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STRICT INEQUALITY IN THE BOX-COUNTING DIMENSION PRODUCT FORMULAS

Abstract

We supplement the well known upper and lower box-counting product inequalities to give the new product formula

$$\begin{aligned}\dim_{LB} F + \dim_{LB} G &\leq \dim_{LB}(F \times G)\\ &\leq \min\left(\dim_{LB} F + \dim_{B} G, \dim_{B} F + \dim_{LB} G\right)\\ &\leq \max\left(\dim_{LB} F + \dim_{B} G, \dim_{B} F + \dim_{LB} G\right)\\ &\leq \dim_{B}(F \times G)\\ &\leq \dim_{B} F + \dim_{B} G\end{aligned}$$

for subsets of metric spaces. We develop a procedure for constructing sets so that the upper and lower box-counting dimensions of these sets and their product can take arbitrary values satisfying the above product formula. In particular we illustrate how badly behaved both the lower and upper box-counting dimensions can be on taking products.

1 Introduction.

In a metric space X the Hausdorff dimension of a set $F \subset X$ is defined as

$$\dim_H(F) = \sup\left\{d \geq 0 | \mathcal{H}^d(F) = \infty\right\}$$

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where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. The Hausdorff dimension takes values in the non-negative reals and extends the elementary integer-valued topological dimension in the sense that for a large class of 'reasonable' sets these two values coincide. Sets with non-coinciding Hausdorff and topological dimensions are called 'fractal', a term coined by Mandelbrot in his original study of such sets [9]. Hausdorff introduced this generalised dimension in [7] and its subsequent extensive use in geometric measure theory is developed by Federer [6] and Falconer [4]. The fact that the Hausdorff dimension satisfies $\dim_H(F \times G) \geq \dim_H(F) + \dim_H(G)$ for the Cartesian product of subsets of Euclidean space was established by Marstrand [10] after some partial results: the inequality was proved by Besicovitch and Moran [1] with the restriction that $0 < \mathcal{H}^s(F), \mathcal{H}^t(G) < \infty$ for some s, t and was extended to a larger class of Euclidean subsets by Eggleston [3]. This inequality also holds for sets $F \subset X$ and $G \subset Y$ with X, Y abstract metric spaces: first shown by Wegmann [14] with the restriction that the product space $X \times Y$ has the metric

$$\mathrm{d}_{X\times Y} := \max(\mathrm{d}_X, \mathrm{d}_Y)$$

where $\mathrm{d}_X, \mathrm{d}_Y$ are the metrics on X and Y respectively. This result was extended by Howroyd [8] to hold for metrics $\mathrm{d}_{X\times Y}$ on $X \times Y$ that satisfy

$$c \max(\mathrm{d}_X, \mathrm{d}_Y) \leq \mathrm{d}_{X\times Y} \tag{1.1}$$

for some constant $c > 0$. Without this restriction Howroyd notes that the triple

$$(\dim_H(F), \dim_H(G), \dim_H(F \times G))$$

can take arbitrary values by remetrising the spaces, so the inequality does not hold more generally. Besicovitch and Moran [1] also provide an example for which there is a strict inequality in the product formula which is summarised in §7.1 of [5].

In this paper we prove similar product inequalities for the upper and lower box-counting dimensions (treated briefly by Falconer [5]; see Robinson [13] for a more detailed exposition) which have applications to dynamical systems (see, for example, [12]). In the second section our main results are an upper bound on $\dim_{LB}(F \times G)$ and a lower bound on $\dim_B(F \times G)$ provided that the metric on $X \times Y$ satisfies a condition similar to (1.1). Together with more familiar product formulas (the first and last inequalities below) these new bounds give the chain of inequalities

$$\begin{aligned}\dim_{LB} F + \dim_{LB} G &\leq \dim_{LB}(F \times G) \\ &\leq \min(\dim_{LB} F + \dim_B G, \dim_B F + \dim_{LB} G) \\ &\leq \max(\dim_{LB} F + \dim_B G, \dim_B F + \dim_{LB} G) \\ &\leq \dim_B(F \times G) \\ &\leq \dim_B F + \dim_B G,\end{aligned}$$

which is the content of Theorem 2.4. In the third section we define a variant of generalised Cantor sets (see, for example, §4.11 in [11]) which we use in Theorem 3.6 to construct explicitly sets $F, G \subset \mathbb{R}$ such that the upper and lower box-counting dimensions of F, G and $F \times G$ have arbitrary values satisfying the above product formula, demonstrating that these bounds are sharp. In light of this theorem it is trivial to construct two sets for which all of the inequalities in the above product formula are strict, which is analogous to the example in [1] of strict inequality in the Hausdorff product formula.

Example 3.7 gives an extreme case to show how badly behaved the box-counting dimensions can be on taking products. In [2], Edgar shows that Falconer's example in [4] of two subsets $F, G \subset \mathbb{R}$ has $\dim_{LB} F = \dim_{LB} G = 0$ and $\dim_{LB}(F \times G) \geq 1$. However, in light of the above product formula, their product must have lower box-counting dimension equal to 1. Consequently, this example illustrates the extreme case for the lower box-counting dimension. The upper box-counting dimension is less familiar and, as far as we are aware, there is no example of strict inequality for the upper box-counting dimension product formula in the literature. Example 3.7 consists of two subsets $F, G \subset \mathbb{R}$ for which, like Falconer's example, $\dim_{LB} F = \dim_{LB} G = 0$ and $\dim_{LB}(F \times G) = 1$ yet also $\dim_B F = \dim_B G = \dim_B(F \times G) = 1$. In particular, despite the subsets having the maximum possible upper box-counting dimension (as subsets of $\mathbb{R}$), taking their product does not increase the upper box-counting dimension.

2 Box-counting dimensions and product sets.

Let (X, d_X) be a metric space. We say that a set $F \subset X$ is *totally bounded* if for all $\delta > 0$ there exists a finite collection of closed balls B_i of radius δ such that $F \subset \bigcup B_i$; recall that in the Euclidean case $X = \mathbb{R}^n$ a set is totally bounded if and only if it is bounded. The *upper* and *lower box-counting dimensions* of a totally bounded set $F \subset X$ are defined by

$$\dim_B F = \limsup_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log \delta} \qquad \text{and} \tag{2.1}$$

$$\dim_{LB} F = \liminf_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log \delta} \tag{2.2}$$

respectively, where $N(F,\delta)$ is the minimum number of closed balls of radius δ such that F is contained in the union of these balls. Throughout we take $0 < \delta < 1$ so that $\log \delta \neq 0$. The restriction to totally bounded sets is necessary to ensure that $N(F,\delta)$ is finite for all $\delta > 0$. Essentially, if $N(F,\delta)$ scales like $\delta^{-\varepsilon}$ as $\delta \to 0$ then the box-counting dimensions capture ε which gives an indication of how 'spread out' the set F is at small length-scales. These limits are unchanged if we replace $N(F,\delta)$ with one of many similar geometric quantities (discussed by Falconer in [5] §3.1 'Equivalent Definitions' which we adapt below in Lemma 2.1). Two such quantities are the minimum number of sets of diameter at most δ that cover F, which we denote $N'(F,\delta)$, and the largest number of disjoint closed balls of radius δ with centres in F, which we denote $\tilde{N}(F,\delta)$. If the metric space is Euclidean then a further equivalent quantity is the number of δ-mesh boxes, that is sets of the form

$$[m_1\delta, (m_1+1)\delta] \times \ldots \times [m_n\delta, (m_n+1)\delta]$$

for integers $m_1, \ldots, m_n$, that intersect F, which we denote $M(F,\delta)$.

