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ABEL DERIVATIVE AND ABEL CONTINUITY

Abstract

Abel derivative of order k is introduced and the first order Abel derivative is studied. Using Abel derivative some monotonicity results are obtained.

1 Introduction.

Abel derivative was in a dormant state in the work of Zygmund [8] and Verblunsky [7]. Following them Abel derivative and Abel continuity were introduced by S. J. Taylor in [5]. In [5] the author introduced Abel continuity and second order Abel derivative of a 2π-periodic Lebesgue integrable function to define the Abel-Perron integral which is useful for Abel summable trigonometric series. Since then Abel derivative and Abel continuity remain unattended though some work is done in [2, 3]. We have introduced for $k \geq 1$, the kth order Abel derivative and studied the first order Abel derivative. It helps to determine not only the monotonicity of a function f but also the Abel summability of the Fourier series and the differentiated series of f. It is shown that the Abel derivative is symmetric in nature (see [6]) and that the first order Abel derivative is more general than the first order symmetric derivative (Theorem 4). Some monotonicity theorems are obtained. It may be noted that Estrada and Vindas [1] have recently obtained monotonicity results which are related to our Theorems 10 and 11.

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2 Definitions and Notations

Let f be a 2π-periodic Lebesgue integrable function and let

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

be the Fourier coefficients of f. So the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)r^n \tag{1}$$

converges uniformly and absolutely for $0 < r < 1$. Let

$$f(r,x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)r^n. \tag{2}$$

Then by a standard calculation

$$f(r,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)P(r,t)dt \tag{3}$$

where $P(r,t)$ is the Abel Poisson kernel defined by

$$P(r,t) = \tfrac{1}{2} + \sum_{n=1}^{\infty} r^n \cos nt.$$

It is well known that

$$P(r,t) = \frac{1}{2}\left[\frac{1-r^2}{1-2r\cos t + r^2}\right] = \frac{1}{2}\left[\frac{1-r^2}{(1-r)^2 + 4r\sin^2 \frac{t}{2}}\right]. \tag{4}$$

It can be proved that for $0 < r < 1$

$$P(r,t) \geq 0, \quad P(r,-t) = P(r,t), \tag{5}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} P(r,t)dt = 1, \tag{6}$$

$$P(r,t) \leq C\frac{1-r}{t^2} \text{ if } \frac{1}{2} \leq r < 1 \text{ and } 0 <| t |\leq \pi, C \text{ being a constant.} \tag{7}$$

For proofs of (4) - (7) see [8; p.96].

Let k be a fixed positive integer. Differentiating (2) term by term k times with respect to x we have

$$\frac{\partial^k f(r,x)}{\partial x^k} = (-1)^{\frac{k}{2}} \sum_{n=1}^{\infty} (a_n n^k \cos nx + b_n n^k \sin nx) r^n, \text{if k is even}$$

$$= (-1)^{\frac{k+1}{2}} \sum_{n=1}^{\infty} (a_n n^k \sin nx - b_n n^k \cos nx) r^n, \text{ if } k \text{ is odd.}$$

The upper and lower Abel derivates of f at x of order k are defined by

$$\overline{AD}_k f(x) = \limsup_{r \to 1-} \frac{\partial^k f(r,x)}{\partial x^k}$$

$$\underline{AD}_k f(x) = \liminf_{r \to 1-} \frac{\partial^k f(r,x)}{\partial x^k}$$

respectively. Thus $\overline{AD}_k f(x)$ and $\underline{AD}_k f(x)$ are the upper and lower Abel sums of the $k-$times differentiated series of the Fourier series of f at x. If $\overline{AD}_k f(x) = \underline{AD}_k f(x)$ then this common value is called the Abel derivative of f at x of order k and will be denoted by $AD_k f(x)$.

The kth symmetric de la Vallée Poussin (d.l.V.P.) derivative and upper and lower (d.l.V.P.) derivates at a point x are defined in [2] and here we shall denote them by $f_{(k)}^{(s)}$, $\overline{f}_{(k)}^{(s)}$ and $\underline{f}_{(k)}^{(s)}$ respectively. Throughout the paper $\Re$ and μ denote the set of real numbers and the Lebesgue measure respectively.

