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ON INTERNALLY STRONG ŚWIĄTKOWSKI FUNCTIONS

Abstract

In this paper we introduce some notions which are related to quasi-continuity and a strong Świątkowski property. We examine the basic properties, the uniform closure, and several varieties of maximal classes for the family of internally strong Świątkowski functions. Moreover we study when the function f can be expressed as the sum of two internally quasi-continuous functions and the sum of two internally strong Świątkowski functions.

1 Preliminaries

We use mostly standard terminology and notation. The letters $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of real numbers, integers, and positive integers, respectively. For all $a, b \in \mathbb{R}$ we define $\mathrm{I}[a, b) = [a, b)$, if $a < b$, and $\mathrm{I}[a, b) = (b, a]$ otherwise. The symbols $\mathrm{I}(a, b)$ and $\mathrm{I}[a, b]$ are defined analogously. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, $\operatorname{card} A$, $\mu(A)$, and χ_A to denote the interior, the closure, the boundary, the cardinality, the Lebesgue measure, and the characteristic function of A, respectively.

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Let I be a nondegenerate interval and $f\colon I \to \mathbb{R}$ (the interval I can be also equal to $\mathbb{R}$). The symbols $\mathcal{C}(f)$ and $\mathcal{A}(f)$ will stand for the set of all points of continuity of f and the set of all local maxima (not necessarily strict) of f, respectively. We say that f is a *Darboux function* ($f \in \mathcal{D}$), if it maps connected sets onto connected sets. We say that f is *quasi-continuous* in the sense of Kempisty [3] ($f \in \mathcal{Q}$), if for all $x \in I$ and open sets $U \ni x$ and $V \ni f(x)$, the set $\operatorname{int}(U \cap f^{-1}(V))$ is nonempty. We will say that f is *internally quasi-continuous* ($f \in \mathcal{Q}_i$), if it is quasi-continuous and its set of points of discontinuity is nowhere dense. We say that f is a *strong Świątkowski function* [5] ($f \in \acute{\mathcal{S}}_s$), if whenever $a, b \in I$, $a < b$, and $y \in \mathrm{I}(f(a), f(b))$, there is an $x_0 \in (a,b) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. We will say that f is an *internally strong Świątkowski function* ($f \in \acute{\mathcal{S}}_{si}$), if whenever $a, b \in I$, $a < b$, and $y \in \mathrm{I}(f(a), f(b))$, there is an $x_0 \in (a,b) \cap \operatorname{int}\mathcal{C}(f)$ such that $f(x_0) = y$. (Clearly $\acute{\mathcal{S}}_{si} \subset \acute{\mathcal{S}}_s \subset \mathcal{D} \cap \mathcal{Q}$.) We say that $f \in \mathcal{C}onst$ if and only if $f[I]$ is a singleton. We will identify functions with their graphs. Later in this paper the word function will denote a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. For each nonempty set $A \subset I$ we define $\operatorname{osc}(f, A)$ as the *oscillation of f on A*, i.e., $\operatorname{osc}(f, A) = \sup\bigl\{|f(x) - f(t)| : x, t \in A\bigr\}$. For each $x \in I$ we define $\operatorname{osc}(f, x)$ as the *oscillation of f at x*, i.e., $\operatorname{osc}(f, x) = \lim_{\delta \to 0^+} \operatorname{osc}\bigl(f, I \cap (x - \delta, x + \delta)\bigr)$.

If $\mathcal{L}$ is a family of real functions, then we define:

$$\mathcal{M}_a(\mathcal{L}) = \bigl\{f : \forall\, g \in \mathcal{L} \;\; f + g \in \mathcal{L}\bigr\},$$
$$\mathcal{M}_m(\mathcal{L}) = \bigl\{f : \forall\, g \in \mathcal{L} \;\; f \cdot g \in \mathcal{L}\bigr\},$$
$$\mathcal{M}_{\max}(\mathcal{L}) = \bigl\{f : \forall\, g \in \mathcal{L} \;\; \max\{f, g\} \in \mathcal{L}\bigr\}.$$

The above classes are called the maximal additive class for $\mathcal{L}$, the maximal multiplicative class for $\mathcal{L}$, and the maximal class with respect to maximums for $\mathcal{L}$, respectively.

2 Introduction

In 1988 Z. Grande [2] proved that a function f is quasi-continuous if and only if $f{\restriction}\mathcal{C}(f)$ is dense in f. Similarly we can write an equivalent definition of internally quasi-continuity, namely, f is internally quasi-continuous if and only if $f{\restriction}\operatorname{int}\mathcal{C}(f)$ is dense in f. It turns out that this stronger condition is fulfilled by some subclasses of Darboux functions, e.g., Baire one star Darboux functions [8] or finitely continuous Darboux functions [7]. In 1988 E. Strońska [9] constructed an approximately continuous quasi-continuous function which is almost everywhere discontinuous. Clearly such a function is not internally quasi-continuous. On the other hand every internally quasi-continuous function is quasi-continuous and it easy to prove that every internally strong Świątkowski function is internally quasi-continuous.

In 1966 A.M. Bruckner, J.G. Ceder and M.L. Weiss defined the class of real functions U in the following way [1]: $f \in U$ if and only if for all $a < b$ and each set $A \subset [a, b]$ with $\operatorname{card} A < \mathfrak{c}$ (where $\mathfrak{c} = \operatorname{card} \mathbb{R}$) the set $f([a, b] \setminus A)$ is dense in $\mathrm{I}[f(a), f(b)]$. They showed that the class U is the uniform closure of the family of all Darboux function, i.e., $f \in U$, if f is the limit of some uniformly convergent sequence of Darboux functions. In 1995 A. Maliszewski [5] proved that the function f is the limit of some uniformly convergent sequence of strong Świątkowski functions if and only if $f \in U$ and it is quasi-continuous.