Lemma 2.1. *Let $F \subset X$ be a totally bounded set and $\delta > 0$. The geometric quantities N, N' and $\tilde{N}$ are related by*

$$N'(F,\delta) \geq \tilde{N}(F,\delta) \geq N(F,2\delta) \geq N'(F,4\delta) \tag{2.3}$$

and further in the Euclidean case $X = \mathbb{R}^n$

$$N\left(F, \delta\sqrt{n}/2\right) \leq M(F,\delta) \leq 3^n N(F,\delta).$$

Proof. Let $x_1, \ldots, x_{\tilde{N}(F,\delta)} \in F$ be the centres of disjoint closed balls of radius δ. If sets $U_1, \ldots, U_k$ of diameter δ form a cover of F then each point x_i must lie in some U_j. However, since $U_j \subset B_\delta(x_i)$ as for all $y \in U_j$

$$d_X(x_i, y) \leq \sup\{d_X(z_1,z_2) \,|\, z_1, z_2 \in U_j\} = \operatorname{diam}(U_j) \leq \delta$$

and since the $B_\delta(x_i)$ are disjoint we conclude that there must be at least $\tilde{N}(F,\delta)$ sets U_j, yielding $N'(F,\delta) \geq \tilde{N}(F,\delta)$.

With the same points $\{x_i\}$ observe that for each $x \in F$ the distance $\mathrm{d}_X(x, x_i) \leq 2\delta$ for some $i = 1, \ldots, \tilde{N}(F, \delta)$ otherwise the additional closed ball $B_\delta(x)$ would be disjoint from each of the closed balls $B_\delta(x_i)$ contradicting the maximality of the set of disjoint balls. Consequently, the balls $B_{2\delta}(x_i)$ cover the set F, yielding $\tilde{N}(F, \delta) \geq N(F, 2\delta)$.

Next, a closed ball of radius 2δ has diameter at most 4δ so a cover of F consisting of balls of radius 2δ is a cover consisting of sets of diameter at most 4δ, yielding $N(F, 2\delta) \geq N'(F, 4\delta)$.

Finally, in the Euclidean case $X = \mathbb{R}^n$ each δ-mesh box is contained in a closed ball of radius $\delta\sqrt{n}/2$: these balls form a cover of F with $M(F, \delta)$ members, yielding $N(F, \delta\sqrt{n}/2) \leq M(F, \delta)$. Further, each ball of radius δ is contained in 3^n mesh boxes of side length δ, yielding $M(F, \delta) \leq 3^n N(F, \delta)$. $\square$

As a result of the above lemma we can interchange the geometric quantities $N, N', \tilde{N}$ and M in the definitions of the box-counting dimension (2.1) and (2.2) allowing us to take the most convenient quantity in each particular situation: in the remainder we derive an upper bound on the dimension of product sets using the quantity N; a lower bound on the dimension of product sets using the quantity $\tilde{N}$; and we calculate the dimensions of the sets constructed in Section 3 using the quantity N'.

For the remainder of this section, let (X, d_X) and (Y, d_Y) be metric spaces and endow the product space $X \times Y$ with a metric $\mathrm{d}_{X\times Y}$ that satisfies

$$c_1 \max(\mathrm{d}_X, \mathrm{d}_Y) \leq \mathrm{d}_{X\times Y} \leq c_2 \max(\mathrm{d}_X, \mathrm{d}_Y) \tag{2.4}$$

for some constants $c_1, c_2 > 0$. We remark that the familiar metrics

$$\begin{aligned} \mathrm{d}_{X\times Y,p} &:= (\mathrm{d}_X^p + \mathrm{d}_Y^p)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty), \quad \text{and} \\ \mathrm{d}_{X\times Y,\infty} &:= \max(\mathrm{d}_X, \mathrm{d}_Y) \end{aligned}$$

satisfy (2.4). It is well known that if $F \subset X$ and $G \subset Y$ are totally bounded subsets then the lower and upper box-counting dimensions of the Cartesian product $F \times G$ satisfy

$$\dim_{LB}(F \times G) \geq \dim_{LB} F + \dim_{LB} G \qquad \text{and} \tag{2.5}$$

$$\dim_B(F \times G) \leq \dim_B F + \dim_B G \tag{2.6}$$

respectively. These inequalities follow from the good behaviour of the above geometric quantities on taking products: for each of these geometric quantities we can derive a relationship between the values taken for the sets F, G and the set $F \times G$, which is the content of the following lemma. Further, we require these geometric relationships in order to establish the useful equivalent definitions for the lower and upper box-counting dimensions of products sets given in Lemma 2.3.

Lemma 2.2. *If $F \subset X$ and $G \subset Y$ are totally bounded sets then for all $\delta > 0$*

$$N(F \times G, c_2\delta) \leq N(F,\delta)\, N(G,\delta) \tag{2.7}$$

$$\tilde{N}(F \times G, c_1\delta/2) \geq \tilde{N}(F,\delta)\, \tilde{N}(G,\delta). \tag{2.8}$$

Further, if $X = \mathbb{R}^n, Y = \mathbb{R}^m$ and $X \times Y = \mathbb{R}^{n+m}$ then for all $\delta > 0$

$$M(F \times G, \delta) = M(F,\delta)\, M(G,\delta). \tag{2.9}$$

The proof of (2.7) is standard (see, for example, Exercise 7.5 of [5] or [13] pp. 35), the inequality (2.8) is less familiar (again, see [13] pp. 35) and the equality (2.9) is elementary. We include a proof for completeness:

Proof. Fix $\delta > 0$ and let $x_1, \ldots, x_{N(F,\delta)}$ and $y_1, \ldots, y_{N(G,\delta)}$ be the centres of closed balls of radius δ whose unions contain F and G respectively. Observe that for each $(x,y) \in F \times G$ the point $x \in B_\delta(x_i)$ for some i and $y \in B_\delta(y_j)$ for some j and so, from (2.4),

$$\mathrm{d}_{X\times Y}((x,y),(x_i,y_j)) \leq c_2 \max(\mathrm{d}_X(x,x_i), \mathrm{d}_Y(y,y_j)) \leq c_2\delta.$$

Consequently, $F \times G$ is contained in the union of $N(F,\delta)\, N(G,\delta)$ closed balls of radius $c_2\delta$, yielding (2.7).

Next, let $x_1, \ldots, x_{\tilde{N}(F,\delta)} \in F$ be the centres of disjoint closed balls of radius δ and $y_1, \ldots, y_{\tilde{N}(G,\delta)} \in G$ be the centres of disjoint closed balls of radius δ. As the balls are disjoint $\mathrm{d}_X(x_i,x_k) \leq \delta$ implies that $x_i = x_k$, and $\mathrm{d}_Y(y_j,y_l) \leq \delta$ implies that $y_j = y_l$. Let (x_i,y_j) and (x_k,y_l) be distinct elements of the set

$$\left\{(x_i,y_j)\,|\, i = 1,\ldots,\tilde{N}(F,\delta), \quad j = 1,\ldots,\tilde{N}(G,\delta)\right\} \subset F \times G.$$

in which case either $\mathrm{d}_X(x_i,x_k) > \delta$ or $\mathrm{d}_Y(y_j,y_l) > \delta$ holds. Consequently from (2.4)

$$\mathrm{d}_{X\times Y}((x_i,y_j),(x_k,y_l)) \geq c_1 \max(\mathrm{d}_X(x_i,x_k), \mathrm{d}_Y(y_j,y_l)) > c_1\delta. \tag{2.10}$$

In particular, the two balls of radius $c_1\delta/2$ with centres (x_i, y_j) and (x_k, y_l) are disjoint: indeed if the point z lies in both balls then

$$\mathrm{d}_{X\times Y}\left(\left(x_i, y_j\right), \left(x_k, y_l\right)\right) \leq \mathrm{d}_{X\times Y}\left(z, \left(x_i, y_j\right)\right) + \mathrm{d}_{X\times Y}\left(z, \left(x_k, y_l\right)\right) \leq c_1\delta$$

contradicting (2.10). We conclude that the $\tilde{N}\left(F,\delta\right)\tilde{N}\left(G,\delta\right)$ points of the form $(x_i, y_j) \in F\times G$ are the centres of disjoint balls of radius $c_1\delta/2$, yielding (2.8).