3 Main Results.

Theorem 1. *If* $\overline{AD}_k f(x_0)$ *and* $\underline{AD}_k f(x_0)$ *are finite and* $k \geq 2$ *then* $AD_{k-2} f(x_0)$ *exists and is finite.*

PROOF. Let k be even and let

$u_0 = \frac{1}{2} a_0$

$u_n = (-1)^{\frac{k}{2}} (a_n n^k \cos nx_0 + b_n n^k \sin nx_0), n \geq 1.$

So the power series $\sum_{n=0}^{\infty} u_n r^n$ has radius of convergence at least 1. Let

$$g(r) = \sum_{n=0}^{\infty} u_n r^n, \quad G(r) = \sum_{n=1}^{\infty} \frac{u_n}{n} r^n.$$

So,

$$\int_0^r \frac{g(t)-u_0}{t}dt = \int_0^r \sum_{n=1}^{\infty} u_n t^{n-1} dt = \sum_{n=1}^{\infty} \frac{u_n}{n} r^n.$$

Since $\overline{AD}_k f(x_0)$ and $\underline{AD}_k f(x_0)$ are finite, $g(r)$ is bounded as $r \to 1-$, and so $\frac{g(t)-u_0}{t}$ is bounded as $t \to 1-$. Hence for $0 < r_1 < r_2 < 1$

$$| G(r_2) - G(r_1) | = \left| \int_{r_1}^{r_2} \frac{g(t)-u_0}{t} dt \right| \to 0 \text{ as } r_1, r_2 \to 1-.$$

So, $\lim_{r\to 1-} G(r)$ exists finitely which shows that

$$\lim_{r\to 1-} \sum_{n=1}^{\infty} \frac{u_n}{n} r^n \text{ is finite.}$$

Repeating this argument, we conclude that

$$\lim_{r\to 1-} \sum_{n=1}^{\infty} \frac{u_n}{n^2} r^n \text{ is finite.}$$

Since

$$AD_{k-2} f(x_0) = \lim_{r\to 1-} \left. \frac{\partial^{k-2} f(r,x)}{\partial x^{k-2}} \right|_{x=x_0} = \lim_{r\to 1-} (-1) \sum_{n=1}^{\infty} \frac{u_n}{n^2} r^n$$

the proof is complete in this case. When k is odd the proof is similar. □

While the finiteness of $\overline{AD}_k f(x)$ and $\underline{AD}_k f(x)$ imply the existence of $AD_{k-2} f(x)$ finitely, nothing can be said about the existence of $AD_{k-1} f(x)$. For let f be defined by

$$\begin{aligned} f(x) &= \tfrac{1}{2}(\pi - x)\ , \text{ for } 0 < x < 2\pi \\ &= 0\ , \text{ for } x = 0, 2\pi \end{aligned}$$

and f is 2π-periodic. Then f being odd, $a_n = 0$ while $b_n = \frac{1}{n}$ for all n. Therefore (2) becomes

$$f(r,x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} r^n$$

and so

$$\frac{\partial f(r,x)}{\partial x} = \sum_{n=1}^{\infty} \cos nx\ r^n, \quad \frac{\partial^2 f(r,x)}{\partial x^2} = -\sum_{n=1}^{\infty} n \sin nx\ r^n$$

and hence

$$AD_2 f(0) = \lim_{r\to 1-} \frac{\partial^2 f(r,0)}{\partial x^2} = 0.$$

But

$$AD_1 f(0) = \lim_{r\to 1-} \frac{\partial f(r,0)}{\partial x} = \lim_{r\to 1-} \sum_{n=1}^{\infty} r^n = \infty.$$

So, Abel derivative behaves as a symmetric derivative. From Theorem 1 it follows that if $\overline{AD}_k f(x)$ and $\underline{AD}_k f(x)$ are finite then $AD_0 f(x)$ or $AD_1 f(x)$ exist finitely according as k is even or odd where

$$AD_0 f(x) = \lim_{r\to 1-} f(r,x).$$

Definition. Let f be 2π-periodic Lebesgue integrable function. If for some x

$$\lim_{r\to 1-} f(r,x) = f(x)$$

then f is said to be Abel continuous at x (see [5]).

So, f is Abel continuous at x if and only if the Fourier series of f is Abel summable at x to $f(x)$.

