RESEARCH

Real Analysis Exchange
Vol. 38(1), 2012/2013, pp. 33–62

Janusz Matkowski, Institute of Mathematics, Informatics and Econometrics, University of Zielona Góra, Podgórna 50, PL-65246 Zielona Góra, Poland.
email: J.Matkowski@uz.wmie.zgora.pl

POINTWISE MONOTONIC FUNCTIONS AND GENERALIZED SUBADDITIVITY

Abstract

A criterion of continuity and monotonicity of one-to-one pointwise monotonic functions is proved. We apply them in the theory of generalized subadditive functions. Some open problems are presented.

1 Introduction.

A real function f defined in an open real interval I is called increasing (or quasi-increasing [11]) at a point $x_0 \in I$, if

$$\limsup_{x\to x_0-} f(x) \le f(x_0) \le \liminf_{x\to x_0+} f(x),$$

and f is called pointwise increasing (quasi-increasing) in I, if it is increasing at every point of I. Analogously we define the pointwise decreasing function.

In section 1 we prove that *if f is one-to-one pointwise decreasing and*

$$\liminf_{t\to\sup I} f(t) = \sup f(I) \qquad \text{or} \qquad \limsup_{t\to\inf I} f(t) = \inf f(I),$$

then f is strictly increasing and continuous (Theorem 2).

It is well known (Hille & Phillips [1], Kuczma [2]) that for every subadditive function $f:(0,\infty)\to\mathbb{R}$, such that $f(0+) := \lim_{x\to 0+} f(x) = 0$, we have

$$f(x+) \le f(x) \le f(x-), \qquad x > 0,$$

Mathematical Reviews subject classification: Primary: 39B62; Secondary: 26A48, 26A51
Key words: linear inequality, pointwise monotonic function, subadditive functions, convex functions, generalizations
Received by the editors May 20, 2011
Communicated by: Miroslav Zeleny

which implies that f is pointwise decreasing. In section 2 we show that this feature is characteristic of a much larger class of functions satisfying the inequality

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in J,$$

for some positive numbers α and a, where J stands for $\mathbb{R}$ or $(0, \infty)$ or $(-\infty, 0)$.

If $f : J \to \mathbb{R}$ is subadditive and $\alpha, \beta \in \mathbb{N}$, $a = \alpha$, $b = \beta$ then, clearly,

$$f(\alpha x + \beta y) \leq af(x) + bf(y), \qquad x, y \in J.$$

However in the case when $\alpha = a > 1$ and $\beta = b > 1$, the function f satisfying this inequality with $J = (0, \infty)$ and condition $f(0+) = 0$ need not be pointwise decreasing (cf. Example 4). Thus a function satisfying the above inequality in the case when $\alpha = a = 1$ or $\beta = b = 1$ is, in a sense, "close" to the class of subadditive functions and, for this reason, we call it *generalized subadditive*.

Using a result of Raikov [14] or Świątkowski (cf. [9]), we obtain some refinements of the classical results on subadditive functions.

In section 3 we apply the results on the pointwise monotonicity in the proof of the continuity of one-to-one generalized subadditive functions (important in the theory of converses of Minkowski inequality). In particular we obtain a considerable generalization of the result on subadditive functions proved in [11] and [12]. A very special case of Corollary 1 says that any one-to-one bounded from below generalized subadditive function $f : (0, \infty) \to \mathbb{R}$ such that $f(0+) = 0$ is continuous and strictly increasing.

In section 4 we deal with periodicity and monotonicity of the generalized subadditive functions $f : \mathbb{R} \to \mathbb{R}$. In particular we show that some simple structure of the set $\{x : f(x) \leq 0\}$ forces the periodicity of f (Theorem 8).

Section 5 is devoted to the injective generalized subadditive functions defined in $\mathbb{R}$. As a corollary from Theorem 11 we obtain that every one-to-one generalized subadditive function $f : \mathbb{R} \to \mathbb{R}$, continuous at 0 and such that $f(0) = 0$, is continuous and strictly monotonic.

