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LOCALLY DEFINED OPERATORS IN THE SPACE OF FUNCTIONS OF BOUNDED φ–VARIATION

Abstract

We prove that every locally defined operator mapping the space of continuous and bounded φ-variation functions into itself is a Nemytskij composition operator.

1 Introduction.

Let $I = [a, b]$ be a closed interval of the real line $\mathbb{R}$ ($a, b \in \mathbb{R},\ a < b$) and let $\mathcal{G} = \mathcal{G}(I)$, $\mathcal{H} = \mathcal{H}(I)$ be function spaces $\varphi\colon I \to \mathbb{R}$. An operator $K\colon \mathcal{G} \to \mathcal{H}$ is called *locally defined*, or $(\mathcal{G}, \mathcal{H})$-*local* or briefly *local*, if for every open interval $J \subset \mathbb{R}$ and for all functions $f, g \in \mathcal{G}$, the implication

$$f|_{J\cap I} = g|_{J\cap I} \Longrightarrow K(f)|_{J\cap I} = K(g)|_{J\cap I}$$

holds true. For some pairs $(\mathcal{G}, \mathcal{H})$ of function spaces the forms of local operators $K\colon \mathcal{G} \to \mathcal{H}$ (or their representation theorems) have been established. For instance, in [6] it was done in the case when $\mathcal{G} = \mathcal{C}^n(I)$ and $\mathcal{H} = \mathcal{C}(I)$ or $\mathcal{H} = \mathcal{C}^1(I)$, in [9, 10, 15] in the case when $\mathcal{G}$ and $\mathcal{H}$ are the spaces of n-times and k-times, respectively, Whitney differentiable functions, in [17] in the case when $\mathcal{G}$ is the space of Hölder functions and $\mathcal{H} = \mathcal{C}(I)$ (cf. also [16] and [1]).

Note that if a local operator K maps the space of continuous functions $\mathcal{C}(I)$ into itself, then

$$K(\varphi)(x) = h\big(x, \varphi(x)\big), \quad \varphi \in \mathcal{C}(I), \quad x \in I,$$

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for a uniquely determined continuous function $h\colon I \times \mathbb{R} \to \mathbb{R}$, that is K is a Nemytskij composition operator ([6]). The same representation formula remains true when $\mathcal{G} = \mathcal{H} = \mathcal{C}^1(I)$ but, surprisingly enough, in this case the function h need not be even continuous ([6], see also [1] p. 325).

In the present paper we give a representation formula for local operators which are self-mappings of a Banach space of continuous functions of (generalized) bounded φ-variation.

2 Preliminaries.

Let $I = [a, b]$ be a closed interval of the real axis. As usually, $\mathbb{R}^I$ stands for the set of all real functions $f\colon I \to \mathbb{R}$.

Denote by $\mathcal{F}$ the set of all convex functions $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u > 0$. Note that ([4, Remark 2.1]), if $\varphi \in \mathcal{F}$, then it is continuous, strictly increasing, $\lim\limits_{u\to\infty} \varphi(u) = \infty$ and superadditive, i.e.,

$$\varphi(u) + \varphi(v) \leq \varphi(u + v); \quad u \geq 0,\ v \geq 0. \tag{1}$$

Indeed, the convexity of φ implies that the function $(0,\infty) \ni t \mapsto \frac{\varphi(t)}{t}$ is nondecreasing, whence, for all $u \geq 0$ and $v \geq 0$, we have

$$\varphi(u+v) = \frac{\varphi(u+v)}{u+v}u + \frac{\varphi(u+v)}{u+v}v \leq \frac{\varphi(u)}{u}u + \frac{\varphi(v)}{v}v = \varphi(u) + \varphi(v),$$

and

$$\varphi(tu) \leq t\varphi(u); \quad u \geq 0, \quad t \in [0, 1]. \tag{2}$$

Given a function $\varphi \in \mathcal{F}$, we say that $f\colon I \to \mathbb{R}$ is a *function of bounded φ-variation in I*, and write $f \in BV_\varphi(I)$, if the *φ-variation of f on I*, defined by

$$V_\varphi(f, I) := \sup\{\sigma(f, P)\colon P \in \mathcal{P}(I)\}, \tag{3}$$

is finite; here

$$\sigma(f, P) := \sum_{i=1}^{m} \varphi\big(|f(t_i) - f(t_{i-1})|\big),$$

and $\mathcal{P}(I)$ stands for the set of all partitions

$$P = (t_0, t_1, \ldots, t_{m-1}, t_m),\ a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b,$$

of the closed interval I.

Clearly, $BV_\varphi(I)$ coincides with the classical concept of variation in the sense of Jordan ([12, Chapter 8]) if $\varphi = \mathrm{id}_I$, and in the sense of Wiener [14] if $\varphi(u) = u^p$, $u \geq 0$, $p > 1$. The general definition was introduced by Young [18].