Theorem 2. *If f is 2π-periodic and Lebesgue integrable then for a point x*

$$\liminf_{h\to 0} \frac{f(x+h)+f(x-h)}{2} \le \liminf_{r\to 1-} f(r,x) \le \limsup_{r\to 1-} f(r,x) \le \limsup_{h\to 0} \frac{f(x+h)+f(x-h)}{2}.$$

PROOF. We prove the right hand side. We may suppose that $\limsup\limits_{h\to 0} \dfrac{f(x+h)+f(x-h)}{2} < \infty$. Choose M such that

$$\limsup_{h\to 0} \frac{f(x+h)+f(x-h)}{2} < M < \infty.$$

Then there is $\delta, 0<\delta<\pi$, such that

$$f(x+h)+f(x-h) < 2M \text{ for } 0<h<\delta.$$

So, from (5) and (6)

$$\frac{1}{\pi}\int_0^\delta \big[f(x+t)+f(x-t)\big]P(r,t)dt \le \frac{2M}{\pi}\int_0^\delta P(r,t)dt < \frac{M}{\pi}\int_{-\pi}^{\pi} P(r,t)dt = M. \tag{8}$$

Also from (7) taking $\frac{1}{2}\le r<1$

$$\left|\frac{1}{\pi}\int_\delta^\pi \big[f(x+t)+f(x-t)\big]P(r,t)dt\right| \le \frac{1}{\pi}\int_\delta^\pi \mid f(x+t)+f(x-t) \mid P(r,t)dt \le C\frac{1-r}{\pi}\int_\delta^\pi \frac{|f(x+t)+f(x-t)|}{t^2}dt$$

and hence

$$\lim_{r\to 1-} \frac{1}{\pi}\int_\delta^\pi [f(x+t)+f(x-t)]P(r,t)dt = 0. \tag{9}$$

From (3) and (5)

$$f(r,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)P(r,t)dt = \frac{1}{\pi}\int_0^\pi [f(x+t)+f(x-t)]P(r,t)dt$$

$$= \frac{1}{\pi}\left(\int_0^\delta + \int_\delta^\pi\right)[f(x+t)+f(x-t)]P(r,t)dt.$$

From this and from (8) and (9)

$$\limsup_{r\to 1-} f(r,x) \le M.$$

Since M is arbitrary,

$$\limsup_{r\to 1-} f(r,x) \le \limsup_{h\to 0} \frac{f(x+h)+f(x-h)}{2}.$$

The left hand inequality is similar. □

Corollary 3. *If f is continuous at x, then f is Abel continuous at x.*

The converse is not true. For let

$f(x) = 1$, for $0 < x < \pi$,

$f(x) = -1$, for $-\pi < x < 0$,

and $f(x) = 0$, for $x = 0, \pi$; $\quad f(x+2\pi) = f(x)$.

Then f is not continuous at $x = 0$ but by Theorem 2

$$\lim_{r\to 1-} f(r,0) = 0 = f(0)$$

and therefore f is Abel continuous at $x = 0$.

Remark. Theorem 2 generalizes a well-known result [8; p.97, Theorem 6.11].

Theorem 4. *If f is 2π-periodic and Lebesgue integrable then for all x*

$$\underline{f}^{(s)}_{(1)}(x) \le \underline{AD}_1 f(x) \le \overline{AD}_1 f(x) \le \overline{f}^{(s)}_{(1)}(x)$$

where $\underline{f}^{(s)}_{(1)}(x)$ and $\overline{f}^{(s)}_{(1)}(x)$ denote the first order lower and upper symmetric derivates of f at x, respectively.

Proof. We prove the right hand inequality with $x = x_0$; the proof for left hand inequality being similar.

We may suppose that $\overline{f}^{(s)}_{(1)}(x_0) < \infty$. Let

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

be the Fourier series of f. For convenience we shall write

$$1 - 2r\cos t + r^2 = \Delta(r,t)$$

and when there is no confusion $\Delta(r,t)$ will be written Δ or $\Delta(t)$. Then we have from (4)

$$P(r,t) = \tfrac{1}{2}\left(\tfrac{1-r^2}{\Delta}\right).$$

Hence writing $P'(r,x) = \frac{\partial}{\partial x} P(r,x)$ we have

$$P'(r,x) = \frac{-r(1-r^2)\sin x}{\Delta^2(x)}. \tag{10}$$

We shall use this notation for differentiation with respect to the second variable of P. Then as in (2)

$$f(r,x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)r^n.$$

So, by (3)

$$f(r,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)P(r,t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)P(r,t-x)dt. \tag{11}$$

Hence from (11) and (10)