In the last section we present some remarks and examples, explaining among others the relations between the classical subadditive functions and generalized subadditive functions and we pose some open problems. Clearly, every subadditive function $f : (0, \infty) \to \mathbb{R}$ satisfies the inequality

$$f(2x + y) \leq 2f(x) + f(y), \qquad x, y > 0,$$

and we show that there are functions satisfying this inequality which are not subadditive. A special case of a more general open problem reads as follow. Suppose that $f : (0, \infty) \to \mathbb{R}$ satisfies this inequality and $f(0+) = 0$. Is then f subadditive?

2 One-to-one pointwise monotonic functions

We begin with the following

Definition 1. *Let $I \subset \mathbb{R}$ be an open interval and let $x_0 \in I$. A function $f : I \to \mathbb{R}$ is said to be*

(i) left-increasing at x_0, if

$$\limsup_{x \to x_0 -} f(x) \leq f(x_0);$$

(ii) right-increasing at x_0, if

$$f(x_0) \leq \liminf_{x \to x_0 +} f(x);$$

and increasing at x_0 (cf. [11]), if it is both left- and right-increasing at x_0.

We say that f is left-decreasing (right-decreasing; decreasing) at x_0, if $(-f)$ is left-increasing (respectively, right-increasing; increasing) at x_0.

Remark 1. *A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in \operatorname{int} I$ if, and only if, it is both increasing and decreasing at x_0.*

Making use of an idea of [11] we prove the following

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval. If a function $f : I \to \mathbb{R}$ is one-to-one, increasing at every point of I, and*

$$\limsup_{t \to \sup I} f(t) = \inf f(I) \tag{1}$$

or

$$\liminf_{t \to \inf I} f(t) = \sup f(I), \tag{2}$$

then f is strictly decreasing and continuous.

Proof. Assume first that condition (1) is satisfied.

To show that f is decreasing assume, on the contrary, that there are $x_1, x_2 \in I$ such that

$$x_1 < x_2 \quad \text{and} \quad f(x_1) < f(x_2).$$

The pointwise monotonicity of f at the points x_1, x_2 and injectivity of f imply that, without any loss of generality, we may assume that

$$\inf f(I) < f(x_1) \quad \text{and} \quad f(x_2) < \sup f(I).$$

Put

$$A=\{x\in I: x>x_2\wedge\forall t\ \ [x_2<t<x\Rightarrow f(t)>f(x_1)]\}.$$

The set A is nonempty as, by the right-increasing monotonicity of f at x_2, there is $\delta>0$ such that $f(t)>f(x_1)$ for all $t\in(x_2,x_2+\delta)$. Note that

$$\sup A=\sup I. \tag{3}$$

Indeed, in the opposite case we would have $x_0:=\sup A\in I$. The injectivity of f implies that either $f(x_0)>f(x_1)$ or $f(x_0)<f(x_1)$. In the first case the right-increasing monotonicity of f at x_0 implies that there is $\delta>0$ such that $f(t)>f(x_1)$ for all $t\in(x_0,x_0+\delta)$. It follows that $f(t)>f(x_1)$ for all $t\in(x_2,x_0+\delta)$, contradicting the definition of x_0. In the second case the left-increasing monotonicity of f at x_0 implies that there is $\delta>0$ such that $f(t)<f(x_1)$ for all $t\in(x_0-\delta,x_0)$, that also contradicts the definition of x_0.

Since $\inf f(I)<f(x_1)$, equality (3) contradicts condition (1). Thus the function f must be decreasing in I.