The main goal of this paper is to characterize the uniform closure of the family of internally strong Świątkowski functions (Corollary 4.6). Moreover we show that the maximal additive class, the maximal multiplicative class, and the maximal class with respect to maximums for the family of internally strong Świątkowski functions consist of constant functions only (Theorems 4.7–4.9). Finally we examine when the function f can be expressed as the sum of two internally quasi-continuous functions and the sum of two internally strong Świątkowski functions (Theorem 4.10).

3 Auxiliary lemmas

Definition 3.1. We will say that the function f is internally quasi-continuous at a point $x \in \mathbb{R}$, if there is a sequence $(x_n) \subset \operatorname{int} \mathcal{C}(f)$ such that $x_n \to x$ and $f(x_n) \to f(x)$.

Lemma 3.2 is evident.

Lemma 3.2. *The function f is internally quasi-continuous if and only if it is internally quasi-continuous at every point.*

Lemmas 3.3 and 3.4 can be easily proved using [4, Theorem 12].

Lemma 3.3. *Let $g\colon [\alpha, \beta] \to \mathbb{R}$ and $x \in (\alpha, \beta)$. If $g{\restriction}[\alpha, x) \in \acute{\mathcal{S}}_s$, $g{\restriction}(x, \beta] \in \acute{\mathcal{S}}_s$, and $x \in \mathcal{C}(g)$, then $g \in \acute{\mathcal{S}}_s$.*

Lemma 3.4. *Let $g\colon [\alpha, \beta] \to \mathbb{R}$ and $x \in (\alpha, \beta)$. If $g{\restriction}[\alpha, x] \in \acute{\mathcal{S}}_s$, $g{\restriction}(x, \beta] \in \acute{\mathcal{S}}_s$, and $g(x) \in g\big[[x, t] \cap \mathcal{C}(g)\big]$ for each $t \in (x, \beta)$, then $g \in \acute{\mathcal{S}}_s$.*

Now we will prove two analogous lemmas for the family $\acute{\mathcal{S}}_{si}$.

Lemma 3.5. *Let $g\colon [\alpha, \beta] \to \mathbb{R}$ and $x \in (\alpha, \beta)$. If $g{\restriction}[\alpha, x) \in \acute{\mathcal{S}}_{si}$, $g{\restriction}(x, \beta] \in \acute{\mathcal{S}}_{si}$, and $x \in \operatorname{int} \mathcal{C}(g)$, then $g \in \acute{\mathcal{S}}_{si}$.*

Proof. Let $a, b \in [\alpha, \beta]$, $a < b$, and $y \in \mathrm{I}(g(a), g(b))$. We may assume that $a \le x \le b$. If $y \in \mathrm{I}(g(a), g(x))$, then we can choose a point $c \in (a, x)$ such that

$$|g(c) - g(x)| < |y - g(x)|.$$

Hence $y \in \mathrm{I}(g(a), g(c))$ and since $g{\restriction}[\alpha, x) \in \acute{\mathcal{S}}_{si}$, we have $g(x_0) = y$ for some $x_0 \in (a, c) \cap \operatorname{int} \mathcal{C}(g) \subset (a, b) \cap \operatorname{int} \mathcal{C}(g)$. Similarly we proceed if $y \in \mathrm{I}(g(x), g(b))$. If $y = g(x)$, then $a < x < b$ and $x \in \operatorname{int} \mathcal{C}(g)$. So, we have proved that $g \in \acute{\mathcal{S}}_{si}$. $\square$

Lemma 3.6. *Let $g\colon [\alpha, \beta] \to \mathbb{R}$ and $x \in (\alpha, \beta)$. If $g{\restriction}[\alpha, x] \in \acute{\mathcal{S}}_{si}$, $g{\restriction}(x, \beta] \in \acute{\mathcal{S}}_{si}$, and $g(x) \in g\big[[x, t] \cap \operatorname{int} \mathcal{C}(g)\big]$ for each $t \in (x, \beta)$, then $g \in \acute{\mathcal{S}}_{si}$.*

Proof. Let $a, b \in [\alpha, \beta]$, $a < b$, and $y \in \mathrm{I}(g(a), g(b))$. We may assume that $x \notin \operatorname{int} \mathcal{C}(g)$ (see Lemma 3.5) and $a \le x < b$. If $y \in \mathrm{I}(g(a), g(x))$, then since $g{\restriction}I \cap [\alpha, x] \in \acute{\mathcal{S}}_{si}$, we have $g(x_0) = y$ for some $x_0 \in (a, x) \cap \operatorname{int} \mathcal{C}(g) \subset (a, b) \cap \operatorname{int} \mathcal{C}(g)$. If $y \in \mathrm{I}[g(x), g(b))$, then we can choose a $z \in (x, b) \cap \operatorname{int} \mathcal{C}(g)$ such that $g(x) = g(z)$. Hence $y \in \mathrm{I}[g(z), g(b))$, and since $g{\restriction}(x, \beta] \in \acute{\mathcal{S}}_{si}$, there is an $x_0 \in [z, b) \cap \operatorname{int} \mathcal{C}(g) \subset (a, b) \cap \operatorname{int} \mathcal{C}(g)$ with $g(x_0) = y$. So, we proved that $g \in \acute{\mathcal{S}}_{si}$. $\square$

The proof of Lemma 3.7 is immediate.

Lemma 3.7. *Let $I \subset \mathbb{R}$ be an interval, $g\colon I \to \mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$. If $g \in \acute{\mathcal{S}}_{si}$ and f is continuous, then $f \circ g \in \acute{\mathcal{S}}_{si}$.*

The next lemma is due to A. Maliszewski [5, Lemma 3].