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial x}f(r,x)\bigg|_{x=x_0} &= -\frac{1}{\pi r}\int_{-\pi}^{\pi} f(t)P'(r,t-x_0)dt \\
&= -\frac{1}{\pi r}\int_{-\pi}^{\pi} f(x_0+t)P'(r,t)dt \\
&= \frac{1}{\pi r}\int_{-\pi}^{\pi} f(x_0-t)P'(r,t)dt \\
&= -\frac{1}{\pi r}\int_{-\pi}^{\pi} \frac{f(x_0+t)-f(x_0-t)}{2}P'(r,t)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)K(r,t)dt
\end{aligned} \tag{12}$$

where

$$g(t) = g_{x_0}(t) = \frac{f(x_0+t)-f(x_0-t)}{2\sin t}$$

and

$$K(r,t) = \frac{-P'(r,t)\sin t}{r} = \frac{(1-r^2)\sin^2 t}{\Delta^2(t)}.$$

Since $g(t) = g(-t)$ and $K(r,t) = K(r,-t)$ we have from (12)

$$\frac{1}{r}\frac{\partial}{\partial x}f(r,x)\bigg|_{x=x_0} = \frac{2}{\pi}\int_0^{\pi} g(t)K(r,t)dt. \tag{13}$$

The relation (13) holds for all $2\pi-$ periodic Lebesgue integrable functions and for all choices of x_0 and hence we can put $f(x) = \sin x$ and $x_0 = 0$ and so in this case $g(t) = 1$ for all t. Hence for this substitution in (13) we get

$$\frac{2}{\pi}\int_0^{\pi} K(r,t)dt = \frac{1}{r}\frac{\partial}{\partial x}(r\sin x)\bigg|_{x=0} = 1. \tag{14}$$

Let $\epsilon > 0$ be arbitrary. Since $\limsup_{t\to 0} g(t) = \overline{f}^{(s)}_{(1)}(x_0)$, there is $\delta, 0 < \delta < \pi$, such that

$$g(t) < \overline{f}^{(s)}_{(1)}(x_0) + \epsilon \text{ for } t \in (0,\delta). \tag{15}$$

Since $K(r,t) > 0$, by (14) and (15)

$$\begin{aligned}\frac{2}{\pi}\int_0^{\delta} g(t)K(r,t)dt &\le \frac{2}{\pi}\left(\overline{f}^{(s)}_{(1)}(x_0) + \epsilon\right)\int_0^{\delta} K(r,t)dt \\ &\le \frac{2}{\pi}\left(\overline{f}^{(s)}_{(1)}(x_0) + \epsilon\right)\int_0^{\pi} K(r,t)dt \\ &= \overline{f}^{(s)}_{(1)}(x_0) + \epsilon.\end{aligned} \tag{16}$$

Also

$$\frac{2}{\pi}\left|\int_{\delta}^{\pi} g(t)K(r,t)dt\right| \le \frac{2}{\pi}\sup_{\delta\le t\le\pi} K(r,t)\int_{\delta}^{\pi}|g(t)|dt. \tag{17}$$

For $\delta \le t \le \pi$, $(\Delta(t))^{-1} \le (1 - 2r\cos\delta + r^2)^{-1}$ and so
$|P'(r,t)| = \left|\frac{-r(1-r^2)\sin t}{\Delta^2(t)}\right| \le \frac{r(1-r^2)}{(1-2r\cos\delta+r^2)^2}$.
Hence

$$\sup_{\delta\le t\le\pi}|K(r,t)| = \sup_{\delta\le t\le\pi}\left|\frac{P'(r,t)\sin t}{r}\right| \le \frac{1-r^2}{(1-2r\cos\delta+r^2)^2}. \tag{18}$$

Since δ is independent of r, from (18)

$$\lim_{r\to 1-}\sup_{\delta\le t\le\pi}|K(r,t)| = 0. \tag{19}$$

From (17) and (19)

$$\lim_{r\to 1-}\frac{2}{\pi}\int_{\delta}^{\pi} g(t)K(r,t)dt = 0. \tag{20}$$

From (13), (16) and (20)

$$\limsup_{r\to 1-} \frac{1}{r}\frac{\partial}{\partial x} f(r,x)\Big|_{x=x_0} \le \overline{f}^{(s)}_{(1)}(x_0) + \epsilon.$$

Since ϵ is arbitrary,

$$\limsup_{r\to 1-} \frac{1}{r}\frac{\partial}{\partial x} f(r,x)\Big|_{x=x_0} \le \overline{f}^{(s)}_{(1)}(x_0)$$

which implies $\overline{AD}_1 f(x_0) \le \overline{f}^{(s)}_{(1)}(x_0)$ completing the proof. □

The following corollary gives Fatou's Theorem (see [8; Vol.I, p.99]) in our context.