To show that f is decreasing in the case when condition (2) is satisfied, we argue similarly, putting

$$B=\{x\in I: x<x_1\wedge\forall t\ \ [x<t<x_1\Rightarrow f(t)<f(x_2)]\}$$

and showing that

$$\inf B=\inf I.$$

To end the proof it is enough to apply Remark 1. □

Note that conditions (1) and (2) are equivalent, respectively, to

$$\lim_{t\to\sup I} f(t)=\inf f(I),\qquad \lim_{t\to\inf I} f(t)=\sup f(I).$$

Replacing in this result f by $(-f)$ we obtain the following

Theorem 2. *Let $I\subset\mathbb{R}$ be an open interval. If a function $f: I\to\mathbb{R}$ is one-to-one, decreasing at every point of I, and*

$$\liminf_{t\to\sup I} f(t)=\sup f(I)\qquad \text{or}\qquad \limsup_{t\to\inf I} f(t)=\inf f(I),$$

then f is strictly increasing and continuous.

To see that in Theorem 1 condition (1) or (2) is indispensable consider the following

Example 1. *The function $f:(0,\infty)\to(0,\infty)$ defined by*

$$f(x)=\begin{cases} x & \text{for } x\in\left(0,\frac{1}{2}\right)\cup(2,\infty)\\ \frac{5}{2}-x & \text{for } x\in[\frac{1}{2},2]\end{cases}$$

is one-to-one and increasing at every point, but it is not continuous.

3 Pointwise monotonicity of generalized subadditive functions

Let $a, b, \alpha, \beta > 0$ and let $f : (0, \infty) \to \mathbb{R}$. The linear functional inequality

$$f(\alpha x + \beta y) \leq af(x) + bf(y), \qquad x, y > 0,$$

under the condition

$$\limsup_{x \to 0+} f(x) \leq 0,$$

was considered by Pycia [13] (cf. also [4], [10]) where some special cases where treated). In the case when f is defined on a real interval (or on a convex subset of $\mathbb{R}^k$) and $\alpha + \beta = 1$, $a = \alpha, b = \beta$, this inequality has a rich theory (cf. for instance Kuczma [2]).

The case $a = b = \alpha = \beta = 1$ is treated in Hille and Phillips [1], Kuczma [2] (cf. also Rosenbaum [15]).

In this section we consider this inequality with $b = \beta = 1$. Since in this case the solutions of the functional inequality behave similarly as subadditive functions (cf. Theorem 3), they are referred to as generalized subbaditive functions.

We shall need the following

Remark 2. *Let $\alpha > 0$ and $a > 0$ be and let $J = (0, \infty)$ or $J = \mathbb{R}$. If $f : J \to \mathbb{R}$ satisfies the inequality*

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in J,$$

then

$$f\left(\alpha \sum_{j=1}^{n} x_j + y\right) \leq a \sum_{j=1}^{n} f(x_j) + f(y), \qquad n \in \mathbb{N}; \quad x_1, \ldots, x_n, y \in J.$$

(We omit an easy inductive proof.)

We need the following

Theorem 3. *Let $\alpha > 0$ and $a > 0$ and let $J = (0, \infty)$ or $J = \mathbb{R}$. If a function $f : J \to \mathbb{R}$ satisfies*

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in J, \tag{4}$$

and

$$\limsup_{x \to 0+} f(x) \leq 0, \tag{5}$$

then $f(0+) = 0$*; for every* $x \in J$*, there exist* $f(x+)$ *and* $f(x-)$*, the left and right limits of* f *at the point* x*, and*

$$f(x+) \le f(x) \le f(x-), \qquad x \in J; \tag{6}$$

in particular, f *is pointwise decreasing at every point of* J*.*

Proof. Assume first that $J = (0, \infty)$. Taking $y = x$ in (4) we get $f((\alpha+1)x) \le (a+1) f(x)$ for all $x > 0$, whence, by induction,

$$f((\alpha+1)^n x) \le (a+1)^n f(x), \quad x > 0, \ n \in \mathbb{N}.$$

Condition (5) implies that there exist $M > 0$ and $\delta > 0$ such that $f(x) \le M$ for all $x \in (0, \delta)$. It follows that

$$f(x) \le (a+1)^n M, \qquad x \in (0, (\alpha+1)^n \delta), \ n \in \mathbb{N},$$

and, consequently, f is bounded from above on every bounded interval contained in $(0, \infty)$.

