \int d(x, \bigcup_{\sigma \in \Gamma} \varphi_\sigma(\alpha_\sigma \cup \beta_\sigma))^r d\mu(x) \\ &\leq \sum_{\sigma \in \Gamma} \Big[p_0 K \Big(\hat{\nu}[\sigma_1 \cdots \sigma_{|\sigma|}] + p_{\sigma_1} \hat{\nu}[\sigma_2 \cdots \sigma_{|\sigma|}] + \cdots + p_{\sigma_1 \cdots \sigma_{|\sigma|-1}} \hat{\nu}[\sigma_{|\sigma|}] \Big) \\ &\qquad \int d(x, \varphi_\sigma(\alpha_\sigma))^r d(\nu \circ \varphi_\sigma^{-1})(x) + p_{\sigma_1 \cdots \sigma_{|\sigma|}} \\ &\qquad \int d(x, \varphi_\sigma(\beta_\sigma))^r d(\mu \circ \varphi_\sigma^{-1})(x) \Big] \\ &\leq \tilde{C}^r \sum_{\sigma \in \Gamma} \Big[p_0 K \|\varphi'_\sigma\|^r \Big(\hat{\nu}[\sigma_1 \cdots \sigma_{|\sigma|}] + p_{\sigma_1} \hat{\nu}[\sigma_2 \cdots \sigma_{|\sigma|}] + \cdots \\ &\qquad + p_{\sigma_1 \cdots \sigma_{|\sigma|-1}} \hat{\nu}[\sigma_{|\sigma|}] \Big) V_{n_\sigma, r}(\nu) + p_{\sigma_1 \cdots \sigma_{|\sigma|}} \|\varphi'_\sigma\|^r V_{n_\sigma, r}(\mu) \Big]. \end{aligned}$$

Hence the lemma. □

Lemma 19. *There exists a constant $D > 1$ such that $V_{n,r}(\nu) \leq D V_{n,r}(\mu)$ for all $n \geq 1$.*

Proof. Let α be an optimal set for $V_{n,r}(\mu)$ and $0 < r < +\infty$. Since $\mu =$

$p_0\nu + \sum_{j=1}^N p_j\mu\circ\varphi_j^{-1}$, by Lemma 2, we have

$$\begin{aligned} V_{n,r}(\mu) &= \int d(x,\alpha)^r d\mu(x) \\ &= p_0\int d(x,\alpha)^r d\nu(x) + \sum_{j=1}^N p_j \int d(x,\alpha)^r d(\mu\circ\varphi_j^{-1})(x) \\ &\geq p_0\int d(x,\alpha)^r d\nu(x) + \tilde{C}^{-r}\sum_{j=1}^N p_j\|\varphi_j'\|^r \int d(x,\varphi_j^{-1}\alpha)^r d\mu(x) \\ &\geq p_0 V_{n,r}(\nu) + \tilde{C}^{-r}\sum_{j=1}^N p_j\|\varphi_j'\|^r V_{n,r}(\mu). \end{aligned}$$

Since φ_j are conformal mappings, $\varphi_j \in \mathcal{C}^1$, $|\varphi_j'(x)| \neq 0$ for all $1 \leq j \leq N$ and all $x \in X$. Moreover, X is compact. Therefore, there exists a number $R \in \mathbb{R}$, such that $0 < R \leq |\varphi_j'(x)| \leq \|\varphi_j'\| \leq s < 1$ for all $1 \leq j \leq N$ and all $x \in X$. Thus we have,

$$V_{n,r}(\mu) \geq p_0 V_{n,r}(\nu) + \tilde{C}^{-r}\sum_{j=1}^N p_j R^r V_{n,r}(\mu).$$

Take $D = (1 - \tilde{C}^{-r}\sum_{j=1}^N p_j R^r)/p_0$, and note that $D > (1 - \sum_{j=1}^N p_j)/p_0 = 1$ to obtain the assertion of the lemma. □