Corollary 5. *Let f be 2π-periodic Lebesgue integrable function. If $f^{(s)}_{(1)}(x_0)$ exists then $AD_1 f(x_0)$ exists and equals $f^{(s)}_{(1)}(x_0)$.*

But the converse is not true. For, if

$$f(x) = \tfrac{1}{n}, \text{ if } x = \tfrac{1}{n}, n = 1, 2, 3, ...$$
$$= 0, \text{ otherwise}$$

then $f^{(s)}_{(1)}(0)$ does not exist but $AD_1 f(0)$ exists and equals to 0.

Theorem 4 has not been extended to higher order for Abel and d.l.V.P. derivates. Rajchman and Zygmund proved that $\overline{AD}_2 f(x_0) \ge \underline{f}^{(s)}_{(2)}(x_0)$ and $\underline{AD}_2 f(x_0) \le \overline{f}^{(s)}_{(2)}(x_0)$ (see [8, p.353] and [7, p.445]). However it is known that if the d.l.V.P. derivative of order $k, f^{(s)}_{(k)}(x)$ exists finitely then the Abel derivative $AD_k f(x)$ also exists finitely and equals $f^{(s)}_{(k)}(x)$ [2; Corollaries of Theorems 1 and 2].

Theorem 6. *Let f be 2π-periodic and Lebesgue integrable. Then for a point x*

$$\liminf_{h\to 0+} \frac{f(x+h) - f(x-h)}{2} \le \frac{\pi}{2} \liminf_{r\to 1-} (1-r)\frac{\partial f(r,x)}{\partial x}$$
$$\le \frac{\pi}{2} \limsup_{r\to 1-} (1-r)\frac{\partial f(r,x)}{\partial x} \le \limsup_{h\to 0+} \frac{f(x+h) - f(x-h)}{2}.$$

PROOF. From (10) we have $P'(r,-t) = -P'(r,t)$ and so as in (12)

$$\begin{aligned}
\frac{\partial f(r,x)}{\partial x} &= -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt\\
&= -\frac{1}{\pi}\int_{0}^{\pi}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt - \frac{1}{\pi}\int_{-\pi}^{0}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt\\
&= -\frac{1}{\pi}\int_{0}^{\pi}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt - \frac{1}{\pi}\int_{0}^{\pi}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt\\
&= -\frac{2}{\pi}\int_{0}^{\pi}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt.
\end{aligned} \tag{21}$$

We prove the right hand inequality. We may suppose that $\limsup_{h\to 0+}\frac{f(x+h)-f(x-h)}{2} < \infty$. Choose $\limsup_{h\to 0+}\frac{f(x+h)-f(x-h)}{2} < M < \infty$, where M is arbitrary. Then there is δ, $0<\delta<\pi$, such that

$$\frac{f(x+h)-f(x-h)}{2} < M \text{ for } 0<h<\delta.$$

Then since by (10) $P'(r,t) < 0$ for $t \in (0,\pi)$, we have using (4)

$$\begin{aligned}
&-\frac{2}{\pi}\int_{0}^{\delta}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt\\
&\le -\frac{2M}{\pi}\int_{0}^{\delta}P'(r,t)dt \le -\frac{2M}{\pi}\int_{0}^{\pi}P'(r,t)dt\\
&= -\frac{2M}{\pi}(P(r,\pi)-P(r,0)) = -\frac{M}{\pi}\left[\frac{1-r}{1+r}-\frac{1+r}{1-r}\right].
\end{aligned}$$

Hence

$$\limsup_{r\to 1-}\left[(1-r)\left(-\frac{2}{\pi}\right)\int_{0}^{\delta}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt\right] \le \frac{2M}{\pi}. \tag{22}$$

Also since for $\delta \le t \le \pi$, $\frac{1}{\Delta(t)} \le \frac{1}{1-2r\cos\delta+r^2}$ from (10) we have

$$|P'(r,t)| \le \frac{r(1-r^2)}{(1-2r\cos\delta+r^2)^2} \text{ for } \delta\le t\le\pi$$

and hence

$$\left|-\frac{2}{\pi}\int_{\delta}^{\pi}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt\right| \le \frac{2}{\pi}\frac{r(1-r^2)}{(1-2r\cos\delta+r^2)^2}\int_{\delta}^{\pi}\left|\frac{f(x+t)-f(x-t)}{2}\right|dt.$$