It is known that $BV_\varphi(I)$ is convex, but it is not necessarily a linear space. Define the space $W_\varphi(I)$ as follows:

$$f \in W_\varphi(I) \text{ if there exists } \lambda > 0 \text{ such that } \frac{f}{\lambda} \in BV_\varphi(I).$$

The set $W_\varphi(I)$ is a linear space. Indeed, if $f_j \in W_\varphi(I)$, then $V_\varphi\left(\frac{f_j}{\lambda}\right) < \infty$ for some $\lambda_j > 0$, $j = 1, 2$, and, by the convexity of the functional V_φ, we get

$$V_\varphi\left(\frac{f_1 + f_2}{\lambda_1 + \lambda_2}\right) \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} V_\varphi\left(\frac{f_1}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} V_\varphi\left(\frac{f_2}{\lambda_2}\right),$$

whence $f_1 + f_2 \in W_\varphi(I)$. Obviously, $\lambda f \in W_\varphi(I)$ for all $\lambda \in \mathbb{R}$ and $f \in W_\varphi(I)$.

There is a vast literature concerning the space of functions of generalized φ-variation (see, for instance, [2, 3, 8, 13]).

Recall that Maligranda and Orlicz [8] proved that the space $W_\varphi(I)$ endowed with the norm $\|f\|_\varphi = |f(a)| + p_\varphi(f)$, where

$$p_\varphi(f) = p_\varphi(f, I) = \inf\left\{\lambda > 0 \colon V_\varphi\left(\frac{f}{\lambda}\right) \leq 1\right\}, \quad f \in W_\varphi(I), \tag{4}$$

is a Banach algebra. The seminorm defined by (4) is called the *Luxemburg–Nakano–Orlicz seminorm* [7, 11, 12].

In order to establish some relations between the spaces $W_\varphi(I)$, generated by different functions $\varphi \in \mathcal{F}$, let us recall the following definition from [3]. Given $\varphi, \psi \in \mathcal{F}$, we write $\psi \ll \varphi$ and say that φ *dominates* ψ *near zero* if there exist $r > 0$, $c > 0$ and $t_0 > 0$ such that $\psi(t) \leq r\varphi(ct)$ for all $t \in [0, t_0]$.

3 Local operators.

From now on, let $CW_\varphi(I) = W_\varphi(I) \cap C(I)$, where $C(I)$ stands for the space of continuous functions defined on I.

Recall the following properties of the functional V_φ:

1. V_φ is nondecreasing, i.e., if J_1, J_2 are sub-intervals of J and $J_1 \subset J_2$, then $V_\varphi(f, J_1) < V_\varphi(f, J_2)$;

2. V_φ is semi-additive, i.e., if J_1, J_2 are sub-intervals of J such that $J_1 \cap J_2$ is a singleton, then $V_\varphi(f, J_1) + V_\varphi(f, J_2) \leq V_\varphi(f, J_1 \cup J_2)$;

3. V_φ is sequentially lower semicontinuous, i.e.,

$$V_\varphi(f,I) \le \liminf_{n\to\infty} V_\varphi(f_n,I),$$

if $f_n \in \mathbb{R}^I$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in I$.

Remark 1. For $P, Q \in \mathcal{P}(I)$, $P = (p_0, p_1, \dots, p_m)$, $Q = (q_0, q_1, \dots, q_n)$, we say that $P \preceq Q$ if $\{p_0, p_1, \dots, p_m\} \subset \{q_0, q_1, \dots, q_n\}$. Of course, $(\mathcal{P}(I), \preceq)$ is partially ordered.

Chistyakov ([2], p. 1459) observed that if f is an arbitrary bounded monotonic function on I and $P, Q \in \mathcal{P}(I)$, $P \preceq Q$, then

$$\sigma(f,Q) \le \sigma(f,P),$$

and, consequently,

$$V_\varphi\left(\frac{f}{\lambda}, I\right) = V_\varphi\left(\frac{|f(b)-f(a)|}{\lambda}\right),$$

for all $\lambda > 0$.

Lemma 2. *Let $I = [a,b] \subset \mathbb{R}$ ($a, b \in \mathbb{R}$, $a < b$), $n \in \mathbb{N}$, $(x_k, y_k) \in I \times \mathbb{R}$, $k = 0, \dots, n$, such that*

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

be fixed. If the function $f\colon I \to \mathbb{R}$ is defined by

$$f(t) = \begin{cases} y_i & \textit{if} \quad t = x_i, \quad i = 0, \dots, n, \\ \textit{affine} & \textit{otherwise}, \end{cases}$$

then

$$V_\varphi(f,I) = \max\Big\{ \sum_{s=1}^{l} \varphi(|y_{j_s} - y_{j_{s-1}}|)\colon l \in \{1, \dots, n\};\ y_{j_0} = y_0,\ y_{j_l} = y_m \Big\}.$$

