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ON REPRESENTATIONS OF BAIRE ONE FUNCTIONS AS THE SUM OF LOWER AND UPPER SEMICONTINUOUS FUNCTIONS

Abstract

According to the Vitali-Carathéodory theorem, the integral of a finite summable function f on a measurable set may be approximated by the integral of a sum of lower and upper semicontinuous functions. In the case, that f is a Baire one function, we give the answer to the following question: is there a lower semicontinuous function l and a upper semicontinuous function u such that $f = l+u$ almost everywhere? The answer is in general negative.

We deal with the classes of real functions defined on the interval $[0,1]$. The symbols C, B_1, D, lsc and usc stand for the class of continuous, Baire 1, Darboux, lower and upper semicontinuous functions, respectively. DB_1 denotes $D \cap B_1$ and f/F denotes the restriction of the function f on the set F. We use a notation $d\,(F, x_0)$ for the density of the set F at the point x_0. Let $A \subset_d B$ denote $A \subset B$ and $d\,(B, x) = 1$ for all $x \in A$ and $A \subset_c B$ (A *is billateraly* $c-dense$ *in* B) means that for each $x \in A$, the sets $(x, x+\delta) \cap B$, $(x-\delta, x) \cap B$ are nondenumerable for every $\delta > 0$.

Let $I = [0,1]$, F_i, $i = 1, 2, \ldots$ be perfect nowhere dense subsets of I,

$$F_1 \subset_d F_2 \subset_d F_3 \subset_d \ldots,$$

such that the set $F = \bigcup_{i=1}^{\infty} F_i$ has the Lebesgue measure $\lambda\,(F) = 1$. Then we define the function f^* in the following way:

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$$f^*(x) = \begin{cases} 1, & x \in F_1 \\ \frac{(-1)^{k-1}}{k}, & x \in F_k \setminus F_{k-1}, k = 2, 3, \dots \\ 0, & x \in I \setminus F \end{cases}$$

Lemma 1. *The function $f^* \in B_1$.*

Proof. It is sufficient to show by [2], that for each $\alpha \in R$ the sets $\{f^* > \alpha\} = \{x \in I; f^*(x) > \alpha\}$ and $\{f^* < \alpha\} = \{x \in I; f^*(x) < \alpha\}$ are sets of type F_σ . It is easy to see that every open set is of F_σ type and closed set of G_δ type. Then the following statement can be made: if A and B are closed subsets of I, then the set $A \setminus B = A \cap (I \setminus B)$ is of type F_σ and G_δ as well as any finite union of such sets.

We show that the sets $\{f^* > \alpha\}$ and $\{f^* < \alpha\}$ are of type F_σ for each $\alpha \in R$.

If $\alpha > 1$, then the set $\{f^* > \alpha\} = \emptyset$ and $\{f^* < \alpha\} = [0, 1]$.

If $\alpha = 1$, then the set $\{f^* > \alpha\} = \emptyset$ and $\{f^* < \alpha\} = [0, 1] \setminus F_1$. All of these sets are of type F_σ.

If $0 < \alpha < 1$, then there exists an odd natural number k such that $\frac{1}{k+2} \leq \alpha < \frac{1}{k}$. From the definition of the function f^* it follows that

$$\{f^* > \alpha\} = (F_k \setminus F_{k-1}) \cup (F_{k-2} \setminus F_{k-3}) \cup \dots \cup (F_3 \setminus F_2) \cup F_1$$

and thus the set $\{f^* > \alpha\}$ is of type F_σ and G_δ too. Moreover, the same it holds for the set $\{f^* \leq \alpha\}$ and from there the set

$$\{f^* < \alpha\} = \begin{cases} \left\{f^* \leq \frac{1}{k+2}\right\}, \ for \ \frac{1}{k+2} < \alpha < \frac{1}{k}. \\ \left\{f^* \leq \frac{1}{k+4}\right\}, \ for \ \alpha = \frac{1}{k+2} \end{cases}$$

is again the set of type F_σ. The analogous assertion is valid for $\alpha < 0$. If $\alpha = 0$, then the sets

$$\{f^* > 0\} = \bigcup_{k=1}^{\infty} \left\{f^* > \frac{1}{k}\right\} \ and \ \{f^* < 0\} = \bigcup_{k=1}^{\infty} \left\{f^* < -\frac{1}{k}\right\}$$

are sets of type F_σ too thus the function $f^* \in B_1$. □

We will say that a function $g \in lsc + usc$, iff there exist any functions $l \in lsc$ and $u \in usc$ such that $g = l + u$. W. Sierpiński in [5] constructed a bounded Baire one function which cannot be written as sum of lower and upper semicontinuous functions and A. Maliszewski in [4] proved that there is a bounded Darboux Baire one function which does not belong to $lsc + usc$. Additionally we find the following:

Proposition 2. *There is a bounded function $f \in DB_1$ such that for arbitrary function $g \in lsc + usc$, the Lebesgue measure of the set $\{x \in I; f(x) \neq g(x)\}$ is positive.*

Proof. Let f^* be the real function defined above. According to Proposition 1 in [3] there exists a function $f \in DB_1$ such that the set $\{x \in I; f(x) \neq f^*(x)\}$ is a first category subset of the set $[0,1] \setminus F$. We prove that the function f satisfies the assertion of Proposition 2 by contradiction. Assume that there exist a lower semicontinuous function l and upper semicontinuous function u such that the function $g = l + u$ and the Lebesgue measure $\lambda(\{x \in I; f(x) \neq g(x)\}) = 0$. Without loss of generality we may assume that $l \geq 0$ and $u \leq 0$. Otherwise there exists a positive real number K such that $l \geq -K$ and $u \leq K$, because the functions l and u are defined on the compact set $[0,1]$. Then the function l can be replaced by $l + K$ and u can by replaced by $u - K$. If we denote $d = -u$, then the solution of the functional equation $g = l + u$ on interval $I = [0,1]$ is equivalent to a solution of the equation

$$l = g + d,$$

where the functions $l \geq 0,\ d \geq 0$ are lower semicontinuous.

Let $J \subset I$ be an arbitrary open interval and let

$$\sum_{n=0}^{\infty} \alpha_n, \quad (\alpha_n > 0 \; for \; each \; n = 0, 1, 2, \ldots)$$

be any convergent series of positive real numbers and the set

$$A = \{x \in I; f(x) \neq g(x)\}.$$

From the definition of the function f follows the existence of $x_0 \in (J \cap F_{k_0}) \setminus A$, such that $f(x_0) = -\frac{1}{k_0}$ for some even natural number k_0. Because $f(x_0) = g(x_0)$, by the assumption $l(x_0) \geq 0$ we have $d(x_0) \geq \frac{1}{k_0}$. Since the function $d \in lsc$ then there exists an open neighborhood $U_0 \subset J$ of the point x_0 such that

$$d(U_0) \geq \frac{1}{k_0} - \alpha_0 \qquad (0).$$

Again by the definition of the function f, because $F_{k_0} \subset_d F_{k_0+1}$, we choose $x_1 \in (U_0 \cap F_{k_0+1}) \setminus A$ such that

$$f(x_1) = g(x_1) = \frac{1}{k_0 + 1}.$$

Then from (0)

$$l(x_1) \geq \frac{1}{k_0} - \alpha_0 + \frac{1}{k_0+1}$$

and there exists an open neighborhood $U_1 \subset U_0$ of the point x_1 such that

$$l(U_1) \geq \frac{1}{k_0} - \alpha_0 + \frac{1}{k_0+1} - \alpha_1 \qquad (1).$$

Repeating this cycle, we choose $x_2 \in (U_1 \cap F_{k_0+2}) \setminus A$ such that

$$f(x_2) = g(x_2) = -\frac{1}{k_0+2}.$$

From (1) follows

$$l(x_2) = g(x_2) + d(x_2) \geq \frac{1}{k_0} - \alpha_0 + \frac{1}{k_0+1} - \alpha_1$$

and consequently

$$d(x_2) \geq \frac{1}{k_0} - \alpha_0 + \frac{1}{k_0+1} - \alpha_1 + \frac{1}{k_0+2}.$$

There exists an open neighborhood $U_2 \subset U_1$ of the point x_2 such that

$$d(U_2) \geq \frac{1}{k_0} - \alpha_0 + \frac{1}{k_0+1} - \alpha_1 + \frac{1}{k_0+2} - \alpha_2 \qquad (2).$$

Next we choose $x_3 \in (U_2 \cap F_{k_0+3}) \setminus A$ such that

$$f(x_3) = g(x_3) = \frac{1}{k_0+3}.$$

From (2) we have

$$l(x_3) \geq \frac{1}{k_0} - \alpha_0 + \frac{1}{k_0+1} - \alpha_1 + \frac{1}{k_0+2} - \alpha_2 + \frac{1}{k_0+3}$$

and again we obtain the existence of an open neighborhood $U_3 \subset U_2$ of the point x_3 such that

$$l(U_3) \geq \frac{1}{k_0} - \alpha_0 + \frac{1}{k_0+1} - \alpha_1 + \frac{1}{k_0+2} - \alpha_2 + \frac{1}{k_0+3} - \alpha_3 \qquad (3).$$

It is necessary to note that the selection of such points x_0, x_1, x_2, x_3 is possible because the set F is dense in $[0,1]$ and $f/F = f^*/F$.