Take an arbitrary $r > 0$. According to what we have just shown, there is $M \in \mathbb{R}$ such that

$$f(x) \le M, \qquad x \in (0, \alpha r).$$

Since $\alpha(r-x) \in (0, \alpha r)$ for $x \in (0, r)$, applying (4) with $y = \alpha(r-x)$, we have

$$\begin{aligned} f(x) &\ge \frac{1}{a} f(\alpha x + \alpha(r-x)) - f(\alpha(r-x)) \\ &= \frac{1}{a} f(\alpha r) - f(\alpha(r-x)) \ge \frac{1}{a} f(\alpha r) - M \end{aligned}$$

for all $x \in (0, r)$. Thus f is bounded from below on any interval $(0, r)$ and, consequently, f is bounded on any bounded subinterval of $J = (0, \infty)$.

Assume that $J = \mathbb{R}$. Take an arbitrary $r > 0$. According to what we have just shown, there is $M \in \mathbb{R}$ such that $f(x) \le M$ for all $x \in (0, r)$. Since $(-\alpha r, 0) = (\alpha r, 0) + (-\alpha r)$, every $z \in (-\alpha r, 0)$ can be written in the form $z = \alpha x + (-\alpha r)$ for some $x \in (0, r)$, and, by (4),

$$f(z) = f(\alpha x + (-\alpha r)) \le a f(x) + f(-\alpha r) \le aM + f(-\alpha r).$$

Thus f is bounded from above on arbitrary interval $(-\alpha r, 0)$. Now, similarly as in the previous case, we can show that f is bounded from below on every bounded subinterval of $(-\infty, 0)$.

In the sequel of the proof we can assume that $J = (0, \infty)$ or $J = \mathbb{R}$.

Let us fix $x \in J \cup \{0\}$ and take arbitrary two reals sequences (x_n), (y_n) such that $x < x_n < y_n$, $n \in \mathbb{N}$, $\lim_{n\to\infty} y_n = x$ and

$$\liminf_{u\to x+} f(u) = \lim_{n\to\infty} f(x_n), \qquad \limsup_{u\to x+} f(u) = \lim_{n\to\infty} f(y_n).$$

From (4) we have, for all $n \in \mathbb{N}$,

$$f(y_n) = f\left(\alpha\left(\alpha^{-1}(y_n - x_n)\right) + x_n\right) \le af\left(\alpha^{-1}(y_n - x_n)\right) + f(x_n).$$

From (5), letting $n \to \infty$, we get

$$\limsup_{u\to x+} f(u) \le \liminf_{u\to x+} f(u).$$

The boundedness of f in any neighborhood of x implies that the right limit $f(x+)$ exists and is finite. In particular $f(0+)$ exists, is finite and, in view of (5), we have $f(0+) \le 0$. On the other hand, letting x and y tend to 0 from the right in inequality (4), we get $f(0+) \le (1+a) f(0+)$, whence $f(0+) \ge 0$. This proves that $f(0+) = 0$.

Since, for all $n \in \mathbb{N}$,

$$f(y_n) = f\left(\alpha\left(\alpha^{-1}(y_n - x)\right) + x\right) \le af\left(\alpha^{-1}(y_n - x)\right) + f(x),$$

letting n tend to infinity, we get $f(x+) \le f(x)$.