Proof. Let $(y_{j_s})_{s=0}^{l_0}$, $l_0 \in \{1, \dots, n\}$, be a subsequence of $(y_k)_{k=0}^{n}$ such that $y_{j_0} = y_0$, $y_{j_{l_0}} = y_n$, and

$$\begin{aligned} &\sum_{s=1}^{l_0} \varphi(|y_{j_s} - y_{j_{s-1}}|) \\ &\qquad = \max\Big\{ \sum_{s=1}^{l} \varphi(|y_{j_s} - y_{j_{s-1}}|)\colon l \in \{1, \dots, n\};\ y_{j_0} = y_0,\ y_{j_l} = y_n \Big\}. \end{aligned}$$

Since $(x_{js})_{s=0}^{l_0}$ is a partition of I, we have

$$\sum_{s=1}^{l_0} \varphi(|y_{j_s} - y_{j_{s-1}}|) \le V_\varphi(f, I).$$

To prove the inverse inequality, take an arbitrary partition $P = (t_i)_{i=0}^m \in \mathcal{P}(I)$ and assume that $f(t_0) \le f(t_1)$. Let k_1 be the larger of the numbers $1, \ldots, m$ such that

$$y_0 = f(t_0) \le f(t_1) \le \ldots \le f(t_{k_1}). \tag{5}$$

Then there exists $j_1 \in \{1, \ldots, n\}$ such that $x_{j_1-1} < t_{k_1} \le x_{j_1}$ or $x_{j_1} < t_{k_1} \le x_{j_1+1}$ and

$$f(t_{k_1}) \le y_{j_1}. \tag{6}$$

Thus, by inequality (1) and the monotonicity of φ, we get

$$\begin{aligned} \sum_{s=1}^{k_1} \varphi\big(\big|f(t_s) - f(t_{s-1})\big|\big) &\le \varphi\left(\sum_{s=1}^{k_1} \big(f(t_s) - f(t_{s-1})\big)\right) \\ &= \varphi\big(f(t_{k_1}) - f(t_0)\big) \le \varphi(y_{j_1} - y_0), \end{aligned} \tag{7}$$

whence, if $k_1 = m$, then

$$\sum_{s=1}^{m} \varphi\big(\big|f(t_s) - f(t_{s-1})\big|\big) \le \varphi(y_n - y_0) = \varphi\big(\big|y_n - y_0\big|\big). \tag{8}$$

If $k_1 < m$, then $f(t_{k_1}) > f(t_{k_1+1})$. Analogously, denoting by k_2 the larger of numbers $k_1 + 1, \ldots, m$ such that

$$f(t_{k_1}) > f(t_{k_1+1}) \ge \ldots \ge f(t_{k_2}), \tag{9}$$

by the definition of f, there exists j_2 such that

$$\begin{array}{ll} j_2 \in \{j_1 + 1, \ldots, n\} & \text{if} \quad t_{k_1} \in (x_{j_1-1}, x_j]; \\ j_2 \in \{j_1, \ldots, n\} & \text{if} \quad t_{k_1} \in (x_{j_1}, x_{j_1+1}], \end{array}$$

$$f(t_{k_2}) \ge y_{j_2}, \tag{10}$$

and either $t_{k_2} \in (x_{j_2-1}, x_{j_2}]$ or $t_{k_2} \in (x_{j_2}, x_{j_2+1}]$.

Hence, applying (6), (9), (10), inequality (1) and the monotonicity of φ, we obtain

$$\begin{aligned} \sum_{s=k_1}^{k_2} \varphi\big(\big|f(t_s) - f(t_{s-1})\big|\big) &\le \varphi\left(\sum_{s=k_1}^{k_2} \big(f(t_{s-1}) - f(t_s)\big)\right) \\ &= \varphi\big(f(t_{k_1}) - f(t_{k_2})\big) \le \varphi(y_{j_1} - y_{j_2}), \end{aligned}$$

as $y_{j_1} > y_{j_2}$, and consequently, by (7),

$$\begin{aligned}\sum_{s=1}^{k_2} \varphi\big(|f(t_s) - f(t_{s-1})|\big) &\le \varphi(y_{j_1} - y_0) + \varphi\,(y_{j_1} - y_{j_2}) \\ &= \varphi\big(|y_{j_1} - y_0|\big) + \varphi\big(|y_{j_2} - y_{j_1}|\big).\end{aligned}$$

Therefore, if $k_2 = m$, then

$$\sum_{s=1}^{m} \varphi\big(|f(t_s) - f(t_{s-1})|\big) \le \varphi\big(|y_1 - y_0|\big) + \varphi\big(|y_{j_n} - y_{j_1}|\big).$$

If $k_2 < m$, then $f(t_{k_2}) < f(t_{k_2+1})$, and we can repeat the above procedure for k_2. By the definition of f, after l steps, we obtain

$$\sum_{s=1}^{m} \varphi\big(|f(t_s) - f(t_{s-1})|\big) \le \varphi\big(|y_{j_1} - y_0|\big) + \varphi\big(|y_{j_1} - y_{j_2}|\big) + \ldots + \varphi\big(|y_{j_{l-1}} - y_n|\big),$$

where $l \in \{j_2 + 1, \ldots, n\}$ and $y_{j_l} = y_n$; which completes the proof. □