Next, let $U_1, \ldots, U_{M(F,\delta)}$ be the δ-mesh boxes of $\mathbb{R}^n$ that intersect F and let $V_1, \ldots, V_{M(G,\delta)}$ be the set of δ-mesh boxes of $\mathbb{R}^m$ that intersect G. Clearly each $U_i \times V_j$ is a δ-mesh box in $\mathbb{R}^n \times \mathbb{R}^m$ and further there exists $x \in U_i \cap F$, $y \in V_j \cap G$ so the product box $U_i \times V_j$ intersects $F \times G$.

Further, an arbitrary point $(x, y) \in F\times G$ has $x \in U_i$ for some i and $y \in V_j$ for some j, so that $(x, y) \in U_i \times V_j$. Consequently, the set of δ-mesh boxes of $\mathbb{R}^{n+m}$ that intersect $F \times G$ is precisely the set

$$\left\{U_i \times V_j | i = 1, \ldots, M\left(F,\delta\right),\ j = 1, \ldots, M\left(G,\delta\right)\right\},$$

which has exactly $M\left(F,\delta\right)M\left(G,\delta\right)$ members, yielding (2.9). □

The inequalities of Lemma 2.2 and the equivalence of the geometric quantities proved in Lemma 2.1 allow us to derive the following equivalent definitions for the box-counting dimensions of a product set:

Lemma 2.3. *For totally bounded sets $F \subset X$ and $G \subset Y$*

$$\dim_{LB}\left(F\times G\right) = \liminf_{\delta\to 0+}\left(\frac{\log N\left(F,\delta\right)}{-\log\delta} + \frac{\log N\left(G,\delta\right)}{-\log\delta}\right), \qquad \textit{and} \qquad (2.11)$$

$$\dim_{B}\left(F\times G\right) = \limsup_{\delta\to 0+}\left(\frac{\log N\left(F,\delta\right)}{-\log\delta} + \frac{\log N\left(G,\delta\right)}{-\log\delta}\right), \qquad (2.12)$$

and further the above equalities hold if N is replaced with any of the geometric quantities N', $\tilde{N}$ or M.

Proof. It is immediate from Lemma 2.1 that the right hand sides of (2.11) and (2.12) are invariant under the choice of geometric quantity. Next, from the geometric inequality (2.7)

$$\begin{aligned}\liminf_{\delta\to 0+}\left(\frac{\log N\left(F,\delta\right)}{-\log\delta} + \frac{\log N\left(G,\delta\right)}{-\log\delta}\right) &= \liminf_{\delta\to 0+}\frac{\log\left(N\left(F,\delta\right)N\left(G,\delta\right)\right)}{-\log\delta} \\ &\geq \liminf_{\delta\to 0+}\frac{\log N\left(F\times G,\delta\right)}{-\log\delta} \\ &= \dim_{LB}\left(F\times G\right),\end{aligned}$$

which, using the equivalent definition in terms of $\tilde{N}$,

$$= \liminf_{\delta\to 0+} \frac{\log \tilde{N}\left(F\times G,\delta\right)}{-\log\delta} \geq \liminf_{\delta\to 0+} \frac{\log\left(\tilde{N}\left(F,\delta\right)\tilde{N}\left(G,\delta\right)\right)}{-\log\delta}$$

using the inequality (2.8). Finally, from the inequality (2.3), the latter is

$$\geq \liminf_{\delta\to 0+} \frac{\log\left(N\left(F,2\delta\right)N\left(G,2\delta\right)\right)}{-\log\delta} = \liminf_{\delta\to 0+} \frac{\log\left(N\left(F,\delta\right)N\left(G,\delta\right)\right)}{-\log\delta}$$

so there is equality throughout, yielding (2.11). The upper box-counting equivalence (2.12) follows similarly. □

This observation simplifies the proof of the main theorem and the calculation of the box-counting dimensions in the subsequent examples. Note that in the Euclidean case the proof of the above lemma is immediate as

$$\dim_{LB}\left(F\times G\right) = \liminf_{\delta\to 0+}\left(\frac{\log M\left(F,\delta\right)}{-\log\delta} + \frac{\log M\left(G,\delta\right)}{-\log\delta}\right) \qquad \text{and}$$
$$\dim_{B}\left(F\times G\right) = \limsup_{\delta\to 0+}\left(\frac{\log M\left(F,\delta\right)}{-\log\delta} + \frac{\log M\left(G,\delta\right)}{-\log\delta}\right)$$

follow from the equality (2.9).

Theorem 2.4. *For totally bounded sets $F\subset X$ and $G\subset Y$ the upper and lower box-counting dimensions of the product set $F\times G$ satisfy*

$$\begin{aligned}\dim_{LB} F + \dim_{LB} G &\leq \dim_{LB}\left(F\times G\right)\\ &\leq \min\left(\dim_{LB} F + \dim_{B} G, \dim_{B} F + \dim_{LB} G\right)\\ &\leq \max\left(\dim_{LB} F + \dim_{B} G, \dim_{B} F + \dim_{LB} G\right)\\ &\leq \dim_{B}\left(F\times G\right)\\ &\leq \dim_{B} F + \dim_{B} G.\end{aligned}$$

Proof. The result follows immediately from the equivalent definitions (2.11) and (2.12) together with the elementary analytic inequalities

$$\liminf A + \liminf B \leq \liminf\left(A+B\right), \tag{2.13}$$
$$\liminf\left(A+B\right) \leq \liminf A + \limsup B, \tag{2.14}$$
$$\liminf A + \limsup B \leq \limsup\left(A+B\right), \qquad \text{and} \tag{2.15}$$
$$\limsup\left(A+B\right) \leq \limsup A + \limsup B. \tag{2.16}$$

The inequalities (2.13) and (2.14) yield

$$\begin{aligned}\dim_{LB} F + \dim_{LB} G &= \liminf_{\delta\to 0+} \frac{\log N(F,\delta)}{-\log\delta} + \liminf_{\delta\to 0+} \frac{\log N(G,\delta)}{-\log\delta} \\ &\leq \liminf_{\delta\to 0+} \left(\frac{\log N(F,\delta)}{-\log\delta} + \frac{\log N(G,\delta)}{-\log\delta} \right) \qquad (2.17) \\ &\leq \liminf_{\delta\to 0+} \frac{\log N(F,\delta)}{-\log\delta} + \limsup_{\delta\to 0+} \frac{\log N(G,\delta)}{-\log\delta} \\ &= \dim_{LB} F + \dim_{B} G\end{aligned}$$

and the result follows as (2.17) is equal to $\dim_{LB}(F\times G)$ from (2.11). The remaining inequalities are proved similarly. □

It is possible to derive similar product formulas for the product of m totally bounded sets $F_1,\ldots,F_m$ by introducing the bounds

$$\dim_{LB}(F_1\times\ldots\times F_m) \leq \min_{i=1,\ldots,m} \left(\dim_{LB}(F_i) + \sum_{\substack{j=1\\ j\neq i}}^{m} \dim_B(F_j) \right) \qquad (2.18)$$

and

$$\dim_{B}(F_1\times\ldots\times F_m) \geq \max_{i=1,\ldots,m} \left(\dim_{B}(F_i) + \sum_{\substack{j=1\\ j\neq i}}^{m} \dim_{LB}(F_j) \right) \qquad (2.19)$$

which follow from the analytic inequalities

$$\liminf (A_1+\ldots+A_m) \leq \liminf A_1 + \sum_{j=2}^{m} \limsup A_j \qquad \text{and}$$

$$\limsup (A_1+\ldots+A_m) \geq \limsup A_1 + \sum_{j=2}^{m} \liminf A_j.$$

These bounds also follow from inductively applying Theorem 2.4. As it is possible for the right hand side of (2.18) to be greater than the left hand side of (2.19) we cannot write this as a single formula as in the statement of Theorem 2.4.