Continuing this process, we construct a sequence of open sets

$$[0,1] \supset J \supset U_0 \supset U_1 \supset U_2 \supset \dots$$

associated with a sequence of points $x_0, x_1, x_2, x_3, \dots,\ x_n \in F,\ n = 0, 1, 2, \dots,$ such that for every even n the following hold:

$$d\,(U_n) \geq \sum_{i=0}^{n} \left(\frac{1}{k_0 + i} - \alpha_i \right) \qquad (n)$$

$$l\,(U_{n+1}) \geq \sum_{i=0}^{n+1} \left(\frac{1}{k_0 + i} - \alpha_i \right) \qquad (n+1)\,.$$

The functions d and l are real functions defined on $I = [0,1]$, then

$$\bigcup_{M=1}^{\infty} \{d < M\} = \bigcup_{M=1}^{\infty} \{l < M\} = [0,1]\,.$$

The series

$$\sum_{i=0}^{\infty} \left(\frac{1}{k_0 + i} - \alpha_i \right)$$

diverges to $+\infty$. From the foregoing it follows that, for an arbitrary open interval $J \subset [0,1]$ and each $M > 0$, there exists an open interval $U \subset J$ such that $d\,(U) > M$. That is, each of the sets $\{d < M\}$, $M = 1, 2, \dots$, is nowhere dense in $[0,1]$. Therefore the closed interval $[0,1]$ is a countable union of nowhere dense sets, which contradicts the Category Theorem of Baire. It was shown that the assumption $\lambda(\{x \in I; f\,(x) \neq g\,(x)\}) = 0$ is not true. $\square$

Let the class B_1 of Baire 1 functions defined on interval $[0,1]$ be furnished with the sup norm. In the next theorem it will be shown that the class $lsc{+}usc$ is dense in the class B_1.

The authors of the article [1] define the class of functions $[C]$ and prove the following Theorem 4.

Definition 3. $f \in [C]$ iff there exists a sequence of closed sets $A_n, n = 1, 2, \dots$ such that $\cup A_n = R$ and f/A_n is continuous for every $n = 1, 2, \dots$.

Theorem 4. *Let $f \in B_1$. Then there are $f_n \in [C]\,, n = 1, 2, \dots$ such that $f_n \to f$ uniformly.*

Lemma 5. *$[C] \subset lsc + usc$.*

PROOF. Let a function $f \in [C]$. According to Definition 3 there exists a sequence of closed sets $A_n, n = 1, 2, \dots$ such that $\bigcup A_n = [0,1]$ and f/A_n is continuous for every $n = 1, 2, \dots$. The function f/A_n is bounded. Therefore there exists an increasing sequence of real numbers $\beta_n, n = 1, 2, \dots$ such that

$$| f(x) | \leq \beta_n, \ for\ each\ x \in \bigcup_{i=1}^{n} A_i.$$

We define the functions u and l:

$$\begin{array}{lcl} u(x) & = & f^{-}(x) - \beta_1 \\ l(x) & = & f^{+}(x) + \beta_1 \end{array}, \ for\ x \in A_1$$

$$\begin{array}{lcl} u(x) & = & f^{-}(x) - n\beta_n \\ l(x) & = & f^{+}(x) + n\beta_n \end{array}, \ for\ x \in A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

where as usually

$$f^{-} = \min\{0, f\} \wedge f^{+} = \max\{0, f\}.$$

We prove that $l \in lsc$. Let $x_0 \in [0,1]$ and let $x_n, n = 1, 2, \dots$ to be any sequence of points, $x_n \to x_0$. There exists n_0 such that

$$x_0 \in A_{n_0} \setminus \bigcup_{i=1}^{n_0-1} A_i$$

and $l(x_0) = f^{+}(x_0) + n_0\beta_{n_0}$. Because $\bigcup A_i, 1 \leq i \leq n_0 - 1$ is a closed set, it is sufficient to consider

$$x_n \in [0,1] \setminus \bigcup_{i=1}^{n_0-1} A_i.$$

If $x_n \in A_{n_0}$ for every $n = 1, 2, \dots$, then from continuity of f^{+}/A_{n_0} we have

$$l(x_n) = f^{+}(x_n) + n_0\beta_{n_0} \to f^{+}(x_0) + n_0\beta_{n_0} = l(x_0).$$

If $x_n \in A_{k_n}, k_n > n_0$ for every $n = 1, 2, \dots$ then

$$l(x_n) = f^{+}(x_n) + k_n\beta_{k_n} \geq \beta_{k_n} + (k_n - 1)\beta_{k_n} \geq \beta_{n_0} + n_0\beta_{n_0} \geq f^{+}(x_0) + n_0\beta_{n_0} = l(x_0).$$

Consequently

$$\liminf_{x_n \to x_0} l(x_n) \geq l(x_0),$$

which means $l \in lsc$ and analogically $u \in usc$. The function $f \in lsc + usc$, since $f = l + u$. □

The assertion of Theorem 6 is an immediate consequence of Theorem 4 and Lemma 5.

Theorem 6. *Let $f \in B_1$. Then there are $f_n \in lsc + usc, n = 1, 2, \ldots$ such that $f_n \to f$ uniformly.*

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