Theorem 3. *Let $\varphi \in \mathcal{F}$ and $I = [a, b]$, $a, b \in \mathbb{R}$, $a < b$, be a closed interval. If a locally defined operator K maps $CW_\varphi(I)$ into $C(I)$, then there exists a unique function $h\colon I \times \mathbb{R} \to \mathbb{R}$ such that, for all $f \in CW_\varphi(I)$,*

$$K(f)(s) = h(s, f(s)), \quad s \in I. \tag{11}$$

Proof. First we show that for every $f, g \in CW_\varphi(I)$ and for every $s_0 \in \operatorname{int} I$ the condition

$$f(s_0) = g(s_0) \tag{12}$$

implies that

$$K(f)(s_0) = K(g)(s_0). \tag{13}$$

To this end take an arbitrary pair of functions $f, g \in CW_\varphi(I)$ fulfilling condition (12) and choose $s_0 \in \operatorname{int} I$ arbitrarily. Define

$$\gamma(t) = \begin{cases} f(t) & \text{for} \quad t \in [a, s_0] \\ g(t) & \text{for} \quad t \in (s_0, b] \end{cases}.$$

To prove that $\gamma \in CW_\varphi(I)$, define $f_1, g_1 \colon I \to \mathbb{R}$ by

$$f_1(t) = \begin{cases} f(t) - f(s_0) & \text{for} \quad t \in [a, s_0] \\ 0 & \text{for} \quad t \in (s_0, b] \end{cases},$$

$$g_1(t) = \begin{cases} 0 & \text{for} \quad t \in [a, s_0] \\ g(t) - g(s_0) & \text{for} \quad t \in (s_0, b] \end{cases}.$$

Since $f, g \in CW_\varphi(I)$, we get $V_\varphi\left(\frac{f}{\lambda}\right)$ and $V_\varphi\left(\frac{g}{\mu}\right)$ are finite for some $\lambda, \mu > 0$. Let $P = (t_i)_{i=0}^m$ be a partition of I such that $t_{k-1} \le s_0 \le t_k$ for some $1 \le k \le m$. Then, by the property of φ that $\varphi(0) = 0$,

$$\sum_{i=1}^m \varphi\left(\left|\frac{f_1}{\lambda}(t_i) - \frac{f_1}{\lambda}(t_{i-1})\right|\right) = \sum_{i=1}^{k-1} \varphi\left(\left|\frac{f}{\lambda}(t_i) - \frac{f}{\lambda}(t_{i-1})\right|\right) + \varphi\left(\left|\frac{f}{\lambda}(s_0) - \frac{f}{\lambda}(t_{k-1})\right|\right)$$

and

$$\sum_{i=1}^m \varphi\left(\left|\frac{g_1}{\mu}(t_i) - \frac{g_1}{\mu}(t_{i-1})\right|\right) = \varphi\left(\left|\frac{g}{\mu}(t_k) - \frac{g}{\mu}(s_0)\right|\right) + \sum_{i=k+1}^m \varphi\left(\left|\frac{g}{\mu}(t_i) - \frac{g}{\mu}(t_{i-1})\right|\right).$$

Hence, applying the monotonicity of V_φ, we have that $V_\varphi\left(\frac{f_1}{\lambda}\right)$ and $V_\varphi\left(\frac{g_1}{\mu}\right)$ are finite, and, finally, that $f_1 + g_1 \in CW_\varphi(I)$ as $CW_\varphi(I)$ is a linear space. Thus

$$V_\varphi\left(\frac{f_1 + g_1}{\tau}\right) < \infty \tag{14}$$

for some $\tau > 0$. Since $(f_1 + g_1)(s) - (f_1 + g_1)(t) = \gamma(s) - \gamma(t)$ for all $s, t \in [a, b]$, condition (14) implies that $\gamma \in CW_\varphi(I)$.

Since

$$f|_{(-\infty, s_0) \cap I} = \gamma|_{(-\infty, s_0) \cap I}, \quad g|_{(-\infty, s_0) \cap I} = \gamma|_{(-\infty, s_0) \cap I},$$

by the definition of a local operator, we get

$$K(f)|_{(-\infty, s_0) \cap I} = K(\gamma)|_{(-\infty, s_0) \cap I}, \quad K(g)|_{(-\infty, s_0) \cap I} = K(\gamma)|_{(-\infty, s_0) \cap I}.$$

Therefore, by the continuity of $K(f)$, $K(\gamma)$ and $K(g)$ at s_0, we get

$$K(f)(s_0) = K(\gamma)(s_0) = K(g)(s_0).$$

Suppose now that s_0 is the left endpoint of the interval I (i.e., $s_0 = a$). By the continuity of f and g at s_0, there exists a sequence $(s_n)_{n \in \mathbb{N}}$ such that

$$s_0 < s_{n+1} < s_n, \quad \left|s_n - s_0\right| < \frac{b - s_0}{n}, \quad n \in \mathbb{N},$$

and

$$\left|f(s_n) - f(s_0)\right| < \frac{1}{n^2}, \quad \left|g(s_n) - g(s_0)\right| < \frac{1}{n^2}, \quad n \in \mathbb{N}. \tag{15}$$