It is known that there are sets with unequal upper and lower box-counting dimension (see Exercise 3.8 of [5] or §3.1 in [13]), however if these values coincide for a set F we define the box-counting dimension of F to be this common value. If sets F and G have well-defined box-counting dimension then the box-counting dimension of their product is also well behaved:

Corollary 2.5. *If* $\dim_B F = \dim_{LB} F$ *and* $\dim_B G = \dim_{LB} G$ *then*

$$\dim_B (F \times G) = \dim_{LB} (F \times G) = \dim_B F + \dim_B G.$$

Proof. As $\dim_{LB} F + \dim_{LB} G = \dim_B F + \dim_B G$ we clearly have equality throughout the statement of Theorem 2.4. □

3 Compatible generalised Cantor sets.

A generalised Cantor set (see §4.11 in [11]) is a variation of the well known Cantor middle-third set that permits the proportion removed from each interval to vary throughout the iterative process. Formally, for $b > 1$ we define the application of the generator gen_b to a set of disjoint intervals $\mathcal{I}$ as the procedure in which the open middle $1 - 2^{1-b}$ proportion of each interval in $\mathcal{I}$ is removed.

With generators of this form, we can produce sets F of arbitrary box-counting dimension in the range $(0, 1)$ through the repeated application of a single generator.

Lemma 3.1. *Fix* $b > 1$. *Starting from the initial set* $F_0 = [0, 1]$ *let* $F_j = \mathrm{gen}_b (F_{j-1})$ *for all* $j \in \mathbb{N}$. *The resulting set* $\mathrm{Cantor}\,(b) := \bigcap F_j$ *has upper and lower box-counting dimension equal to* $\frac{1}{b}$.

Proof. See §4.10 in [11]. This is also a consequence of Corollary 3.3 and Lemma 3.5. □

The intermediary sets F_j provide a convenient cover of the resulting Cantor set, so it is natural to use the minimum number of sets of diameter at most δ in our calculations. In the remainder we refer to the function

$$\delta \mapsto \frac{\log N' (F, \delta)}{- \log \delta}$$

as the *box-counting function* of F.

In the following we detail a method to construct a generalised Cantor set F from an arbitrary sequence of generators gen_{b_j}. Roughly, the intermediary set F_{j-1} will consist of a finite number of disjoint intervals and we define the intermediary set F_j by iteratively applying the generator gen_{b_j} to the set F_{j-1}. As we repeatedly apply the generator gen_{b_j} to a finite number of disjoint intervals the set F_j has the appearance of a finite number of disjoint scaled copies of $\text{Cantor}\,(b_j)$ at appropriate length-scales. Consequently, by applying the generator gen_{b_j} a sufficient number of times the box-counting function of F approaches $\frac{1}{b_j}$ for δ approximately the length of the intervals of F_j. It is relatively straightforward to calculate the number of iterations required, the length scales δ and the value of $N'(F,\delta)$; but it is prohibitively difficult to calculate these quantities for the set $F \times G$ where G is another arbitrary generalised Cantor set.

We rectify this incompatibility by constructing the generalised Cantor sets F and G in parallel from two arbitrary sequences of generators $\left\{\text{gen}_{b_j}\right\}$ and $\left\{\text{gen}_{c_j}\right\}$. At the j^{th} stage of the construction, as above, we iteratively apply the generators gen_{b_j} and gen_{c_j} respectively to the intermediary sets F_{j-1} and G_{j-1} a sufficient number of times for each set to ensure that

- the intermediary sets F_j and G_j consist of intervals of the same length,
- for δ equal to the common length of the intervals of F_j and G_j the box-counting functions

$$\frac{\log N'(F,\delta)}{-\log\delta} \qquad \text{and} \qquad \frac{\log N'(G,\delta)}{-\log\delta} \tag{3.1}$$

  approach $\frac{1}{b_j}$ and $\frac{1}{c_j}$ respectively, which is the content of Corollary 3.3, and that

- for all δ the box-counting functions (3.1) are tightly controlled, which is the content of Lemma 3.4.

As a consequence, for a given length scale δ we have good bounds on the values of the box-counting functions (3.1) and, from Lemma 2.3, these yield good bounds on the box counting function of the set $F \times G$.

In the remainder we assume that the b_i and c_i are rational numbers greater than 1. Let $K_0 = 0$ and for $j \in \mathbb{N}$ define $K_j := 2^{2^j} \prod_{i=1}^{j+1} \text{num}\,(b_i)\,\text{num}\,(c_i)$, where $\text{num}\, b_i$ is the numerator of the rational number b_i. Observe that the K_j are positive integers that increase with j,

$$\frac{K_j - K_{j-1}}{b_j}, \frac{K_j - K_{j-1}}{c_j} \in \mathbb{N} \tag{3.2}$$

for all $j \in \mathbb{N}$, and

$$\frac{\sum_{i=1}^{j-1} K_i}{K_j} \to 0 \qquad \text{as} \quad j \to \infty. \tag{3.3}$$

We define $F \coloneqq \bigcap_{j\in\mathbb{N}} F_j$ where $F_0 = [0,1]$ and the set at the j^{th} stage of the construction, F_j, is formed by applying the generator gen_{b_j} a total of $(K_j - K_{j-1})/b_j$ times to the set of disjoint intervals F_{j-1}. This is well defined as, from (3.2), $(K_j - K_{j-1})/b_j$ is a positive integer. Similarly, we define $G \coloneqq \bigcap_{j\in\mathbb{N}} G_j$ where $G_0 = [0,1]$ and G_j is formed by applying the generator gen_{c_j} a total of $(K_j - K_{j-1})/c_j$ times to the set G_{j-1}.

To find bounds on the box-counting functions of F and G we must first calculate $N'(F,\delta)$ and $N'(G,\delta)$ for a given length scale δ, which we postpone until the appendix. From this calculation we can prove the following:

Lemma 3.2. *For all $j \in \mathbb{N}$ and $n = 1, \ldots, (K_j - K_{j-1})/b_j$ if δ is in the range*

$$2^{-K_{j-1}-b_j n} \le \delta < 2^{-K_{j-1}-b_j(n-1)}$$

then the box-counting function of F satisfies

$$\begin{aligned}\frac{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n - 1}{K_{j-1} + b_j n} &\le \frac{\log N'(F,\delta)}{-\log\delta} \\ &< \frac{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n}{K_{j-1} + b_j(n-1)}.\end{aligned} \tag{3.4}$$

PROOF. Immediate from (A.2), (A.3) and Lemma A.1. □

Replacing b_i with c_i throughout the above lemma gives the corresponding result for the set G.

Corollary 3.3. *With F and G constructed as above*

$$\begin{aligned}\dim_{LB} F &\le \liminf_{j\to\infty}\frac{1}{b_j} & \limsup_{j\to\infty}\frac{1}{b_j} &\le \dim_B F\\ \dim_{LB} G &\le \liminf_{j\to\infty}\frac{1}{c_j} & \limsup_{j\to\infty}\frac{1}{c_j} &\le \dim_B G\\ \dim_{LB}(F\times G) &\le \liminf_{j\to\infty}\frac{1}{b_j}+\frac{1}{c_j} & \limsup_{j\to\infty}\frac{1}{b_j}+\frac{1}{c_j} &\le \dim_B(F\times G)\end{aligned}$$

PROOF. Consider the sequence $\delta_j := 2^{-K_j}$ and apply Lemma 3.2 with $n = (K_j - K_{j-1})/b_j$ to yield

$$\frac{\sum_{i=1}^{j}(K_i - K_{i-1})/b_i - 1}{K_j} \le \frac{\log N'(F,\delta_j)}{-\log\delta_j} < \frac{\sum_{i=1}^{j}(K_i - K_{i-1})/b_i}{K_j - b_j}.$$