Proposition 20. *Let $0 < r < +\infty$ be fixed and κ_r be as in Lemma 13. Then under the assumption of Remark 16, we have*

$$\limsup_{n\to\infty} n V_{n,r}^{\frac{\kappa_r}{r}}(\mu) < +\infty.$$

PROOF. Let $0 < A < 1$ be the constant as defined in Corollary 15, and C_r be the constant as defined in Remark 16. Fix $m \in \mathbb{N}$. Choose any $n \in \mathbb{N}$ so that $\frac{m}{n} < AC_r^{-1}$, and set $\epsilon = C_r A^{-1}\frac{m}{n}$. Then $0 < \epsilon < 1$. Let

$$\Gamma = \Gamma(\epsilon) = \{\sigma \in \Omega_* : (\mu(E_\sigma)\|\varphi_\sigma'\|^r)^{\frac{\kappa_r}{r+\kappa_r}} < \epsilon \leq (\mu(E_{\sigma^-})\|\varphi_{\sigma^-}'\|^r)^{\frac{\kappa_r}{r+\kappa_r}}\}.$$

Then by Remark 16 and Corollary 15, we have

$$C_r \geq \sum_{\sigma\in\Gamma}(\mu(E_\sigma)\|\varphi_\sigma'\|^r)^{\frac{\kappa_r}{r+\kappa_r}} \geq A^{\frac{\kappa_r}{r+\kappa_r}}\sum_{\sigma\in\Gamma}(\mu(E_{\sigma^-})\|\varphi_{\sigma^-}'\|^r)^{\frac{\kappa_r}{r+\kappa_r}} > A\epsilon|\Gamma|,$$

which implies $|\Gamma| < C_r(A\epsilon)^{-1} = \frac{n}{m} < \infty$, i.e., Γ is a finite maximal antichain and $n > |\Gamma|m$. Then by Lemma 18 and Lemma 19, and then by (11), we have

$$\begin{aligned}
&V_{2n,r}(\mu)\\
&\leq \tilde{C}^r \sum_{\sigma\in\Gamma} \Big[p_0 K \|\varphi'_\sigma\|^r \Big(\hat{\nu}[\sigma_1\cdots\sigma_{|\sigma|}] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_{|\sigma|}] + \cdots\\
&\qquad\qquad + p_{\sigma_1\cdots\sigma_{|\sigma|-1}}\hat{\nu}[\sigma_{|\sigma|}]\Big) V_{m,r}(\nu) + p_{\sigma_1\cdots\sigma_{|\sigma|}} \|\varphi'_\sigma\|^r V_{m,r}(\mu)\Big]\\
&\leq \tilde{C}^r D \sum_{\sigma\in\Gamma} \Big[p_0 K \Big(\hat{\nu}[\sigma_1\cdots\sigma_{|\sigma|}] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_{|\sigma|}] + \cdots + p_{\sigma_1\cdots\sigma_{|\sigma|-1}}\hat{\nu}[\sigma_{|\sigma|}]\Big)\\
&\qquad\qquad + p_{\sigma_1\cdots\sigma_{|\sigma|}}\Big] \|\varphi'_\sigma\|^r V_{m,r}(\mu)\\
&\leq \tilde{C}^r D K \sum_{\sigma\in\Gamma} \Big[p_0 \Big(\hat{\nu}[\sigma_1\cdots\sigma_{|\sigma|}] + p_{\sigma_1}\hat{\nu}[\sigma_2\cdots\sigma_{|\sigma|}] + \cdots + p_{\sigma_1\cdots\sigma_{|\sigma|-1}}\hat{\nu}[\sigma_{|\sigma|}]\Big)\\
&\qquad\qquad + p_{\sigma_1\cdots\sigma_{|\sigma|}}\Big] \|\varphi'_\sigma\|^r V_{m,r}(\mu)\\
&\leq \tilde{C}^r D K^2 \sum_{\sigma\in\Gamma} \mu(E_\sigma) \|\varphi'_\sigma\|^r V_{m,r}(\mu)\\
&\leq \tilde{C}^r D K^2 \sum_{\sigma\in\Gamma} \left(\mu(E_\sigma)\|\varphi'_\sigma\|^r\right)^{\frac{\kappa_r}{r+\kappa_r}} \left(\mu(E_\sigma)\|\varphi'_\sigma\|^r\right)^{\frac{r}{r+\kappa_r}} V_{m,r}(\mu),
\end{aligned}$$