So,

$$\limsup_{r\to 1-}\left[(1-r)\left(-\frac{2}{\pi}\right)\int_{\delta}^{\pi}\frac{f(x+t)-f(x-t)}{2}P'(r,t)dt\right] = 0. \tag{23}$$

From (21), (22) and (23)

$$\limsup_{r\to 1-}(1-r)\frac{\partial f(r,x)}{\partial x} \leq \frac{2M}{\pi}.$$

Since M is arbitrary, the result follows. $\square$

Proposition 7. *Let f be 2π-periodic Lebesgue integrable on [0, 2π]. Then if f is Abel continuous at $x_0 \in (0, 2\pi)$ and*

$$F(x) = \int_0^x (f(t) - \frac{1}{2}a_0)dt$$

then $AD_1F(x_0)$ exists and $AD_1F(x_0) = f(x_0) - \frac{1}{2}a_0$, where $\frac{1}{2}a_0$ is the constant term in the Fourier expansion of f.

Proof. Let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{24}$$

be the Fourier series of f. Since F is 2π-periodic and absolutely continuous, the Fourier series of F converges to F everywhere. Let

$$F(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx). \tag{25}$$

Then it can be verified that

$$A_0 = 2\sum_{n=1}^{\infty}\frac{b_n}{n}, A_n = -\frac{b_n}{n}, B_n = \frac{a_n}{n}.$$

So, from (24) and (25)

$$\frac{\partial F(r,x)}{\partial x} = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)r^n = f(r,x) - \frac{1}{2}a_0. \tag{26}$$

Since f is Abel continuous at x_0, letting $r \to 1-$

$$AD_1F(x_0) = f(x_0) - \tfrac{1}{2}a_0.$$

$\square$

Proposition 8. *If f is 2π-periodic and Lebesgue integrable and if the first Abel derivative AD_1f exists (possibly infinite) then AD_1f is in Baire class 1.*

Proof. Let $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the Fourier series of f. Then as in (2)

$$f(r,x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)r^n.$$

So,

$$\frac{\partial f(r,x)}{\partial x} = -\sum_{n=1}^{\infty}(na_n \sin nx - nb_n \cos nx)r^n.$$

Since $AD_1 f(x)$ exists,

$$AD_1 f(x) = \lim_{r\to 1-} \frac{\partial f(r,x)}{\partial x} = \lim_{\nu\to\infty}\left[-\sum_{n=1}^{\infty}(na_n \sin nx - nb_n \cos nx)\Big(1-\frac{1}{\nu}\Big)^n\right].$$

Now

$$\left|(na_n \sin nx - nb_n \cos nx)\Big(1-\tfrac{1}{\nu}\Big)^n\right| \le 2Mn\Big(1-\tfrac{1}{\nu}\Big)^n, \text{ for all } n$$

where M is a positive real number such that

$$|a_n| \le M, |b_n| \le M, \text{ for all } n,$$

and since $\sum_{n=1}^{\infty} n\Big(1-\dfrac{1}{\nu}\Big)^n$ is convergent for fixed ν,

$\sum_{n=1}^{\infty}(na_n \sin nx - nb_n \cos nx)\Big(1-\dfrac{1}{\nu}\Big)^n$ converges uniformly for fixed ν, and so is continuous and hence $AD_1 f$ is in Baire class 1. □

Remark. Theorem 4 and Proposition 8 give a short proof of a theorem of Larson (see [6, p.263]), when f is Lebesgue integrable.

We need the following Lemma.

Lemma 9. *(Rajchman and Zygmund). Let F be 2π-periodic and Lebesgue integrable in $[0,2\pi]$ and let the Fourier series of F be Abel summable at x_0 to $F(x_0)$. Then*

$$\underline{AD}_2 F(x_0) \le \overline{F}^{(s)}_{(2)}(x_0), \qquad \overline{AD}_2 F(x_0) \ge \underline{F}^{(s)}_{(2)}(x_0)$$

where $\overline{F}^{(s)}_{(2)}(x_0)$ and $\underline{F}^{(s)}_{(2)}(x_0)$ are the second order upper and lower d.l.V.P. derivates of F at x_0.

For a proof see [7; p.445, Theorem II] and [8; p.353, Lemma 7.6].

The definition of approximate limit is in [8, p.323]. The definitions of approximate lower and upper limits which are used in (iii) of the following theorem are similar.