Consequently,

$$\frac{\log N'(F,\delta_j)}{-\log\delta_j} \geq \frac{1}{b_j} + \frac{-K_{j-1}/b_j + \sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i - 1}{K_j} \quad \text{and} \tag{3.5}$$

$$\frac{\log N'(F,\delta_j)}{-\log\delta_j} < \frac{1}{b_j} + \frac{1 - K_{j-1}/b_j + \sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i}{K_j - b_j} \tag{3.6}$$

and from (3.3) the second terms tend to zero as $j \to \infty$. Consequently,

$$\liminf_{\delta\to 0+} \frac{\log N'(F,\delta)}{-\log\delta} \leq \liminf_{j\to\infty} \frac{\log N'(F,\delta_j)}{-\log\delta_j} = \liminf_{j\to\infty} \frac{1}{b_j} \quad \text{and}$$
$$\limsup_{\delta\to 0+} \frac{\log N'(F,\delta)}{-\log\delta} \geq \limsup_{j\to\infty} \frac{\log N'(F,\delta_j)}{-\log\delta_j} = \limsup_{j\to\infty} \frac{1}{b_j}.$$

The result for the set G follows similarly using the same sequence δ_j. Next, we sum each of (3.5) and (3.6) with their equivalent inequalities for the set G so that at the limit

$$\liminf_{j\to\infty}\left(\frac{\log N'(F,\delta_j)}{-\log\delta_j} + \frac{\log N'(G,\delta_j)}{-\log\delta_j}\right) = \liminf_{j\to\infty} \frac{1}{b_j} + \frac{1}{c_j} \quad \text{and}$$
$$\limsup_{j\to\infty}\left(\frac{\log N'(F,\delta_j)}{-\log\delta_j} + \frac{\log N'(G,\delta_j)}{-\log\delta_j}\right) = \limsup_{j\to\infty} \frac{1}{b_j} + \frac{1}{c_j}$$

and the result for the product set $F \times G$ follows from the equivalent definitions (2.11) and (2.12). □

Finally, we find some bounds on the box-counting function for all δ.

Lemma 3.4. *For δ in the range*

$$2^{-K_j} \leq \delta < 2^{-K_{j-1}} \tag{3.7}$$

the box-counting functions (3.1) *have the following bounds:*

$$\min\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) - \varepsilon_j \leq \frac{\log N'(F,\delta)}{-\log\delta} < \max\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) + \varepsilon_j \tag{3.8}$$

and

$$\min\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right) - \varepsilon_j \leq \frac{\log N'(G,\delta)}{-\log\delta} < \max\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right) + \varepsilon_j \tag{3.9}$$

where $\varepsilon_j \to 0$ as $j \to \infty$.

PROOF. For each $n = 1, \ldots, (K_j - K_{j-1})/b_j$ consider δ in the range

$$2^{-K_{j-1}-b_j n} \leq \delta < 2^{-K_{j-1}-b_j(n-1)}. \tag{3.10}$$

Lemma 3.2 yields the lower bound

$$\begin{aligned}
\frac{\log N'(F,\delta)}{-\log\delta} \geq& \frac{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n - 1}{K_{j-1} + b_j n} \\
=& \frac{1}{b_{j-1}}\frac{K_{j-1} + b_{j-1}n}{K_{j-1} + b_j n} - \frac{K_{j-2}/b_{j-1} - \sum_{i=1}^{j-2}(K_i - K_{i-1})/b_i + 1}{K_{j-1} + b_j n} \\
\geq& \frac{1}{b_{j-1}}\frac{K_{j-1} + b_{j-1}n}{K_{j-1} + b_j n} - \frac{K_{j-2}/b_{j-1} + 1}{K_{j-1}}
\end{aligned}$$

and writing the second term as ε_j^- we consider the separate cases

$$\begin{aligned}
\geq& \begin{cases} \frac{1}{b_{j-1}}\frac{K_{j-1}+b_j n}{K_{j-1}+b_j n} - \varepsilon_j^- & b_{j-1} \geq b_j \\ \frac{1}{b_{j-1}}\frac{b_{j-1}}{b_j}\frac{K_{j-1}+b_j n}{K_{j-1}+b_j n} - \varepsilon_j^- & b_{j-1} < b_j \end{cases} \\
=& \min\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) - \varepsilon_j^-.
\end{aligned}$$

As n is arbitrary this lower bounds holds for all δ in the range (3.7). Further, observe from (3.3) that ε_j^- tends to zero as $j \to \infty$.

Similarly, for δ in the range (3.10), we have the upper bound

$$\begin{aligned}
\frac{\log N'(F,\delta)}{-\log\delta} <& \frac{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n}{K_{j-1} + b_j(n-1)} \\
=& \frac{1}{b_{j-1}}\frac{K_{j-1} + b_{j-1}(n-1)}{K_{j-1} + b_j(n-1)} \\
&+ \frac{1 - K_{j-2}/b_{j-1} + \sum_{i=1}^{j-2}(K_i - K_{i-1})/b_i}{K_{j-1} + b_j(n-1)} \\
\leq& \frac{1}{b_{j-1}}\frac{K_{j-1} + b_{j-1}(n-1)}{K_{j-1} + b_j(n-1)} + \frac{1 + \sum_{i=1}^{j-2}(K_i - K_{i-1})/b_i}{K_{j-1}}
\end{aligned}$$

Again, writing the second term as ε_j^+ we consider the separate cases

$$\begin{aligned}
\leq& \begin{cases} \frac{1}{b_{j-1}}\frac{K_{j-1}+b_j(n-1)}{K_{j-1}+b_j(n-1)} + \varepsilon_j^+ & b_j \geq b_{j-1} \\ \frac{1}{b_{j-1}}\frac{b_{j-1}}{b_j}\frac{K_{j-1}+b_j(n-1)}{K_{j-1}+b_j(n-1)} + \varepsilon_j^+ & b_j < b_{j-1} \end{cases} \\
=& \max\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) + \varepsilon_j^+.
\end{aligned}$$

Again, as n is arbitrary this upper bound holds for all δ in the range (3.7), and from (3.3) ε_j^+ tends to zero as $j \to \infty$.

Similarly, by considering δ in the range

$$2^{-K_{j-1}-c_j m} \leq \delta < 2^{-K_{j-1}-c_j(m-1)}$$

for $m = 1, \ldots, (K_j - K_{j-1})/b_j$ we find non-negative quantities η_j^- and η_j^+ such that

$$\min\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right) - \eta_j^- \leq \frac{\log N'(G,\delta)}{-\log\delta} < \max\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right) + \eta_j^+$$

for δ in the range (3.7) where $\eta_j^-, \eta_j^+ \to 0$ as $j \to \infty$.

Taking $\varepsilon_j := \max\left(\varepsilon_j^-, \varepsilon_j^+, \eta_j^-, \eta_j^+\right)$ completes the proof. □

Lemma 3.5. *With F and G constructed as above*

$$\liminf_{j\to\infty} \frac{1}{b_j} \leq \dim_{LB} F, \qquad \dim_B F \leq \limsup_{j\to\infty} \frac{1}{b_j},$$
$$\liminf_{j\to\infty} \frac{1}{c_j} \leq \dim_{LB} G, \qquad \dim_B G \leq \limsup_{j\to\infty} \frac{1}{c_j},$$

$$\liminf_{j\to\infty}\left(\min\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) + \min\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right)\right) \leq \dim_{LB}(F \times G), \quad \textit{and}$$
$$\limsup_{j\to\infty}\left(\max\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) + \max\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right)\right) \geq \dim_B(F \times G).$$

Proof. Taking the limits of (3.8) as $j \to \infty$ we see that

$$\dim_{LB}(F) = \liminf_{\delta\to 0} \frac{\log N'(F,\delta)}{-\log\delta} \geq \liminf_{j\to\infty}\left(\min\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right)\right) = \liminf_{j\to\infty} \frac{1}{b_j},$$

and

$$\dim_B(F) = \limsup_{\delta\to 0} \frac{\log N'(F,\delta)}{-\log\delta} \leq \limsup_{j\to\infty}\left(\max\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right)\right) = \limsup_{j\to\infty} \frac{1}{b_j},$$

and the results for the set G follow similarly.