and thus,

$$V_{2n,r}(\mu) < \tilde{C}^r D K^2 \sum_{\sigma\in\Gamma} \left(\mu(E_\sigma)\|\varphi'_\sigma\|^r\right)^{\frac{\kappa_r}{r+\kappa_r}} \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu)$$

which implies

$$V_{2n,r}(\mu) < \tilde{C}^r D K^2 C_r \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu) = \tilde{C}^r D K^2 C_r (C_r A^{-1})^{\frac{r}{\kappa_r}} \left(\frac{m}{n}\right)^{\frac{r}{\kappa_r}} V_{m,r}(\mu),$$

yielding $2nV_{2n,r}^{\frac{\kappa_r}{r}}(\mu) \leq 2\Big(\tilde{C}^r D K^2 C_r\Big)^{\frac{\kappa_r}{r}} C_r A^{-1} m V_{m,r}^{\frac{\kappa_r}{r}}(\mu)$. Since for fixed m, this inequality holds for all but a finite number of n, we have

$$\limsup_{n\to\infty} 2nV_{2n,r}^{\frac{\kappa_r}{r}}(\mu) \leq 2\Big(\tilde{C}^r D K^2 C_r\Big)^{\frac{\kappa_r}{r}} C_r A^{-1} m V_{m,r}^{\frac{\kappa_r}{r}}(\mu) < +\infty,$$

and thus the proposition is obtained.

□

Remark 21. *In [13], Roychowdhury determined the quantization dimension $D_r(\nu)$ of the image measure ν. But, from there it is not known whether $\liminf_{n\to\infty} ne_{n,r}^{D_r(\nu)}(\nu) > 0$. Using the parallel arguments as in [14], it can be shown that $\liminf_{n\to\infty} ne_{n,r}^{D_r(\nu)}(\nu) > 0$.*

Let us now prove the following proposition.

Proposition 22. *Let the conformal mappings $\{\varphi_j : 1 \leq j \leq N\}$ satisfy the strong separation condition and let $0 < r < +\infty$. Moreover, let $D_r(\nu)$ be the quantization dimension of order r of the probability measure ν. Then $\liminf_{n\to\infty} ne_{n,r}^{D_r(\nu)}(\mu) > 0$.*

PROOF. Since $D_r(\nu)$ is the quantization dimension of order r of the image measure ν, by Remark 21, we have $\liminf_{n\to\infty} ne_{n,r}^{D_r(\nu)}(\nu) > 0$. By Lemma 19, for any $n \geq 1$, we have $e_{n,r}(\nu) = V_{n,r}^{\frac{1}{r}}(\nu) \leq D^{\frac{1}{r}} V_{n,r}^{\frac{1}{r}}(\mu) = D^{\frac{1}{r}} e_{n,r}(\mu)$. Hence,

$$\liminf_{n\to\infty} ne_{n,r}^{D_r(\nu)}(\mu) \geq D^{-\frac{D_r(\nu)}{r}} \liminf_{n\to\infty} ne_{n,r}^{D_r(\nu)}(\nu) > 0.$$

□

Proof of Theorem 12.

By Proposition 11.3 of [3], we know:

(a) If $0 \leq t < \underline{D}_r < s$ then

$$\lim_{n\to\infty} ne_{n,r}^t = +\infty \text{ and } \liminf_{n\to\infty} ne_{n,r}^s = 0.$$

(b) If $0 \leq t < \overline{D}_r < s$ then

$$\limsup_{n\to\infty} ne_{n,r}^t = +\infty \text{ and } \lim_{n\to\infty} ne_{n,r}^s = 0.$$

From (a) and Proposition 22, we have $D_r(\nu) \leq \underline{D}_r(\mu)$. By ($b$) and Proposition 20, we have $\overline{D}_r(\mu) \leq \kappa_r$. Hence, $D_r(\nu) \leq \underline{D}_r(\mu) \leq \overline{D}_r(\mu) \leq \kappa_r$. Note that if $q_r = \frac{\kappa_r}{r+\kappa_r}$ then by Lemma 13, $\beta(q_r) = rq_r$. Thus it follows that $\kappa_r = \frac{\beta(q_r)}{1-q_r}$. Hence the proof of the theorem.

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