Lemma 3.8. *Assume that the set K is nowhere dense and closed, function $g \in U$ is locally bounded on $\mathbb{R} \setminus K$ and quasi-continuous, and $\eta > 0$. Then there is a nowhere dense closed set F and a continuous function α such that $\alpha = 0$ on K, $|\alpha| \le \eta$ on $\mathbb{R}$, and*

$$[\varliminf_{t \to x} g(t), \varlimsup_{t \to x} g(t)] \subset (g + \alpha)[I \cap F \setminus K]$$

for every $x \in K$ and every open interval $I \ni x$.

The proof of Lemma 3.9 is similar to the proof of [5, Lemma 1].

Lemma 3.9. *Let $g \in U$. If for every $x \notin \operatorname{int} \mathcal{C}(g)$ and every open interval $I \ni x$ the following condition holds:*

$$[\varliminf_{t \to x} g(t), \varlimsup_{t \to x} g(t)] \subset g[I \cap \operatorname{int} \mathcal{C}(g)],$$

then g is an internally strong Świątkowski function.

Proof. Let $a < b$ and $y \in \mathrm{I}(g(a), g(b))$. Without loss of generality we may assume that $g(a) < g(b)$. Since $g \in U$, there is an $a' \in (a, b)$ such that $g(a') < y$. Put

$$x_0 = \sup\{t \in [a', b) : \ g < y \ \text{ on } \ [a', t]\}.$$

Using once more that $g \in U$ we have $x_0 < b$ and

$$\varliminf_{t \to x_0} g(t) \le y \le \varlimsup_{t \to x_0} g(t).$$

So, if $x_0 \in \operatorname{int} \mathcal{C}(g)$, then $g(x_0) = y$. In the other case, since $x_0 \in (a, b)$, by assumption,

$$y \in [\varliminf_{t \to x_0} g(t), \varlimsup_{t \to x_0} g(t)] \subset g[(a, b) \cap \operatorname{int} \mathcal{C}(g)].$$

Hence there is an $x_1 \in (a, b) \cap \operatorname{int} \mathcal{C}(g)$ with $g(x_1) = y$, which proves that $g \in \acute{\mathcal{S}}_{si}$. □

4 Main results

First we present some properties of the families $\acute{\mathcal{S}}_{si}$ and $\mathcal{Q}_i$.

Proposition 4.1. *There is an internally quasi-continuous and strong Świątkowski function which is not internally strong Świątkowski.*

Proof. For $a < b$ and $x \in (a, b)$ define

$$\varphi_{a,b}(x) = (b - a) \sin^2 \frac{\pi(b - a)}{2(x - a)} + a,$$

Then the function $\varphi_{a,b}$ is continuous on (a, b), and for each $\delta \in (0, b - a)$

$$\varphi_{a,b}[(a, a + \delta)] = [a, b]. \tag{1}$$

For each $n \in \mathbb{N}$ put $a_n = \frac{1}{n}$. Define

$$f(x) = \begin{cases} \varphi_{a_{2n}, a_{2n-1}}(x) & \text{if } x \in (a_{2n}, a_{2n-1}),\ n \in \mathbb{N}, \\ \quad x & \text{otherwise.} \end{cases}$$

Note that $\mathbb{R} \setminus \mathcal{C}(f) = \{\frac{1}{2n} : \ n \in \mathbb{N}\}$. So, using condition (1) we obtain that $f{\restriction}\operatorname{int}\mathcal{C}(f)$ is dense in f, hence the function f is internally quasi-continuous. Moreover, using once more condition (1) and Lemma 3.4 we have $f{\restriction}\big[\frac{1}{2n+1}, \frac{1}{2n-1}\big] \in \acute{\mathcal{S}}_s$ for each $n \in \mathbb{N}$. Finally, by Lemma 3.3, $f \in \acute{\mathcal{S}}_s$. Since $f(x) < 0$ for $x < 0$, $f(x) > 0$ for $x > 0$, and $0 \notin \operatorname{int} \mathcal{C}(f)$, the function f is not internally strong Świątkowski. □

Theorem 4.2. *Assume that the function f is quasi-continuous and $\varepsilon > 0$. Then there is an internally quasi-continuous function g such that $|f - g| < \varepsilon$ on $\mathbb{R}$. Additionally, if $f \in U$, then $g \in U$.*

Proof. Let $\varepsilon > 0$ and f is quasi-continuous. Define

$$F = \{x \in \mathbb{R} : \operatorname{osc}(f, x) \geq \varepsilon\}.$$

Note that the set F is closed and boundary, hence nowhere dense. Write the set $\mathbb{R} \setminus F$ as the union of a family $\mathcal{I}$, consisting of nonoverlapping compact intervals, such that

$$\text{for each } x \in \mathbb{R} \setminus F, \text{ there are } I_1, I_2 \in \mathcal{I} \text{ with } x \in \operatorname{int}(I_1 \cup I_2),$$
$$\operatorname{osc}(f, I) < \varepsilon \text{ for each } I \in \mathcal{I}.$$

Now fix an $I = [p, q] \in \mathcal{I}$ and put $m_I = \inf\{f(x) : x \in I\}$, and $M_I = \sup\{f(x) : x \in I\}$. Define the function $g_I \colon I \to [m_I, M_I]$ by the formula:

$$g_I(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{bd} I, \\ (M_I - m_I)\sin^2 \frac{\pi(q-p)}{4(x-p)} + m_I & \text{if } x \in (p, (p+q)/2], \\ (M_I - m_I)\sin^2 \frac{\pi(q-p)}{4(q-x)} + m_I & \text{if } x \in ((p+q)/2, q). \end{cases}$$

Observe that the function g_I is continuous on the interval $\operatorname{int} I = (p, q)$ and

$$g_I(I) = [m_I, M_I]. \tag{2}$$

Now define

$$g(x) = \begin{cases} g_I(x) & \text{if } x \in I,\ I \in \mathcal{I}, \\ f(x) & \text{otherwise.} \end{cases}$$

Since $\operatorname{osc}(f, I) = M_I - m_I < \varepsilon$ for each $I \in \mathcal{I}$, and since $f(x) = g(x)$ for each $x \in F$, using condition (2) we obtain that $|f - g| < \varepsilon$ on $\mathbb{R}$. Now we will show that g is internally quasi-continuous.