Define the sequence of functions $\gamma_n \colon [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, in the following way:

$$\gamma_{2k}(t) = \begin{cases} \frac{f(s_{2k})-f(s_0)}{s_{2k}-s_0}(t-s_0)+f(s_0) & \text{for} \quad t\in[s_0,s_{2k}] \\ \frac{g(s_{2i-1})-f(s_{2i})}{s_{2i-1}-s_{2i}}(t-s_{2i})+f(s_{2i}) & \text{for} \quad t\in(s_{2i},s_{2i-1}],\ i\in\{1,\dots,k\} \\ \frac{f(s_{2i})-g(s_{2i+1})}{s_{2i}-s_{2i+1}}(t-s_{2i+1})+g(s_{2i+1}) & \text{for} \quad t\in(s_{2i+1},s_{2i}],\ i\in\{1,\dots,k-1\} \\ g(s_1) & \text{for} \quad t\in(s_1,b] \end{cases}$$

$$\gamma_{2k-1}(t) = \begin{cases} \frac{g(s_{2k-1})-g(s_0)}{s_{2k-1}-s_0}(t-s_0)+g(s_0) & \text{for} \quad t\in[s_0,s_{2k-1}] \\ \frac{f(s_{2i-2})-g(s_{2i-1})}{s_{2i-2}-s_{2i-1}}(t-s_{2i-1})+g(s_{2i-1}) & \text{for} \quad t\in(s_{2i-1},s_{2i-2}],\ i\in\{2,\dots,k\} \\ \frac{g(s_{2i-3})-f(s_{2i-2})}{s_{2i-3}-s_{2i-2}}(t-s_{2i-2})+f(s_{2i-2}) & \text{for} \quad t\in(s_{2i-2},s_{2i-3}],\ i\in\{2,\dots,k\} \\ g(s_1) & \text{for} \quad t\in(s_1,b] \end{cases}$$

for all $k\in\mathbb{N}$.

We show that $\gamma_n \in BV_\varphi(I)$, $n\in\mathbb{N}$. By the definition of γ_{2k}, $k\in\mathbb{N}$, the triangle inequality, (12) and (15), we have

$$\left|\gamma_{2k}(s_i)-\gamma_{2k}(s_0)\right| < \frac{2}{i^2}$$

and

$$\left|\gamma_{2k}(s_i)-\gamma_{2k}(s_j)\right| \le \left|\gamma_{2k}(s_i)-\gamma_{2k}(s_0)\right| + \left|\gamma_{2k}(s_j)-\gamma_{2k}(s_0)\right| \le \frac{1}{i^2}+\frac{1}{j^2} < \frac{2}{i^2},$$

for all $i,j\in\{1,\dots,2k\}$, $i<j$. Hence, applying the monotonicity of φ, inequality (2) with $t=\frac{1}{i^2}$, $i\in\mathbb{N}$, we get

$$\varphi\left(\left|\gamma_{2k}(s_i)-\gamma_{2k}(s_0)\right|\right) \le \varphi\left(\frac{2}{i^2}\right) \le \frac{1}{i^2}\varphi(2),\ i\in\{1,\dots,2k\},$$

and

$$\varphi\left(\left|\gamma_{2k}(s_i)-\gamma_{2k}(s_j)\right|\right) \le \varphi\left(\frac{2}{i^2}\right) \le \frac{1}{i^2}\varphi(2),\ i,j\in\{1,\dots,2k\}, \quad i<j.$$

Therefore, taking into account Lemma 2 with $n = 2k + 1$ and

$$\begin{gathered}(x_0, y_0) = \big(s_0, f(s_0)\big),\\ (x_1, y_1) = \big(s_{2k}, f(s_{2k})\big),\\ (x_2, y_2) = \big(s_{2k-1}, g(s_{2k-1})\big),\\ \dots\\ (x_{n-1}, y_{n-1}) = \big(s_1, g(s_1)\big),\ (x_n, y_n) = \big(b, g(s_1)\big),\end{gathered}$$

we get, for all $k \in \mathbb{N}$,

$$\begin{aligned}V_\varphi(\gamma_{2k}, I) =& \max\Big\{\varphi\big(\big|\gamma_{2k}(s_0) - \gamma_{2k}(s_{j_l})\big|\big) + \varphi\big(\big|\gamma_{2k}(s_{j_l}) - \gamma_{2k}(s_{j_{l-1}})\big|\big)\\ &+ \ldots + \varphi\big(\big|\gamma_{2k}(s_{j_1}) - \gamma_{2k}(s_1)\big|\big) : l \in \{1, \ldots, 2k\}\Big\}\\ \leq& \frac{1}{j_l^2}\ \varphi(2) + \frac{1}{j_{l-1}^2}\ \varphi(2) + \ldots + \frac{1}{j_i^2}\ \varphi(2) + \varphi(2),\end{aligned}$$

and, finally,

$$V_\varphi(\gamma_{2k}, I) \leq \varphi(2) \sum_{i=1}^{2k} \frac{1}{i^2}. \tag{16}$$