We now consider the product set $F \times G$: if δ is in the range (3.7) then by summing (3.8) and (3.9) we see that

$$\begin{aligned}\min\left(\frac{1}{b_j},\frac{1}{b_{j-1}}\right)+\min\left(\frac{1}{c_j},\frac{1}{c_{j-1}}\right)-2\varepsilon_j \\ \leq \frac{\log N'(F,\delta)}{-\log\delta}+\frac{\log N'(G,\delta)}{-\log\delta} \\ < \max\left(\frac{1}{b_j},\frac{1}{b_{j-1}}\right)+\max\left(\frac{1}{c_j},\frac{1}{c_{j-1}}\right)+2\varepsilon_j.\end{aligned}$$

Taking the limits as $j \to \infty$ and using the equivalent definition from Lemma 2.3 we obtain

$$\begin{aligned}\liminf_{j\to\infty}\left(\min\left(\frac{1}{b_j},\frac{1}{b_{j-1}}\right)+\min\left(\frac{1}{c_j},\frac{1}{b_{c-1}}\right)\right) \\ \leq \liminf_{\delta\to 0}\left(\frac{\log N'(F,\delta)}{-\log\delta}+\frac{\log N'(G,\delta)}{-\log\delta}\right)=\dim_{LB}(F\times G)\end{aligned}$$

and

$$\begin{aligned}\limsup_{j\to\infty}\left(\max\left(\frac{1}{b_j},\frac{1}{b_{j-1}}\right)+\max\left(\frac{1}{c_j},\frac{1}{b_{c-1}}\right)\right) \\ \geq \limsup_{\delta\to 0}\left(\frac{\log N'(F,\delta)}{-\log\delta}+\frac{\log N'(G,\delta)}{-\log\delta}\right)=\dim_{B}(F\times G)\end{aligned}$$

as required. □

Using these bounds we can construct sets F and G such that the box-counting dimensions of F, G and $F \times G$ take arbitrary values satisfying the chain of inequalities in Theorem 2.4, which is the content of the following theorem.

Theorem 3.6. *Let* $f_1, f_2, g_1, g_2 \in [0,1]$ *and* $h_1, h_2 \in [0,2]$ *satisfy*

$$\begin{aligned}f_1+g_1 \leq h_1 \leq \min(f_1+g_2, f_2+g_1) \\ \leq \max(f_1+g_2, f_2+g_1) \leq h_2 \leq f_2+g_2. \quad (3.11)\end{aligned}$$

There exists sets $F, G \subset \mathbb{R}$ *such that*

$$\begin{aligned}\dim_{LB} F &= f_1 & \dim_B F &= f_2 \\ \dim_{LB} F &= g_1 & \dim_B G &= g_2 \\ \dim_{LB}(F\times G) &= h_1 & \dim_B(F\times G) &= h_2.\end{aligned}$$

Initially we construct two sequences x and y such that $\liminf x_i = f_1$, $\limsup x_i = f_2$, $\liminf y_i = g_1$, $\liminf y_i = g_2$, $\liminf x_i + y_i = h_1$, and $\limsup x_i + y_i = h_2$, with the additional property that for each sequence the difference between consecutive terms vanishes at the limit. In this case

$$\liminf (\min (x_i, x_{i+1}) + \min (y_i, y_{i+1})) = \liminf x_i + y_i \qquad \text{and} \tag{3.12}$$

$$\limsup (\max (x_i, x_{i+1}) + \max (y_i, y_{i+1})) = \limsup x_i + y_i. \tag{3.13}$$

As a consequence if we construct compatible generalised Cantor sets F and G from the sequences of generators defined by $b_i := x_i^{-1}$ and $c_i := y_i^{-1}$ then the results of Corollary 3.3 and Lemma 3.5 together with the relationships (3.12) and (3.13) demonstrate that the sets F, G and $F \times G$ have the desired dimensions. However, as the generators are only defined if each b_i and c_i is rational and greater than 1 a further approximation argument is necessary.

PROOF OF THEOREM 3.6. By relabelling the f_i and g_i if necessary, we assume without loss of generality that $f_1 + g_2 \leq f_2 + g_1$. Consequently, from (3.11),

$$f_1 + g_1 \leq h_1 \leq f_1 + g_2, \quad \text{and} \tag{3.14}$$

$$f_2 + g_1 \leq h_2 \leq f_2 + g_2. \tag{3.15}$$

For each fixed $n \in \mathbb{N}$ and each $j = 1, \ldots, n$ define

$$x_{n,j} := \begin{cases} f_1 + \frac{j}{n}(f_2 - f_1) & n = 6k-5 \\ f_2 & n = 6k-4, 6k-3 \\ f_2 - \frac{j}{n}(f_2 - f_1) & n = 6k-2 \\ f_1 & n = 6k-1, 6k \end{cases} \tag{3.16}$$

and

$$y_{n,j} := \begin{cases} g_2 - \frac{j}{n}(g_2 - g_1) & n = 6k-5 \\ g_1 + \frac{j}{n}(h_2 - f_2 - g_1) & n = 6k-4 \\ h_2 - f_2 - \frac{j}{n}(h_2 - f_2 - g_1) & n = 6k-3 \\ g_1 + \frac{j}{n}(g_2 - g_1) & n = 6k-2 \\ g_2 - \frac{j}{n}(g_2 - h_1 + f_1) & n = 6k-1 \\ h_1 - f_1 + \frac{j}{n}(g_2 - h_1 + f_1) & n = 6k \end{cases} \tag{3.17}$$

for $k \in \mathbb{N}$, and note that

$$f_2 - f_1,\ g_2 - g_1,\ h_2 - f_2 - g_1,\ g_2 - h_1 + f_1 \geq 0. \tag{3.18}$$

It is immediate that for all $n \in \mathbb{N}$

$$x_{n,j} \in [f_1, f_2] \qquad j = 1, \ldots, n \tag{3.19}$$

and that for $n = 6k-5, 6k-2$

$$y_{n,j} \in [g_1, g_2] \qquad j = 1, \dots, n.$$

Further, for $n = 6k-4, 6k-3$

$$g_1 \le y_{n,j} \le h_2 - f_2 \le f_2 + g_2 - f_2 = g_2 \qquad j = 1, \dots, n$$

from (3.15), and for $n = 6k-1, 6k$

$$g_2 \ge y_{n,j} \ge h_1 - f_1 \ge f_1 + g_1 - f_1 = g_1 \qquad j = 1, \dots, n$$

from (3.14). Consequently, for all $n \in \mathbb{N}$

$$y_{n,j} \in [g_1, g_2] \qquad j = 1, \dots, n. \tag{3.20}$$

Further, the endpoints of these bounds are achieved at

$$x_{6k-1,1} = f_1, \qquad\qquad x_{6k-4,1} = f_2, \tag{3.21}$$
$$y_{6k-5,6k-5} = g_1, \qquad \text{and} \qquad y_{6k-2,6k-2} = g_2 \tag{3.22}$$

for each $k \in \mathbb{N}$.