Fix an $x_0 \in \mathbb{R}$. If $x_0 \in \mathbb{R} \setminus F$, then the function g is internally quasi-continuous at x_0 directly from its construction. So, let $x_0 \in F$. Since f is quasi-continuous, there is a sequence $(x_n) \subset \mathcal{C}(f)$ such that $x_n \to x_0$ and $f(x_n) \to f(x_0)$. Fix an $n \in \mathbb{N}$. The condition $x_n \in \mathcal{C}(f)$ implies that $x_n \in \mathbb{R} \setminus F$. Therefore $x_n \in I_n$ for some $I_n \in \mathcal{I}$. Without loss of generality we can assume that $x_0 < x_n$ and x_n is not the left endpoint of I_n. Then there is a $t_n \in \operatorname{int} I_n \cap (x_0, x_n)$ with $g(t_n) = f(x_n)$ and since g is continuous on $\operatorname{int} I_n$, we obtain that $t_n \in \operatorname{int} \mathcal{C}(g)$. Consequently, there is a sequence $(t_n) \subset \operatorname{int} \mathcal{C}(g)$ such that $t_n \to x_0$ and $g(t_n) \to f(x_0) = g(x_0)$. This proves that $g \in \mathcal{Q}_i$.

To complete the proof assume that $f \in U$. We will show that $g \in U$, too. Let $a < b$ and fix a set $A \subset [a, b]$ with $\operatorname{card} A < \mathfrak{c}$. Observe that $g \restriction (\mathbb{R} \setminus F) \in \mathcal{D}$, hence if $[a, b] \subset \mathbb{R} \setminus F$, then $g([a, b] \setminus A)$ is dense in $\mathrm{I}[g(a), g(b)]$. In the other case suppose that $g([a, b] \setminus A)$ is not dense in $\mathrm{I}[g(a), g(b)]$. Then there is an open interval $J \subset \mathrm{I}(g(a), g(b))$ such that $g([a, b] \setminus A) \cap J = \emptyset$. Hence

$$\big(g([a, b]) \cap J\big) \setminus g(A) = \emptyset. \tag{3}$$

We consider two cases.

Case 1. $J \cap \mathrm{I}(f(a), f(b)) = \emptyset$.

Then $J \subset \mathrm{I}(f(a), g(a))$ or $J \subset \mathrm{I}(f(b), g(b))$. Assume that $J \subset \mathrm{I}(f(a), g(a))$. (The second case is analogous.) Since $f(a) \neq g(a)$, there is an $I \in \mathcal{I}$ such that $a \in \operatorname{int} I$. By assumption, the interval $[a, b] \not\subset I$, hence

$$g(I \cap [a, b]) = [m_I, M_I] \supset \mathrm{I}[g(a), f(a)] \supset J.$$

So, $g([a, b]) \cap J = J$ and finally $\big(g([a, b]) \cap J\big) \setminus g(A) = J \setminus g(A) \neq \emptyset$, which contradicts (3).

Case 2. $J \cap \mathrm{I}(f(a), f(b)) \neq \emptyset$.

Since $f \in U$, we obtain that $f([a, b] \setminus A) \cap J \cap \mathrm{I}[f(a), f(b)] \neq \emptyset$, hence $f([a, b] \setminus A) \cap J \neq \emptyset$. So, there is a $t \in [a, b]$ such that $f(t) \in J$ and $g(t) \notin J$. Therefore $f(t) \neq g(t)$, which implies that $t \in \operatorname{int} I$ for some $I \in \mathcal{I}$. But $[a, b] \not\subset I$, so there is a $z \in \operatorname{int} I \cap [a, b]$ with $g(z) = f(t) \in J$. Hence $g([a, b]) \cap J \supset [m_I, M_I] \cap J$ and $[m_I, M_I] \cap J$ is a nondegenerate interval. Consequently $\big(g([a, b]) \cap J\big) \setminus g(A) \neq \emptyset$, which contradicts (3). This completes the proof. □

In the proof of the next theorem we will use a construction quite similar to the one used in [9].

Theorem 4.3. *Assume that the function f is quasi-continuous and $\varepsilon > 0$. Then there is a quasi-continuous function g with the set of discontinuity points of full Lebesgue measure such that $|f - g| < \varepsilon$ on $\mathbb{R}$.*

Proof. Let f be quasi-continuous and $\varepsilon > 0$. For each $k \in \mathbb{Z}$ put $f_k = f \restriction [k, k + 1]$. Fix a $k \in \mathbb{Z}$. If $\mu\big([k, k + 1] \setminus \mathcal{C}(f)\big) = 1$, then define the function $\alpha^{(k)} = 0$ on $[k, k + 1]$. In the other case $\mu\big([k, k + 1] \setminus \mathcal{C}(f)\big) < 1$. Put $G_k = [k, k + 1] \cap \mathcal{C}(f)$. Then G_k is a G_δ set with positive Lebesgue measure $0 < r \leq 1$. Define $C_0^{(k)} = \emptyset$ and fix an $n \in \mathbb{N}$. Let $C_n^{(k)} \subset G_k \setminus \bigcup_{i=0}^{n-1} C_i^{(k)}$ be a Cantor set such that $\mu(C_n^{(k)}) = r/2^n$ and let $\mathcal{I}_n^{(k)}$ be the family of all components of the set $(k, k+1) \setminus C_n^{(k)}$. For each $I_n^{(k)} = (a, b) \in \mathcal{I}_n^{(k)}$ define the function $\beta_{I_n^{(k)}}\colon \operatorname{cl} I_n^{(k)} \to [0, \varepsilon/2^n]$ as follows:

$$\beta_{I_n^{(k)}}(x) = \begin{cases} 0 & \text{if } x \in \operatorname{bd} I_n^{(k)}, \\ \varepsilon/2^n & \text{if } x = (a+b)/2, \\ \text{linear} & \text{in intervals } [a, (a+b)/2] \text{ and } [(a+b)/2, b]. \end{cases}$$