Similarly, for $x \in J$, taking two sequences (x_n), (y_n) such that $x_n < y_n < x$, $n \in \mathbb{N}$, $\lim_{n\to\infty} x_n = x$ such that

$$\liminf_{u\to x-} f(u) = \lim_{n\to\infty} f(x_n), \qquad \limsup_{u\to x-} f(u) = \lim_{n\to\infty} f(y_n),$$

making use of (4), we get, for all $n \in \mathbb{N}$,

$$f(y_n) = f\left(\alpha\left(\alpha^{-1}(y_n - x_n)\right) + x_n\right) \le af\left(\alpha^{-1}(y_n - x_n)\right) + f(x_n),$$

whence, by (5), letting $n \to \infty$, we obtain

$$\limsup_{u\to x-} f(u) \le \liminf_{u\to x-} f(u),$$

which proves that the limit $f(x-)$ exists. Since, for all $n \in \mathbb{N}$,

$$f(x) = f\left(\alpha\left(\alpha^{-1}(x - x_n)\right) + x_n\right) \le af\left(\alpha^{-1}(x - x_n)\right) + f(x_n),$$

letting $n \to \infty$ and making use of (5), we hence get $f(x) \le f(x-)$. This completes the proof. $\square$

Replacing in this theorem the interval J by $-J$ and the function f by $x \longmapsto f(-x)$, we obtain the following

Theorem 4. *Let $\alpha > 0$ and $a > 0$, and let $J = (-\infty, 0)$ or $J = \mathbb{R}$. If a function $f : J \to \mathbb{R}$ satisfies*

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in J,$$

and

$$\limsup_{x \to 0-} f(x) \leq 0,$$

then

$$\lim_{x \to 0-} f(x) = 0,$$

and, for every $x \in J$, there exist $f(x+)$ and $f(x-)$, and

$$f(x-) \leq f(x) \leq f(x+), \qquad x \in J;$$

in particular, f is pointwise increasing at every point of J.

From Theorem 3 and Theorem 4, as an immediate corollary, we obtain

Theorem 5. *Let $\alpha > 0$ and $a > 0$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality*

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in \mathbb{R},$$

f is upper semicontinuous at 0, and $f(0) \leq 0$, then f is continuous at every point of $\mathbb{R} \setminus \{0\}$; if moreover $f(0) = 0$, then f is continuous.

Taking $a = \alpha = 1$ in Theorem 3, 4 and 5, we obtain the classical regularity theorems for subadditive functions (cf. Hille and Phillips [1], p. 248, Theorem 7.8.3).

To show that in the above results the assumption on the regularity behavior of the function f at 0 can be significantly weakened, introduce the following

Definition 2. *Let $A \subset \mathbb{R}$ be a set. The number*

$$\delta_0^+(A) := \liminf_{r \to 0+} \frac{\lambda(A \cap (0, r))}{r},$$

where λ stands for the one-dimensional inner Lebesgue measure, is called the right density of A at the point 0.

Analogously we define $\delta_0^-(A)$, the left density of A at the point 0.

Note that, for any $a > 0$,

$$\delta_0^+(aA) = \liminf_{r\to 0+} \frac{\lambda(aA\cap(0,r))}{r} = \liminf_{r\to 0+} \frac{\lambda(A\cap(0,r/a))}{r/a} = \delta_0^+(A).$$

Lemma 1. *Let $\alpha > 0$ and $a > 0$.*

(i) *Assume that $J = (0,\infty)$ or $J = \mathbb{R}$ and $f : J \to \mathbb{R}$ satisfies inequality (4). Then existence $A \subset (0,\infty)$ such that $\delta_0^+(A) > 0$ and*

$$\limsup_{x\to 0-} f|_A(x) \le 0,$$

is equivalent to the condition

$$\limsup_{x\to 0+} f(x) \le 0.$$

(ii) *Assume that $J = (-\infty,0)$ or $J = \mathbb{R}$ and $f : J \to \mathbb{R}$ satisfies inequality (4). Then existence $A \subset (-\infty,0)$ such that $\delta_0^-(A) > 0$ and*