Theorem 10. *Let f be 2π-periodic and Lebesgue integrable on $[0,2\pi]$ such that*

(i) $\underline{AD}_1 f \ge 0$ except on a countable set $C \subset (0,2\pi)$;

(ii) $\liminf_{r\to 1-}(1-r)\dfrac{\partial f(r,x)}{\partial x} \ge 0$ for $x \in C$.

Then there exists a set $E \subset (0, 2\pi)$ of measure zero such that f is nondecreasing on $(0, 2\pi) \sim E$.

Moreover if

(iii) $\liminf_{x \to x_0} \text{ap } f(x) \leq f(x_0) \leq \limsup_{x \to x_0} \text{ap } f(x)$*, for every* $x_0 \in (0, 2\pi)$

then f is nondecreasing on $(0, 2\pi)$.

Proof. Let the Fourier series of f be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

and let

$$F(x) = \int_0^x (f(t) - \frac{1}{2}a_0)dt, \ x \in (0, 2\pi). \tag{27}$$

Then as in Proposition 7 the Fourier series of F is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

and as in (26)

$$\frac{\partial^2 F(r,x)}{\partial x^2} = \frac{\partial f(r,x)}{\partial x}. \tag{28}$$

So,

$$\underline{AD}_2 F(x) = \underline{AD}_1 f(x), \ \overline{AD}_2 F(x) = \overline{AD}_1 f(x), \text{ for all } x \in (0, 2\pi). \tag{29}$$

By the given conditions (i) and (ii) and by (28) and (29)

$$\underline{AD}_2 F \geq 0 \text{ except on } C \tag{30}$$

$$\liminf_{r \to 1-}(1-r)\frac{\partial^2 F(r,x)}{\partial x^2} \geq 0 \text{ for } x \in C. \tag{31}$$

Since F is continuous, by Corollary 3 F is Abel continuous and so the Fourier series of F is Abel summable to F and so by Lemma 9 and by (30)

$$\overline{F}^{(s)}_{(2)} \geq 0 \text{ except on } C \tag{32}$$

and by (31) and Lemma 8.5 of [8, p.357]

$$\limsup_{h \to 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h} \geq 0 \text{ for } x \in C. \tag{33}$$

By (32) and (33) and by Lemma 3.20 of [8, p.328] F is convex on $(0, 2\pi)$. So, F' exists except on a countable set in $(0, 2\pi)$ and is nondecreasing on the set of its existence. So, $F' + \frac{1}{2}a_0$ exists and is nondecreasing on the set where F' exists. Since $F' + \frac{1}{2}a_0 = f$ a.e., the first part follows.

For the second part, let E be the set outside which f is nondecreasing. If possible suppose that there are points $c, d, 0 < c < d < 2\pi$ such that, $f(c) > f(d)$. Choose k_1, k_2 such that $f(c) > k_2 > k_1 > f(d)$. Then by the given condition (iii)

$$\limsup_{x \to c} \text{ap } f(x) > k_2 > k_1 > \liminf_{x \to d} \text{ap } f(x). \tag{34}$$

Choose $\delta, 0 < \delta < \frac{d-c}{2}$ such that $(c - \delta, d + \delta) \subset (0, 2\pi)$. Then by (34) there are sets E_1 and E_2 such that $E_1 \subset (c - \delta, c + \delta)$, $E_2 \subset (d - \delta, d + \delta)$, $\mu(E_1) > 0$, $\mu(E_2) > 0$ and $f(x) > k_2$ for $x \in E_1$, $f(x) < k_1$ for $x \in E_2$. Since E is of measure zero, $\mu(E_1 \cap \tilde{E}) > 0$ and $\mu(E_2 \cap \tilde{E}) > 0$ where $\tilde{E}$ is the complement of E. So, there are points $\xi \in E_1 \cap \tilde{E}, \eta \in E_2 \cap \tilde{E}$, and so $f(\xi) > k_2 > k_1 > f(\eta)$. But $\xi < \eta$ and therefore since f is nondecreasing on $\tilde{E}, f(\xi) \le f(\eta)$ which is a contradiction. $\square$

Theorem 11. *Let f be 2π-periodic and Lebesgue integrable on $[0, 2\pi]$. If*
(i) $\underline{AD}_1 f \ge 0$ except on a countable set $C \subset (0, 2\pi)$;
(ii) $\displaystyle\limsup_{h \to 0} \left[\frac{1}{h}\left(\int_x^{x+h} f(t)dt - \int_{x-h}^{x} f(t)dt\right)\right] \ge 0$, *for $x \in C$*
then there exists a set $E \subset (0, 2\pi)$ of measure zero such that f is non-decreasing on $(0, 2\pi) \sim E$.