Now define the function $\alpha_n^{(k)}\colon [k, k+1] \to [0, \varepsilon/2^n]$ by the formula:

$$\alpha_n^{(k)}(x) = \begin{cases} 0 & \text{if } x \in C_n^{(k)} \cup \operatorname{bd}[k, k+1], \\ \beta_{I_n^{(k)}}(x) & \text{if } x \in \operatorname{cl} I_n^{(k)},\ I_n^{(k)} \in \mathcal{I}_n^{(k)}. \end{cases}$$

Observe that $\alpha_n^{(k)}$ is continuous on $[k, k+1] \setminus C_n^{(k)}$ and discontinuous on the Cantor set $C_n^{(k)}$. Put $\alpha^{(k)} = \sum_{n\in\mathbb{N}} \alpha_n{}^{(k)}$ and $C^{(k)} = \bigcup_{n\in\mathbb{N}} C_n^{(k)}$. Observe that $C^{(k)} \subset G_k$ and $\alpha^{(k)}\colon [k, k+1] \to [0, \varepsilon]$ is quasi-continuous. Moreover $\alpha^{(k)}$ is continuous on $[k, k+1] \setminus C^{(k)}$, discontinuous on $C^{(k)}$, and $\mu(G_k \setminus C^{(k)}) = 0$. Finally, put $C = \bigcup_{k\in\mathbb{Z}} C^{(k)}$, $\alpha = \bigcup_{k\in\mathbb{Z}} \alpha^{(k)}$ and $g = f + \alpha$. Then clearly $|f - g| < \varepsilon$ on $\mathbb{R}$, the function g is quasi-continuous and discontinuous on the set $C \cup (\mathbb{R} \setminus \mathcal{C}(f)) = \mathbb{R} \setminus \mathcal{C}(g)$ of full Lebesgue measure. □

An immediate consequence of Theorems 4.2 and 4.3 is the following corollary.

Corollary 4.4. *The set of all bounded internally quasi-continuous functions is dense and boundary in the space of all bounded quasi-continuous functions with the metric of uniform convergence.*

Now we characterize the uniform closure of the family of internally strong Świątkowski functions.

Theorem 4.5. *Assume that $f \in U$ is an internally quasi-continuous function and $\eta > 0$. Then there are an internally strong Świątkowski function g and a continuous function α such that $f = g + \alpha$ and $|\alpha| \le \eta$ on $\mathbb{R}$.*

Proof. Let $K = \mathbb{R} \setminus \operatorname{int} \mathcal{C}(f)$. Since f is an internally quasi-continuous function, f is quasi-continuous and K is nowhere dense closed set. Moreover f is continuous on $\mathbb{R} \setminus K$, hence f locally bounded on this set. So, by Lemma 3.8 there is a nowhere dense closed set F and a continuous function α' such that $\alpha' = 0$ on K, $|\alpha'| \le \eta$ on $\mathbb{R}$, and

$$[\varliminf_{t\to x} f(t), \varlimsup_{t\to x} f(t)] \subset (f + \alpha')[I \cap F \setminus K]$$

for every $x \in K$ and every open interval $I \ni x$. Observe that

$$F \setminus K = F \cap (\mathbb{R} \setminus K) = F \cap \operatorname{int} \mathcal{C}(f) \subset \operatorname{int} \mathcal{C}(f).$$

Let $\alpha = -\alpha'$. Clearly $\alpha = 0$ on K and $|\alpha| \leq \eta$ on $\mathbb{R}$, too. Define $g = f - \alpha$. Then $f = g + \alpha$. We can easily see that $\mathcal{C}(f) = \mathcal{C}(g)$, and for each $x \in K$

$$\varliminf_{t\to x} g(t) = \varliminf_{t\to x} f(t) \quad \text{and} \quad \varlimsup_{t\to x} g(t) = \varlimsup_{t\to x} f(t).$$

Consequently, for every $x \notin \operatorname{int} \mathcal{C}(g)$ and every open interval $I \ni x$ we obtain

$$[\varliminf_{t\to x} g(t), \varlimsup_{t\to x} g(t)] \subset (f - \alpha)[I \cap F \setminus K] \subset g[I \cap \operatorname{int} \mathcal{C}(g)].$$

Moreover $g \in U$ (see [1]). So, using Lemma 3.9 we have $g \in \acute{\mathcal{S}}_{si}$. □

Corollary 4.6. *A quasi-continuous function f belongs to U if and only if there is a sequence (f_n) of internally strong Świątkowski functions such that f is the uniform limit of (f_n).*

Proof. First assume that there is a sequence (f_n) of internally strong Świątkowski functions such that f is the uniform limit of (f_n). Since $\acute{\mathcal{S}}_{si} \subset \acute{\mathcal{S}}_s$, by [5, Corollary 5] the function f is quasi-continuous and $f \in U$.

On the other hand let $f \in U$ be quasi-continuous. Fix an $\varepsilon > 0$. By Theorem 4.2 there is an internally quasi-continuous function $g \in U$ such that $|f - g| < \varepsilon/2$. Moreover, by Theorem 4.5 there is an internally strong Świątkowski function h such that $|g - h| < \varepsilon/2$. So, $|f - h| < \varepsilon$, which proves that f is the uniform limit of the sequence of internally strong Świątkowski functions. □

In 2002 P. Szczuka [10] showed that $\mathcal{M}_a(\acute{\mathcal{S}}_s) = \mathcal{M}_m(\acute{\mathcal{S}}_s) = \mathcal{M}_{\max}(\acute{\mathcal{S}}_s) = \mathcal{C}onst$. In the similar way we will prove three analogous theorems for the family $\acute{\mathcal{S}}_{si}$.