Similar reasoning shows that

$$V_\varphi(\gamma_{2k-1}, I) \leq \varphi(2) \sum_{i=1}^{2k-1} \frac{1}{i^2}, \quad k \in \mathbb{N},$$

which together with (16) implies that $\gamma_n \in BV_\varphi(I)$ and

$$V_\varphi(\gamma_n, I) \leq \varphi(2) \sum_{i=1}^{n} \frac{1}{i^2}, \quad k \in \mathbb{N}. \tag{17}$$

Let us note that, by the definition of γ_n,

$$\gamma_{2k-1}(s_0) = \gamma_{2k}(s_0) = f(s_0) = g(s_0), \quad k \in \mathbb{N}, \tag{18}$$

and, for all $k, i \in \mathbb{N}$,

$$\begin{gathered}\gamma_{2k}(s_{2k}) = f(s_{2k}) = \gamma_{2k+i}(s_{2k}),\\ \gamma_{2k-1}(s_{2k-1}) = g(s_{2k-1}) = \gamma_{2k-1+i}(s_{k-1}).\end{gathered} \tag{19}$$

Moreover, let us observe that for every $t \in I \setminus \{s_n : n \in \mathbb{N}\}$ there exists $n_0 \in \mathbb{N}$ such that

$$\gamma_n(t) = \gamma_{n_0}(t), \quad n \geq n_0,\ n \in \mathbb{N}. \tag{20}$$

Put

$$\gamma(t) := \lim_{n\to\infty} \gamma_n(t), \quad t \in I.$$

By (18), (19) and (20) the function γ is well defined and

$$\big|\gamma(t)-\gamma(s_{2k})\big| \leq \big|g(s_{2k+1})-f(s_{2k})\big| \leq \big|g(s_{2k+1})-g(s_0)\big|+\big|f(s_{2k})-f(s_0)\big| \tag{21}$$

for all $t \in [s_{2k+1}, s_{2k})$, and

$$\big|\gamma(t)-\gamma(s_{2k})\big| \leq \big|g(s_{2k-1})-f(s_{2k})\big| \leq \big|g(s_{2k-1})-g(s_0)\big|+\big|f(s_{2k})-f(s_0)\big| \tag{22}$$

for all $t \in [s_{2k}, s_{2k-1})$.

To show that γ is continuous at s_0, fix $\varepsilon > 0$. By the continuity of f and g at s_0, there exists $n_0 \in \mathbb{N}$ such that

$$\big|g(s_n) - g(s_0)\big| < \frac{\varepsilon}{3}, \quad \big|f(s_n) - f(s_0)\big| < \frac{\varepsilon}{3}; \quad n \in \mathbb{N},\ n \geq n_0. \tag{23}$$

Take an arbitrary $t \in (x_0, s_{n_0})$. There exists $k \in \mathbb{N}$ such that $2k - 1 > n_0$ and either $t \in [s_{2k+1}, s_{2k})$ or $t \in [s_{2k}, s_{2k-1})$.

Since, by the triangle inequality and (18),

$$\begin{aligned}\big|\gamma(t) - \gamma(s_0)\big| &\leq \big|\gamma(t) - \gamma(s_{2k})\big| + \big|\gamma(s_{2k}) - \gamma(s_0)\big| \\ &= \big|\gamma(t) - \gamma(s_{2k})\big| + \big|f(s_{2k}) - f(s_0)\big|,\end{aligned}$$

therefore, by (21) and (23),

$$\big|\gamma(t) - \gamma(s_0)\big| \leq \big|g(s_{2k+1}) - g(s_0)\big| + 2\big|f(s_{2k}) - f(s_0)\big| < \varepsilon$$

in the case when $t \in [s_{2k+1}, s_{2k})$, and, by (22) and (23),

$$\big|\gamma(t) - \gamma(s_0)\big| \leq \big|g(s_{2k-1}) - g(s_0)\big| + 2\big|f(s_{2k}) - f(s_0)\big| < \varepsilon$$

in the case when $t \in [s_{2k}, s_{2k-1})$. As the continuity of γ at the remaining points is obvious, γ is continuous.

Moreover, by the lower semicontinuity of V_φ and (17),

$$V_\varphi(\gamma, I) \leq \liminf_{n\to\infty} \varphi(2) \sum_{i=1}^{n} \frac{1}{i^2},$$

and the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$ implies that $\gamma \in BV_\varphi(I)$.

Thus there exists a function $\gamma \in CW_\varphi(I)$ (with $\lambda = 1$) and a sequence $(s_k)_{k\in\mathbb{N}}$ such that

$$\gamma(s_{2k-1}) = g(s_{2k-1}), \quad \gamma(s_{2k}) = f(s_{2k}), \quad s_k \in I,\ k \in \mathbb{N}.$$

According to the first part of the proof, we have

$$K(\gamma)(s_{2k-1}) = K(g)(s_{2k-1}), \quad K(\gamma)(s_{2k}) = K(f)(s_{2k}),\ k \in \mathbb{N}.$$

Hence, by the continuity of $K(\gamma)$, $K(f)$ and $K(g)$ at s_0, letting $k \to \infty$, we get (13).