If moreover
(iii) f is Abel continuous in $(0, 2\pi)$;
(iv) $\displaystyle\lim_{h \to 0} \left[\frac{1}{h}\left(\int_x^{x+h} f(t)dt - \int_{x-h}^{x} f(t)dt\right)\right] = 0$, *for all $x \in (0, 2\pi)$*
then f is continuous and nondecreasing on $(0, 2\pi)$.

Proof. Proceeding as in Theorem 10 we have

$$\underline{AD}_2 F = \underline{AD}_1 f, \quad \overline{AD}_2 F = \overline{AD}_1 f \tag{35}$$

where F is as in (27). Therefore by condition (i) $\underline{AD}_2 F \ge 0$ except on C and so by Lemma 9

$$\overline{F}^{(s)}_{(2)} \ge 0 \text{ except on } C.$$

Also by condition (ii), (33) holds. So, the first part follows as the first part of Theorem 10.

For the second part (iv) implies that

$$\lim_{h\to 0}\frac{F(x+h)+F(x-h)-2F(x)}{h}=0 \text{ for all } x\in(0,2\pi). \tag{36}$$

Also by (35) and (i) and by Lemma 9 $\overline{F}^{(s)}_{(2)}\geq 0$ except on C and so by (36) and Lemma 3.20 [8; p.328], F is convex on $(0,2\pi)$. So, the right hand and left hand derivatives F'_+ and F'_- exist at each point $x\in(0,2\pi)$ and by (36) $F'_+=F'_-$ i.e., F' exists for each point $x\in(0,2\pi)$. By Theorem 4 we have since F' exists, $F'(x)=AD_1F(x)$. So, by Proposition 7 $F'(x)=f(x)-\frac{1}{2}a_0$ for all $x\in(0,2\pi)$. Since F is convex, F' is nondecreasing and by the Darboux property of F', F' is also continuous. Hence f is nondecreasing and continuous. □

Remark. The condition

$(i)\underline{AD}_1f\geq 0$ except on a countable set $C\subset(0,2\pi)$

in Theorem 10 and Theorem 11 can be relaxed by taking the following two conditions together

$(i)_1$ $\underline{AD}_1f\geq 0$ a.e. in $(0,2\pi)$

$(i)_2$ $\underline{AD}_1f>-\infty$ except on a countable set $C\subset(0,2\pi)$.

For, suppose that $(i)_1$ and $(i)_2$ hold. Let $E=\{x:x\in(0,2\pi);\underline{AD}_1f(x)<0\}$. Then by $(i)_1$ E is of measure zero. Let σ be a function defined on $[0,2\pi]$ such that σ is continuous and nondecreasing on $[0,2\pi]$ and $\sigma'(x)=+\infty$ for $x\in E$ (see [4, Vol. I, p.214]). We take σ on $(0,2\pi]$ and extend it to the whole of $\Re$ by defining $\sigma(x+2\pi)=\sigma(x)$ for all x. Let $\epsilon>0$ be arbitrary and let $g_\epsilon=f+\epsilon\sigma$. Then it can be proved that

$$\frac{\partial g_\epsilon(r,x)}{\partial x}=\frac{\partial f(r,x)}{\partial x}+\epsilon\frac{\partial\sigma(r,x)}{\partial x}$$

and hence

$$\underline{AD}_1g_\epsilon(x)\geq\underline{AD}_1f(x)+\epsilon\underline{AD}_1\sigma(x),\ \text{for all } x. \tag{37}$$

Since σ is nondecreasing, by Theorem 4, $\underline{AD}_1\sigma(x)\geq 0$ for all x. So, by the property of σ and by (37) $\underline{AD}_1g_\epsilon\geq 0$ except on C. Since σ is continuous , by Theorem 6 σ satisfies condition (ii) of Theorems 10 and 11 and so g_ϵ satisfies condition (ii) of these theorems. By Corollary 3 σ satisfies condition (iii) of Theorem 11 and so g_ϵ satisfies it. Hence by (37) g_ϵ satisfies the conditions of Theorems 10 and 11 if f does so. Since ϵ is arbitrary, the result follows.

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