Theorem 4.7. $\mathcal{M}_a(\acute{\mathcal{S}}_{si}) = \mathcal{C}onst.$

Proof. We can easily see that $\mathcal{C}onst \subset \mathcal{M}_a(\acute{\mathcal{S}}_{si})$. So, we need to prove only the opposite inclusion. Let $f \notin \mathcal{C}onst$. We will show that $f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{si})$.

If $f \notin \acute{\mathcal{S}}_{si}$, then $\chi_\emptyset \in \acute{\mathcal{S}}_{si}$ and $f = f + \chi_\emptyset \notin \acute{\mathcal{S}}_{si}$. Therefore $f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{si})$, and we can assume that $f \in \acute{\mathcal{S}}_{si}$.

Since $f \in \acute{\mathcal{S}}_{si} \setminus \mathcal{C}onst$, there is an open interval $J = (a, b) \subset \mathcal{C}(f)$ such that f is nonconstant on J and $f(a) \neq f(b)$. We may assume that $f(a) < f(b)$.

(The case $f(a) > f(b)$ is analogous.) Since the set $f[\mathcal{A}(f)]$ is at most countable (see e.g. [10, Lemma 2.4]), we can take an $x_0 \in J \setminus \mathcal{A}(f)$ such that

$$f(x) < f(x_0) \quad \text{for each } x \in [a, x_0). \tag{4}$$

Choose a sequence $(x_n) \subset (x_0, b)$ such that $x_n \to x_0^+$ and $f(x_n) > f(x_0)$ for each $n \in \mathbb{N}$. Then $x_{n+1} < x_n$ for every $n \in \mathbb{N}$. Define

$$g(x) = \begin{cases} 0 & \text{if } x \in (-\infty, x_0] \cup [x_1, \infty), \\ g_n(x) & \text{if } x \in [x_{n+1}, x_n],\ n \in \mathbb{N}, \end{cases}$$

where

$$g_n(x) = \max\Big\{2\big(f(x_0) - f(x)\big), 1 - \frac{2|x - (x_{n+1} + x_n)/2|}{x_n - x_{n+1}}\Big\}.$$

Observe that $g_n(x_{n+1}) = 0 = g_{n+1}(x_{n+1})$ for each $n \in \mathbb{N}$. So, the function g is well defined. Since x_0 is the only point of discontinuity of g and $g(x_n) = 0$ for each $n \in \mathbb{N}$, by Lemma 3.6, $g \in \acute{\mathcal{S}}_{si}$. Now we will show that $f + g \notin \acute{\mathcal{S}}_{si}$.

Put $\alpha = a$ and $\beta = x_1$. Notice that, by (4), $(f + g)(x) = f(x) < f(x_0)$ for each $x \in [\alpha, x_0)$. Now fix an $x \in (x_0, \beta)$. If $f(x) < f(x_0)$, then

$$(f + g)(x) \geq f(x) + 2f(x_0) - 2f(x) > f(x_0).$$

In the other case if $x = x_{n+1}$ for some $n \in \mathbb{N}$, then $(f + g)(x) = f(x_{n+1}) > f(x_0)$ and if $x \neq x_{n+1}$ for each $n \in \mathbb{N}$, then

$$(f + g)(x) \geq f(x) + 1 - \frac{2|x - (x_{n+1} + x_n)/2|}{x_n - x_{n+1}} > f(x) \geq f(x_0).$$

Consequently $(f + g)(x) > f(x_0)$ for each $x \in (x_0, \beta)$. Hence in particular $f(x_0) \in \big((f + g)(\alpha), (f + g)(\beta)\big)$. Finally observe that

$$\varlimsup_{x \to x_0^+} (f + g)(x) \geq f(x_0) + 1 > f(x_0) = (f + g)(x_0),$$

which implies $x_0 \notin \mathcal{C}(f + g)$. So, $(f + g)(x) \neq f(x_0)$ for each $x \in (\alpha, \beta) \cap \operatorname{int} \mathcal{C}(f + g)$. This proves that $f + g \notin \acute{\mathcal{S}}_{si}$, hence $f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{si})$. □

Theorem 4.8. *$\mathcal{M}_m(\acute{\mathcal{S}}_{si}) = \mathcal{C}onst$.*

Proof. It is easy to show that $\mathcal{C}onst \subset \mathcal{M}_m(\acute{\mathcal{S}}_{si})$. So, we need to prove only the opposite inclusion. Let $f \notin \mathcal{C}onst$. We will show that $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{si})$.

If $f \notin \acute{\mathcal{S}}_{si}$, then $\chi_{\mathbb{R}} \in \acute{\mathcal{S}}_{si}$ and $f = f \cdot \chi_{\mathbb{R}} \notin \acute{\mathcal{S}}_{si}$. Therefore $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{si})$, and we can assume that $f \in \acute{\mathcal{S}}_{si}$.