When x_0 is the right endpoint of I, the argument is similar.

Now, we are in a position to construct the function h. For an arbitrary $y_0 \in \mathbb{R}$ let us define a function $P_{y_0} \colon I \to \mathbb{R}$ by

$$P_{y_0}(t) := y_0, \quad t \in I. \tag{24}$$

Of course, P_{y_0}, as a constant function, belongs to $CW_\varphi(I)$. To define the function $h \colon I \times \mathbb{R} \to \mathbb{R}$, fix $s_0 \in I$, $y_0 \in \mathbb{R}$ arbitrarily and put

$$h(s_0, y_0) := K(P_{y_0})(s_0). \tag{25}$$

Since, by (24), for all functions f,

$$f(s_0) = P_{f(s_0)}(s_0),$$

according to what has already been proved, we have

$$K(f)(s_0) = K(P_{f(s_0)})(s_0) = h\big(s_0, f(s_0)\big).$$

As the uniqueness of the function h is obvious, the proof is completed. $\square$

As a by-product of the proof of the above theorem we obtain the following

Remark 4. By the construction of γ it follows that if $f(a) = g(a)$ (or $f(b) = g(b)$) and f, g are continuous (not necessarily of bounded φ-variation), then there exist a sequence $(x_k)_{k\in\mathbb{N}}$ convergent to a (or b, respectively) and $\gamma \in W_\varphi(I)$ such that

$$\gamma(x_{2k}) = f(x_{2k}) \quad \text{and} \quad \gamma(x_{2k-1}) = g(x_{2k-1}),$$

for all $k \in \mathbb{N}$.

Definition 5. Let $X \subset \mathbb{R}$ and a function $h\colon X \times \mathbb{R} \to \mathbb{R}$ be fixed. The mapping $H\colon \mathbb{R}^X \to \mathbb{R}^X$ given by

$$H(f)(x) := h\big(x, f(x)\big), \quad \varphi \in \mathbb{R}^X,\ (x \in X),$$

is said to be a *composition (Nemytskij or superposition) operator*. The function h is referred to as the *generator of the operator* H. (Here $\mathbb{R}^X$ denotes the set of all functions $f\colon X \to \mathbb{R}$).

As an immediate consequence of Theorem 3 we obtain

Corollary 6. *Let $I = [a,b]$ $(a, b \in \mathbb{R},\ a < b)$ be a closed interval. If a local operator K maps $CW_\varphi(I)$ into $C(I)$, then it is a Nemytskij (composition) operator, i.e., there exists a unique function $h\colon I \times \mathbb{R} \to \mathbb{R}$ such that*

$$K(f)(t) = h\big(t, f(t)\big), \quad t \in I,$$

for all $f \in CW_\varphi(I)$.

Now, let us consider locally defined operators acting between two spaces of continuous functions of generalized bounded variation.

Let $\varphi, \psi \in \mathcal{F}$. Note that if a local operator K maps $CW_\varphi(I)$ into $CW_\psi(I)$, then, obviously, K maps $CW_\varphi(I)$ into $C(I)$. Therefore, by Theorem 3, we have

Theorem 7. *Let $\varphi, \psi \in \mathcal{F}$ and $I = [a,b]$ $(a, b \in \mathbb{R},\ a < b)$ be a closed interval. If a locally defined operator K maps $CW_\varphi(I)$ into $CW_\psi(I)$, then there exists a unique function $h\colon I \times \mathbb{R} \to \mathbb{R}$ such that, for all $f \in CW_\varphi(I)$,*

$$K(f)(s) = h\big(s, f(s)\big), \quad s \in I.$$

Corollary 8. *Let $\varphi, \psi \in \mathcal{F}$ and $I = [a,b]$ be a closed interval. If a locally defined operator K maps $CW_\varphi(I)$ into $CW_\psi(I)$, then it is a Nemytskij operator.*

4 A characterization of some generators of the Nemytskij operator.

Definition 9. Let $\mathcal{G}_1(X,Y) \subset Y^X$ with a norm $\|\cdot\|_1$ and $\mathcal{G}_2(X,Y) \subset Y^X$ with a norm $\|\cdot\|_2$ be two Banach spaces. We say that an operator $H\colon \mathcal{G}_1(X,Y) \to \mathcal{G}_2(X,Y)$ satisfies the (*global*) *Lipschitz condition* if there is a constant $0 < \mu < 1$ such that

$$\|H(f_1) - H(f_2)\|_2 \le \mu \|f_1 - f_2\|_1, \quad f_1, f_2 \in \mathcal{G}_1(X,Y). \tag{26}$$

Let us quote the following result.