$$\limsup_{x\to 0-} f|_A(x) \le 0,$$

is equivalent to the condition

$$\limsup_{x\to 0-} f(x) \le 0.$$

PROOF. To prove (i) assume that there is a set $A \subset (0,\infty)$ such that $\delta_0^+(A) > 0$ and $\limsup_{x\to 0-} f|_A(x) \le 0$. Then, by Raikov theorem [14] (cf. also Świątkowski's result presented in [9]), there are $n \in \mathbb{N}$ and $r > 0$ such that

$$(0,r) \subset A_1 + \ldots + A_n + A,$$

where $A_1 = \ldots = A_n := \alpha A$. Thus any $x \in (0,r)$ can be written in the form

$$x = \alpha t_1 + \ldots + \alpha t_n + t \quad \text{for some} \quad t_1,\ldots,t_n,t \in A.$$

Applying Remark 2, we get

$$f(x) = f(\alpha t_1 + \ldots + \alpha t_n + t) \le af(t_1) + \ldots + af(t_n) + f(t),$$

whence

$$\limsup_{x\to 0+} f(x) \le a\limsup_{t_1\to 0+} f|_A(t_1) + \ldots + a\limsup_{t_n\to 0+} f|_A(t_n) + \limsup_{t\to 0+} f|_A(t) = 0.$$

The proof of (ii) is analogous. □

Remark 3. *Thus Theorem 3 remains true if assumption (5) is replaced by the equivalent condition involving density. Similarly in the case Theorem 4, the assumption* $\limsup_{x\to 0-} f|_A(x) \le 0$ *can be replaced by the respective condition occurring in part (ii) of Lemma 1.*

In particular it follows that assumption (5) can be replaced by $\operatorname{limap}_{x\to 0+} f(x) \le 0$ *where* $\operatorname{limap}_{x\to 0+} f(x)$ *denotes the right approximate limit at* 0.

4 Application to one-to-one generalized subadditive functions in $(0,\infty)$ and $(-\infty,0)$

The main result of this section reads as follows.

Theorem 6. *Let* $\alpha > 0$ *and* $a > 0$. *Suppose that a one-to-one function* $f:(0,\infty)\to\mathbb{R}$ *satisfies inequality (4):*

$$f(\alpha x + y) \le af(x) + f(y), \qquad x, y \in (0,\infty),$$

and there exists a set $A \subset (0,\infty)$ *such that* $\delta_0^+(A) > 0$ *and*

$$\limsup_{x\to 0+} f|_A(x) \le 0.$$

Then

$$f(0+) = 0,$$

and one of the following cases occurs:

(i) *f is continuous, strictly increasing (and positive);*

(ii) *f is strictly decreasing (negative) and* $\lim_{x\to\infty} f(x) = -\infty$;

(iii) *there is* $c \in (0,\infty)$ *such that f is continuous, strictly increasing (positive) either in* $(0,c)$ *or* $(0,c]$, *and f is strictly decreasing (and negative), respectively, in* $[c,\infty)$ *or* (c,∞), *and, in both cases,* $\lim_{x\to\infty} f(x) = -\infty$.

Proof. According to Theorem 3 and Lemma 1 we have $f(0+) = 0$. In view of Theorem 3, the function f is pointwise decreasing in $(0,\infty)$.

Put

$$C = \{x > 0 : f(x) < 0\}.$$

First consider the case $C = \emptyset$. Thus f is nonnegative in $(0,\infty)$. Assume that $f(x) = 0$ for some $x > 0$. Taking $y = x$ in (4) we get

$$0 \le f((\alpha+1)x) \le f(\alpha x + x) \le af(x) + f(x) = (a+1)f(x) = 0,$$

whence $f((\alpha+1)x) = f(x) = 0$, contradicting the injectivity of f. Thus, in this case, f must be positive. It follows that

$$\liminf_{t \to \inf(0,\infty)} f(t) = \lim_{t \to \inf(0,\infty)} f(t) = f(0+) = 0 = \inf((0,\infty)),$$

and, applying Theorem 2, we obtain part (i).