Since $f \in \acute{\mathcal{S}}_{si} \setminus \mathcal{C}onst$, there is an open interval $J \subset \mathcal{C}(f)$ such that f is nonconstant on J. Without loss of generality we may assume that f is positive on $\operatorname{cl} J$. By Theorem 4.7 there is a function $\bar{g}\colon \operatorname{cl} J \to \mathbb{R}$ such that $\bar{g} \in \acute{\mathcal{S}}_{si}$ and $\ln \circ f + \bar{g} \notin \acute{\mathcal{S}}_{si}$ on $\operatorname{cl} J$. Let $J = (a, b)$. Define

$$g(x) = \begin{cases} (\exp \circ \bar{g})(x) & \text{if } x \in \operatorname{cl} J, \\ (\exp \circ \bar{g})(a) & \text{if } x \in (-\infty, a), \\ (\exp \circ \bar{g})(b) & \text{if } x \in (b, \infty). \end{cases}$$

By Lemma 3.7, $\exp \circ \bar{g} \in \acute{\mathcal{S}}_{si}$ on $\operatorname{cl} J$, hence clearly $g \in \acute{\mathcal{S}}_{si}$. But on the interval J we have

$$\ln \circ (fg) = \ln \circ \big(f \cdot (\exp \circ \bar{g})\big) = \ln \circ f + \bar{g} \notin \acute{\mathcal{S}}_{si}.$$

If $fg \in \acute{\mathcal{S}}_{si}$ on J, then by Lemma 3.7, $\ln \circ (fg) \in \acute{\mathcal{S}}_{si}$ on J, a contradiction. So, $fg \notin \acute{\mathcal{S}}_{si}$, which proves that $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{si})$. □

Theorem 4.9. *$\mathcal{M}_{\max}(\acute{\mathcal{S}}_{si}) = \mathcal{C}onst$.*

Proof. We can easily see that $\mathcal{C}onst \subset \mathcal{M}_{\max}(\acute{\mathcal{S}}_{si})$. So, we need to prove only the opposite inclusion. Let $f \notin \mathcal{C}onst$. We will show that $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{si})$.

If $f \notin \acute{\mathcal{S}}_{si}$, then there are $\alpha < \beta$ and $y \in \mathrm{I}(f(\alpha), f(\beta))$ such that $f(x) \neq y$ for each $x \in (\alpha, \beta) \cap \operatorname{int} \mathcal{C}(f)$. Put $g = \min\{f(\alpha), f(\beta)\}$ and $h = \max\{f, g\}$. Then clearly $g \in \mathcal{C}onst \subset \acute{\mathcal{S}}_{si}$. Since $y \in \mathrm{I}(h(\alpha), h(\beta))$ and $h(x) \neq y$ for each $x \in (\alpha, \beta) \cap \operatorname{int} \mathcal{C}(h)$, we obtain that $h \notin \acute{\mathcal{S}}_{si}$. Therefore $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{si})$, and we may assume that $f \in \acute{\mathcal{S}}_{si}$.

Since $f \in \acute{\mathcal{S}}_{si} \setminus \mathcal{C}onst$, there is an open interval $J = (a, b) \subset \mathcal{C}(f)$ such that f is nonconstant on J and $f(a) \neq f(b)$. We may assume that $f(a) < f(b)$. (The case $f(a) > f(b)$ is analogous.) Since the set $f[\mathcal{A}(f)]$ is at most countable, we can take an $x_0 \in J \setminus \mathcal{A}(f)$ such that

$$f(x) < f(x_0) \quad \text{for each } x \in [a, x_0). \tag{5}$$

Choose a sequence $(x_n) \subset (x_0, b)$ such that $x_n \to x_0^+$ and $f(x_n) > f(x_0)$ for each $n \in \mathbb{N}$. Since $(x_n) \subset \mathcal{C}(f)$, for each $n \in \mathbb{N}$, there is a $\delta_n > 0$ such that $f(x) > f(x_0)$ for every $x \in (x_n - \delta_n, x_n + \delta_n)$. Without loss of generality we can assume that $x_{n+1} + \delta_{n+1} < x_n - \delta_n$ for each $n \in \mathbb{N}$. Define

$$g(x) = \begin{cases} f(x) & \text{if } x \in (-\infty, x_0], \\ f(x_0) - 1 & \text{if } x \in \{x_n :\ n \in \mathbb{N}\} \cup [x_1, \infty), \\ f(x_0) + 1 & \text{if } x \in \bigcup_{n=1}^{\infty} [x_{n+1} + \delta_{n+1}, x_n - \delta_n], \\ \text{linear} & \text{in intervals } [x_{n+1}, x_{n+1} + \delta_{n+1}] \text{ and } [x_n - \delta_n, x_n],\ n \in \mathbb{N}. \end{cases}$$

Then x_0 is the only point of discontinuity of g. Since

$$g(x_0) = f(x_0) \in g\big[[x_0, t] \cap \operatorname{int} \mathcal{C}(g)\big] = [f(x_0) - 1, f(x_0) + 1]$$

for each $t \in (x_0, x_1)$, by Lemma 3.6, $g \in \acute{\mathcal{S}}_{si}$. To complete the proof we must show that $h = \max\{f, g\} \notin \acute{\mathcal{S}}_{si}$.

Put $\alpha = a$ and $\beta = x_1 - \delta_1$. Note that by (5), $h(x) < f(x_0)$ for each $x \in [\alpha, x_0)$ Now fix an $x \in (x_0, \beta]$. Then $h(x) > f(x_0)$. Indeed, if there is an $n \in \mathbb{N}$ such that $x \in [x_{n+1}, x_{n+1} + \delta_{n+1})$ or $x \in (x_n - \delta_n, x_n)$, then $h(x) \geq f(x) > f(x_0)$, and if $x \in [x_{n+1} + \delta_{n+1}, x_n - \delta_n]$ for some $n \in \mathbb{N}$, then $h(x) \geq g(x) = f(x_0) + 1 > f(x_0)$. Hence in particular $f(x_0) \in (h(\alpha), h(\beta))$. Finally

$$\varlimsup_{x \to x_0^+} h(x) \geq \varlimsup_{x \to x_0^+} g(x) = f(x_0) + 1 > f(x_0) = h(x_0).$$

It follows that $x_0 \notin \mathcal{C}(h)$, hence $h(x) \neq f(x_0)$ for each $x \in (\alpha, \beta) \cap \operatorname{int} \mathcal{C}(h)$. Consequently $h \notin \acute{\mathcal{S}}_{si}$, which proves that $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{si})$. □

Finally we present the following theorem.