Theorem 10 ([4, Theorem 3.1, Remark 3.6]). *Let $(X, |\cdot|_X), (Y, |\cdot|_Y)$ be real normed spaces and let C be a closed convex subset of X. Suppose that $\varphi, \psi \in \mathcal{F}$ and $h\colon I \times C \to Y$. If a composition operator $H\colon C^I \to Y^I$ generated by h maps $W_\varphi(I, C)$ into $W_\psi(I, Y)$ and is uniformly continuous, then the left regularization of h, i.e., the function $h^-\colon I^- \times X \to Y$, defined by*

$$h^-(t, y) := \lim_{s \uparrow t} h(s, y), \quad t \in I^-; \quad y \in C,$$

exists and

$$h^-(t, y) = A(t)y + B(t), \quad t \in I^-; \quad y \in C,$$

for some $A\colon I^- \to \mathcal{L}(X, Y)^I$ and $B \in BV_\psi(I^-, Y)$. Moreover, the functions A and B are left continuous in I^-. (Here $\mathcal{L}(X, Y)$ stands for the space of all linear mappings $A\colon X \to Y$ and $I^- := I \setminus \{\inf I\}$).

Thus, under the additional assumption that the locally defined operator is uniformly continuous, we get a complete characterization of its generating function h. Namely, we get the following

Theorem 11. *Let $\varphi, \psi \in \mathcal{F}$ and $I = [a, b]$ $(a, b \in \mathbb{R},\ a < b)$ be a closed interval. If a local operator $K\colon CW_\varphi(I) \to CW_\psi(I)$ is uniformly continuous, then there exist $a, b \in CW_\psi(I)$ such that*

$$K(f)(s) = a(s)f(s) + b(s), \quad f \in CW_\varphi(I), \quad (s \in I). \tag{27}$$

Moreover, if $\psi \ll \varphi$ and an operator $K\colon \mathbb{R}^I \to \mathbb{R}^I$ is defined by (27) for some functions $a, b \in CW_\psi(I)$, then the operator K maps $CW_\varphi(I)$ into $CW_\psi(I)$, is locally defined and satisfies the global Lipschitz condition (so it is uniformly continuous).

PROOF. By Theorem 7 there exists a unique function $h\colon I \times \mathbb{R} \to \mathbb{R}$ such that (11) holds for all $f \in CW_\varphi(I)$. Fix $(x_0, y_0) \in I \times \mathbb{R}$, take an arbitrary sequence (x_n), for $n \in \mathbb{N}$ and $x_n \in I$, convergent to x_0 and the function $P_{y_0}\colon I \to \mathbb{R}$ defined by (24). Since, by (25),

$$\begin{aligned}\big|h(x_n, y_0) - h(x_0, y_0)\big| &= \big|h(x_n, P_{y_0}(x_n)) - h(x_0, P_{y_0}(x_0)\big| \\ &= \big|K(P_{y_0})(x_n) - K(P_{y_0})(x_0)\big|,\end{aligned}$$

applying the continuity of $K(P_{y_0})$ at x_0, we get the continuity of h with respect to the first variable. Thus, by Theorem 10,

$$h(t, y) = a(t)y + b(t), \quad t \in I,\ y \in \mathbb{R},$$

for some $a, b\colon I \to \mathbb{R}$. Since $h(\cdot, y_0) = K\big(P_{y_0}(\cdot)\big) \in CW_\varphi(I)$ for all $y \in \mathbb{R}$ (where $P_{y_0}\colon I \to \mathbb{R}$ is defined by (24)) and $b(t) = h(t, 0)$; $a(t) = h(t, 1) - b(t)$, the functions $a, b \in CW_\psi(I)$ which together with (11) gives the required claim.

Since every operator defined by (27) is local, we get the inverse statement by [2, Theorem 7(b)], which completes the proof. $\square$

Remark 12. Every Lipschitzian local operator acting from $CW_\varphi(I)$ into $CW_\psi(I)$ is an affine mapping.

5 An application.

The composition operator plays an important role in the theory of functional equations of the iterative type

$$f(x) = h\big(x, f(\alpha(x))\big), \quad x \in I, \tag{28}$$

where $\alpha\colon I \to I$ and $h\colon I \times \mathbb{R} \to \mathbb{R}$ are given and f is unknown. Note that, taking into account the definition of composition operator H, equation (28) can be written as

$$f = (H \circ F)(f),$$

where the operator $F(f) := f \circ \alpha$, under some fairly general assumptions on α, is linear and $\|F\| < \infty$. Since the existence and uniqueness of solutions of equation (2) depends on the class of the unknown function, the fixed point theory is frequently used in this field (cf. for instance [5]).

An application of the classical Banach principle to equation (28) requires $H \circ F$ to be a contraction mapping. This usually implies that H is Lipschitzian (with a constant μ such that $\mu\|F\| < 1$). In particular, to apply this method to find a solution f in the class $CW_\varphi(I)$ it is necessary to assume that H satisfies condition (26) and, according to our Theorem 11, that the generator h of the operator H is of form (27). Consequently, the study of solutions $f \in CW_\varphi(I)$ of equation (28) with the aid of the Banach metod is possible only if f is linear, i.e., if it is of the form

$$f(x) = a(x) f\big(\alpha(x)\big) + b(x), \quad x \in I.$$

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