Next, consider the sum

$$x_{n,j} + y_{n,j} = \begin{cases} f_1 + g_2 + \frac{j}{n}\left(f_2 + g_1 - (f_1 + g_2)\right) & n = 6k-5 \\ f_2 + g_1 + \frac{j}{n}\left(h_2 - f_2 - g_1\right) & n = 6k-4 \\ h_2 - \frac{j}{n}\left(h_2 - f_2 - g_1\right) & n = 6k-3 \\ f_2 + g_1 - \frac{j}{n}\left(f_2 + g_1 - (f_1 + g_2)\right) & n = 6k-2 \\ f_1 + g_2 - \frac{j}{n}\left(g_2 - h_1 + f_1\right) & n = 6k-1 \\ h_1 + \frac{j}{n}\left(g_2 - h_1 + f_1\right) & n = 6k. \end{cases}$$

Observe that for $n = 6k-5, 6k-2$

$$f_1 + g_2 \le x_{n,j} + y_{n,j} \le f_2 + g_1 \qquad j = 1, \dots, n,$$

for $n = 6k-4, 6k-3$

$$f_2 + g_1 \le x_{n,j} + y_{n,j} \le h_2 \qquad j = 1, \dots, n,$$

and for $n = 6k-1, 6k$

$$h_1 \le x_{n,j} + y_{n,j} \le f_1 + g_2 \qquad j = 1, \dots, n.$$

Consequently, as $h_1 \le f_1 + g_2$ and $f_2 + g_1 \le h_2$, we conclude that for all $n \in \mathbb{N}$

$$x_{n,j} + y_{n,j} \in [h_1, h_2] \qquad j = 1, \ldots, n. \tag{3.23}$$

Further, the endpoints of this bound are achieved at

$$x_{6k-1,6k-1} + y_{6k-1,6k-1} = h_1, \quad \text{and} \quad x_{6k-4,6k-4} + y_{6k-4,6k-4} = h_2 \tag{3.24}$$

for each $k \in \mathbb{N}$.

Define the sequences

$$x := (x_{1,1}, x_{2,1}, x_{2,2}, \ldots, x_{n,1}, \ldots, x_{n,n}, x_{n+1,1}, \ldots) \quad \text{and}$$
$$y := (y_{1,1}, y_{2,1}, y_{2,2}, \ldots, y_{n,1}, \ldots, y_{n,n}, y_{n+1,1}, \ldots).$$

From the bounds (3.19), (3.20) and (3.23), and the periodic achievement of these bounds in (3.21), (3.22) and (3.24) we conclude that

$$\liminf_{i\to\infty} x_i = f_1, \qquad \limsup_{i\to\infty} x_i = f_2, \tag{3.25}$$

$$\liminf_{i\to\infty} y_i = g_1, \qquad \limsup_{i\to\infty} y_i = g_2, \tag{3.26}$$

$$\liminf_{i\to\infty} x_i + y_i = h_1, \qquad \text{and} \qquad \limsup_{i\to\infty} x_i + y_i = h_2. \tag{3.27}$$

Next, write $C := \max(f_2 - f_1, g_2 - g_1, h_2 - f_2 - g_1, g_2 - h_1 + f_1)$. We see from the definitions (3.16) and (3.17) that for each $n \in \mathbb{N}$

$$|x_{n,j} - x_{n,j+1}|, |y_{n,j} - y_{n,j+1}| \le \frac{C}{n} \qquad j = 1, \ldots, n-1, \quad \text{and}$$
$$|x_{n,n} - x_{n+1,1}|, |y_{n,n} - y_{n+1,1}| \le \frac{C}{n+1}.$$

Consequently the difference between consecutive elements of each sequence x and y approaches zero: for each $n \in \mathbb{N}$

$$|x_i - x_{i+1}|, |y_i - y_{i+1}| \le \frac{C}{n+1} \qquad \sum_{j=1}^{n} j < i \le \sum_{j=1}^{n+1} j$$

and, as i in this range satisfies $i \le \sum_{j=1}^{n+1} j \le (n+1)^2$, we conclude that

$$|x_i - x_{i+1}|, |y_i - y_{i+1}| \le \frac{C}{\sqrt{i}} \qquad \forall i \in \mathbb{N}. \tag{3.28}$$

Finally, define the sequences

$$b_i := \begin{cases} i+1 & x_i \le \frac{1}{i+1} \\ \frac{i+1}{k} & \frac{k}{i+1} < x_i \le \frac{k+1}{i+1} \quad k = 1, \dots, i \end{cases}$$

and

$$c_i := \begin{cases} i+1 & y_i \le \frac{1}{i+1} \\ \frac{i+1}{k} & \frac{k}{i+1} < y_i \le \frac{k+1}{i+1} \quad k = 1, \dots, i \end{cases}$$

and observe that $b_i, c_i \in \mathbb{Q}$ and $b_i, c_i > 1$ for all $i \in \mathbb{N}$, so we can construct compatible generalised Cantor sets F and G from the sequences b_i and c_i respectively. Further,

$$\left|b_i^{-1} - x_i\right|, \left|c_i^{-1} - y_i\right| \le \frac{1}{i+1} \qquad \forall i \in \mathbb{N}. \tag{3.29}$$

From Corollary 3.3 and Lemma 3.5 we see that

$$\dim_{LB} F = \liminf_{i\to\infty} \frac{1}{b_i} = \liminf_{i\to\infty} x_i$$

from (3.29) which, from (3.25), yields $\dim_{LB} F = f_1$ as required. Similarly, we see that $\dim_B F = f_2$, $\dim_{LB} G = g_1$ and $\dim_B G = g_2$.

Next, for each $i \in \mathbb{N}$

$$\min\left(\frac{1}{b_i}, \frac{1}{b_{i+1}}\right) \ge x_i - \frac{2}{\sqrt{i}}, \qquad \min\left(\frac{1}{c_i}, \frac{1}{c_{i+1}}\right) \ge y_i - \frac{2}{\sqrt{i}}, \tag{3.30}$$

$$\max\left(\frac{1}{b_i}, \frac{1}{b_{i+1}}\right) \le x_i + \frac{2}{\sqrt{i}}, \quad \text{and} \quad \max\left(\frac{1}{c_i}, \frac{1}{c_{i+1}}\right) \le y_i + \frac{2}{\sqrt{i}}. \tag{3.31}$$

Indeed, from (3.29)

$$\min\left(\frac{1}{b_i}, \frac{1}{b_{i+1}}\right) \ge \min\left(x_i - \frac{1}{i+1}, x_{i+1} - \frac{1}{i+2}\right)$$

which from (3.28)

$$\ge \min\left(x_i - \frac{1}{i+1}, x_i - \frac{1}{i+2} - \frac{C}{\sqrt{i}}\right) \ge x_i - \frac{C+1}{\sqrt{i}}$$

and the remaining inequalities in (3.30) and (3.31) follow similarly.

Consequently, from Lemma 3.5 and (3.30),

$$\begin{aligned}\liminf_{i\to\infty} x_i + y_i &= \liminf_{i\to\infty} x_i - \frac{2}{\sqrt{i}} + y_i - \frac{2}{\sqrt{i}} \\ &\leq \liminf_{i\to\infty} \left(\min\left(\frac{1}{b_i}, \frac{1}{b_{i+1}}\right) + \min\left(\frac{1}{c_i}, \frac{1}{c_{i+1}}\right)\right) \leq \dim_{LB}(F \times G)\end{aligned}$$

and from Corollary 3.3 and (3.29)

$$\begin{aligned}\dim_{LB}(F \times G) &\leq \liminf_{i\to\infty} \frac{1}{b_i} + \frac{1}{c_i} \\ &\leq \liminf_{i\to\infty} x_i + \frac{1}{1+i} + y_i + \frac{1}{1+i} = \liminf_{i\to\infty} x_i + y_i.\end{aligned}$$

We conclude that

$$\dim_{LB}(F \times G) = \liminf_{i\to\infty} x_i + y_i,$$

and so, from (3.27), $\dim_{LB}(F \times G) = h_1$ as required. Similarly, from Corollary 3.3, Lemma 3.5 and the inequalities (3.29) and (3.31), we can demonstrate that

$$\dim_B(F \times G) = \limsup_{i\to\infty} x_i + y_i,$$

and so, from (3.27), that $\dim_B(F \times G) = h_2$ as required. □

Using Theorem 3.6 we are able to construct sets $F, G \subset \mathbb{R}$ such that the upper and lower box-counting dimensions of the sets F, G and $F \times G$ can take arbitrary values subject to the chain of inequalities in Theorem 2.4.