Theorem 4.10. *For every function f the following conditions are equivalent*:

a) *there are functions $g_1, g_2 \in \acute{\mathcal{S}}_{si}$ with $f = g_1 + g_2$,*

b) *there are functions $g_1, g_2 \in \mathcal{Q}_i$ with $f = g_1 + g_2$,*

c) *the set $\operatorname{int} \mathcal{C}(f)$ is dense in $\mathbb{R}$.*

Proof. The implication a) ⇒ b) is evident.

b) ⇒ c). Assume that there are internally quasi-continuous functions g_1 and g_2 such that $f = g_1 + g_2$. Then clearly $\operatorname{int} \mathcal{C}(g_1) \cap \operatorname{int} \mathcal{C}(g_2) \subset \operatorname{int} \mathcal{C}(f)$. Since sets $\operatorname{int} \mathcal{C}(g_1)$ and $\operatorname{int} \mathcal{C}(g_2)$ are dense in $\mathbb{R}$, the set $\operatorname{int} \mathcal{C}(f)$ is dense in $\mathbb{R}$, too.

c) ⇒ a). If the function f is continuous we can set $g_1 = 0$ and $g_2 = f$. Then clearly $f = g_1 + g_2$ and $g_1, g_2 \in \acute{\mathcal{S}}_{si}$. In the opposite case write $\operatorname{int} \mathcal{C}(f)$ as the union of a family $\mathcal{I}$ consisting of nonoverlapping compact intervals, such that for each $x \in \operatorname{int} \mathcal{C}(f)$, there are $I_1, I_2 \in \mathcal{I}$ with $x \in \operatorname{int}(I_1 \cup I_2)$. Fix an $I \in \mathcal{I}$ and let $I = [x_1, x_2]$. Define

$$r_I = \operatorname{dist}(I, \mathbb{R} \setminus \operatorname{int} \mathcal{C}(f)).$$

Observe that $r_I > 0$. Choose elements $x_1 < c_1 < d_1 < c < d_2 < c_2 < x_2$. For $i \in \{1,2\}$ define the function $\varphi_i \colon \mathrm{I}[x_i, c] \to \mathbb{R}$ as follows:

$$\varphi_i(x) = \begin{cases} 1/r_I & \text{if } x = c_i, \\ -1/r_I & \text{if } x = d_i, \\ f(x)/2 & \text{if } x \in \{x_i, c\}, \\ \text{linear} & \text{in intervals } \mathrm{I}[x_i, c_i],\ \mathrm{I}[c_i, d_i], \text{ and } \mathrm{I}[d_i, c]. \end{cases}$$

Now for $i \in \{1,2\}$ define the function $g_{I,i} \colon I \to \mathbb{R}$ by the formula:

$$g_{I,i}(x) = \begin{cases} \varphi_i & \text{if } x \in \mathrm{I}[x_i, c], \\ (f - \varphi_{3-i}) & \text{if } x \in \mathrm{I}[c, x_{3-i}]. \end{cases}$$

Note that functions $g_{I,1}$ and $g_{I,2}$ are continuous, and $f \restriction I = g_{I,1} + g_{I,2}$. Finally, for $i \in \{1,2\}$ define

$$g_i(x) = \begin{cases} g_{I,i}(x) & \text{if } x \in I,\ I \in \mathcal{I}, \\ f(x)/2 & \text{if } x \notin \operatorname{int} \mathcal{C}(f). \end{cases}$$

Then $f = g_1 + g_2$. Moreover $\operatorname{int}\mathcal{C}(f) \subset \operatorname{int}\mathcal{C}(g_1) \cap \operatorname{int}\mathcal{C}(g_2)$. Fix an $i \in \{1,2\}$. To complete the proof we must show that $g_i \in \acute{\mathcal{S}}_{si}$.

Let $\alpha < \beta$ and $y \in \mathrm{I}(g_i(\alpha), g_i(\beta))$. We can assume that $g_i(\alpha) < g_i(\beta)$. (The case $g_i(\alpha) > g_i(\beta)$ is analogous.) If $[\alpha, \beta] \subset \operatorname{int}\mathcal{C}(f)$, then $[\alpha, \beta] \subset \operatorname{int}\mathcal{C}(g_i)$, hence there is an $x_0 \in (\alpha, \beta) \cap \operatorname{int}\mathcal{C}(g_i)$ such that $g_i(x_0) = y$. So, assume that $[\alpha, \beta] \setminus \operatorname{int}\mathcal{C}(f) \neq \emptyset$. Put $\gamma = \max\bigl\{[\alpha, \beta] \setminus \operatorname{int}\mathcal{C}(f)\bigr\}$. Then $\gamma \notin \operatorname{int}\mathcal{C}(f)$. Since the set $\operatorname{int}\mathcal{C}(f)$ is dense in $\mathbb{R}$, for each $n \in \mathbb{N}$ there is an $I_n \in \mathcal{I}$ such that $I_n \subset (\gamma - 1/n, \gamma + 1/n) \cap (\alpha, \beta)$. Moreover, since $\lim_{n\to\infty}\bigl|\frac{1}{r_{I_n}}\bigr| = \infty$, there is an $I_{n_0} \subset (\gamma - 1/n_0, \gamma + 1/n_0) \cap (\alpha, \beta)$ with $y \in g_i[I_{n_0}]$. So, there is an $x_0 \in I_{n_0} \cap \operatorname{int}\mathcal{C}(f) \subset (\alpha, \beta) \cap \operatorname{int}\mathcal{C}(g_i)$ such that $g_i(x_0) = y$. It follows that $g_i \in \acute{\mathcal{S}}_{si}$. $\square$

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