Example 3.7. We apply Theorem 3.6 with the values $f_1 = g_1 = 0$ and $f_2 = g_2 = h_1 = h_2 = 1$, noting that these values satisfy the hypothesis of the theorem. The construction yields two sets F and G such that

$$\dim_{LB} F = \dim_{LB} G = 0 \qquad \text{and}$$
$$\dim_B F = \dim_B G = \dim_{LB}(F \times G) = \dim_B(F \times G) = 1.$$

In particular these sets with zero lower box-counting dimension have a product with positive lower box-counting dimension, and for these same sets the upper box-counting dimension does not increase upon taking the product.

We remark that by inductively applying this construction we are able to produce m compatible generalised Cantor sets F_j such that the upper and lower box-counting dimensions of the sets F_j take arbitrary values subject to the inequalities (2.18), (2.19),

$$\sum_{i=1}^{m} \dim_{LB} F_i \leq \dim_{LB} (F_1 \times \dots F_m) \qquad \text{and}$$

$$\dim_B (F_1 \times \dots F_m) \leq \sum_{i=1}^{m} \dim_B F_i.$$

A Geometry of generalised Cantor sets.

We consider the geometry of the generalised Cantor sets discussed in section 3. Recall that F_{j-1} is the set of disjoint intervals at the $(j-1)^{th}$ stage in the construction of F. For $n = 1, \dots, (K_j - K_{j-1})/b_j$ define $F_{j-1,n}$ to be the result of n successive applications of the generator gen_{b_j} to the set F_{j-1}. The sets $F_{j-1,n}$ are the sets that make up the 'substages' in the construction of F, with one such substage for every application of a generator. Note that $F_j = F_{j-1,(K_j - K_{j-1})/b_j}$ and as the sets are monotonically decreasing in n,

$$F_j = \bigcap_{n=1}^{(K_j - K_{j-1})/b_j} F_{j-1,n}. \tag{A.1}$$

We write $\#(F_{j,n})$ for the number and $l(F_{j,n})$ for the length of the intervals in $F_{j,n}$. It is easy to show that

$$\#(F_{j-1,n}) = 2^{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n} \tag{A.2}$$

$$l(F_{j-1,n}) = 2^{-K_{j-1} - b_j n}. \tag{A.3}$$

Replacing the b_i with c_i throughout the above gives the corresponding result for the intermediary sets $G_{j-1,n}$ used in the construction of G. Note that the intervals in F_{j-1} are the same length as the intervals in G_{j-1} despite the arbitrarily chosen sequences $\{b_i\}_{i=1}^{\infty}$ and $\{c_i\}_{i=1}^{\infty}$. These common lengths greatly simplify the calculation of the box-counting dimensions of the product set $F \times G$, and in this sense the sets F and G are 'compatible'.

We will need to find some explicit points of the generalised Cantor set F. Recall that F is defined by $F := \bigcap_{j \in \mathbb{N}} F_j$ which, in light of (A.1), is the intersection of every intermediary substage $F_{j-1,n}$ and can be written

$$F := \bigcap_{j \in \mathbb{N}} \bigcap_{n=1}^{K_j - K_{j-1}/b_j} F_{j-1,n}.$$

When applying a generator gen_b to an intermediary set of disjoint intervals $\mathcal{I}$, an open proportion is removed from the middle of each interval. Consequently the endpoints of the intervals $\mathcal{I}$ are in the set $\text{gen}_b(\mathcal{I})$ and remain endpoints of intervals. Inductively we see that the endpoints of each intermediary set $F_{j-1,n}$ are in the final set F.

As each intermediary set $F_{j-1,n}$ is a cover of F and the length of the intervals in $F_{j-1,n}$ approach zero as $j \to \infty$ it is natural to use the minimal cover formulation of the box-counting dimension for these sets: we immediately have that if $l(F_{j-1,n}) \leq \delta$ the set $F_{j-1,n}$ is a suitable cover of F. Unfortunately this cover is not always a minimal cover at this length-scale. However, a reasonable lower bound on $N'(F,\delta)$ is easy to find and if we restrict the choice of generators so that the intervals are suitably separated then the sets $F_{j-1,n}$ are minimal covers at the appropriate length-scale.

Lemma A.1. *For δ in the range $l(F_{j-1,n}) \leq \delta < l(F_{j-1,n-1})$ the minimum number of sets of diameter at most δ that cover F satisfies*

$$\#(F_{j-1,n-1}) \leq N'(F,\delta) \leq \#(F_{j-1,n}). \tag{A.4}$$

Further, if the choice of generators is restricted so that $b_i \geq \log(3)/\log(2)$ for all i then

$$N'(F,\delta) = \#(F_{j-1,n}).$$

Proof. The upper bound follows immediately from the fact that $F_{j-1,n}$ is a cover of F consisting of $\#(F_{j-1,n})$ sets of diameter less than δ. For the lower bound consider the following points in F: let E consist of all the left endpoints of the intervals in $F_{j-1,n-1}$ so that E consists of $\#(F_{j-1,n-1})$ points. Now, any two points of E are separated by one of the intervals of $F_{j-1,n-1}$ so no set of diameter $\delta < l(F_{j-1,n-1})$ can intersect two points of E (see Figure 1). Consequently, at least $\#(F_{j-1,n-1})$ sets of diameter δ are required to cover E therefore at least this many are required to cover F, yielding

$$\#(F_{j-1,n-1}) \leq N'(F,\delta) \leq \#(F_{j-1,n}).$$

If we restrict the generators to those gen_b with $b \geq \log(3)/\log(2)$ then with every application of a generator at least the middle third of each interval is removed. Consequently, the intervals in $F_{j-1,n-1}$ are separated by at least the length $l(F_{j-1,n-1})$ so that if E is the set of all (both left and right) endpoints of the intervals in $F_{j-1,n-1}$ then no set of diameter $\delta < l(F_{j-1,n-1})$ can intersect two points of E (see Figure 2).

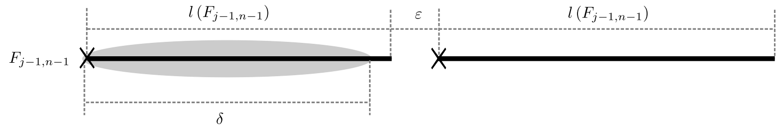

Figure 1: Two intervals in the set $F_{j-1,n-1}$ (black lines) to illustrate that sets with diameter $\delta < l(F_{j-1,n-1})$ (grey ellipses) can not intersect two left endpoints (black crosses). Consequently we require at least one set of diameter δ for each interval in $F_{j-1,n-1}$ to cover all the left endpoints. Generally this cannot be improved as the distance between intervals ε can be arbitrarily small.

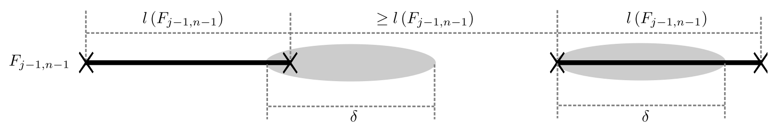

Figure 2: Two intervals in the set $F_{j-1,n-1}$ (black lines) constructed from generators gen_b with $b \geq \log(3)/\log(2)$. As the distance between intervals is at least the length of the interval a set of diameter $\delta < l(F_{j-1,n-1})$ (grey ellipses) can not intersect two endpoints (black crosses). Consequently we require at least two sets of diameter δ for each interval in $F_{j-1,n-1}$ to cover all the endpoints.

As E consists of $2\#(F_{j-1,n-1}) = \#(F_{j-1,n})$ points at least this many sets of diameter δ are required to cover E and hence required to cover F, yielding

$$\#(F_{j-1,n}) \leq N'(F,\delta) \leq \#(F_{j-1,n}).$$

□

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