Now assume that $C \neq \emptyset$ and put

$$c = \inf C.$$

If $c = 0$ then there exists a decreasing sequence $c_k \in C$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} c_k = 0$, and, clearly, the set

$$D = \{(n\alpha+1) c_k : k, n \in \mathbb{N}\}$$

is dense in $(0,\infty)$.

Since $f(c_k) < 0$, in view of Remark 2, for all $k, n \in \mathbb{N}$,

$$f((n\alpha+1)c_k) = f(n\alpha c_k + c_k) \leq naf(c_k) + f(c_k) = (n\alpha+1) f(c_k) < 0,$$

the function f is negative on the set D. This fact and inequalities (6) imply that f is non-positive in $(0,\infty)$. Assuming that $f(x) = 0$ for some $x > 0$ and making use of (4), similarly as at the beginning of the proof, we would get $f((\alpha+1)x) = f(x) = 0$, contradicting the injectivity of f. Thus f is negative in $(0,\infty)$. Hence, for arbitrary $x_1, x_2 > 0$, $x_1 < x_2$, by (4), we have

$$f(x_2) = f\left(\alpha \frac{x_2 - x_1}{\alpha} + x_1\right) \leq af\left(\frac{x_2 - x_1}{\alpha}\right) + f(x_1) < f(x_1),$$

so f is strictly decreasing. The inequality

$$f((n\alpha+1)x) \leq (na+1) f(x), \qquad n \in \mathbb{N},$$

and $f(x) < 0$ imply that $\lim_{n\to\infty} f((n\alpha+1)x) = -\infty$. By the decreasing monotonicity of f we hence get $\lim_{x\to\infty} f(x) = -\infty$.

Assume that $c \in (0,\infty)$. In this case f is positive in the interval $I = (0,c)$. Since

$$\liminf_{t \to \inf I} f(t) = f(0+) = \inf(I),$$

Theorem 2 implies that f is continuous and strictly increasing in $(0,c)$. If $f(c) > 0$, from (6) we have

$$0 < f(c) \leq f(c-).$$

The continuity and injectivity of f in $(0,c]$ imply that $f(c-)=f(c)$ and, consequently, f is continuous in $(0,c]$.

To show that f is negative in (c,∞), assume on the contrary, that the set $E:=\{x>c: f(x)>0\}$ is nonempty and put

$$x_0=\inf E.$$

From the definition of c we have $f(c+)\leq 0$. Therefore there is $r>0$ such that $f(x)<0$ for all $x\in(c,c+r)$. It follows that $c<x_0$ and $f(x)<0$ for all $x\in(c,x_0)$, whence $f(x_0-)<0$. Since $f((0,c))=(0,f(c-))$, the injectivity of f implies that $f(c-)<f(x_0+)$. Thus, from (6), we have

$$0<f(c-)\leq f(x_0+)\leq f(x_0)\leq f(x_0-),$$

whence $0<f(x_0-)$, that is the desired contradiction. Since the remaining statement is obvious, the proof is complete. □

Theorem 7. *Let $\alpha>0$ and $a>0$. Suppose that a one-to-one function $f:(-\infty,0)\to\mathbb{R}$ satisfies the inequality*

$$f(\alpha x+y)\leq af(x)+f(y), \qquad x,y\in(-\infty,0),$$

there exists a set $A\subset(-\infty,0)$ such that $\delta_0^-(A)>0$, and

$$\limsup_{x\to 0-} f|_A(x)\leq 0.$$

Then $f(0-)=0$, and one of the following cases occurs:

(i) *f is continuous, strictly decreasing (and positive);*

(ii) *f is strictly increasing (negative) and $\lim_{x\to-\infty}f(x)=-\infty$;*

(iii) *there is $c\in(-\infty,0)$ such that f is continuous, strictly decreasing positive either in $(c,0)$ or $[c,0)$, and f is strictly increasing and negative, respectively, in $(-\infty,c]$ or $(-\infty,c)$ and, in both cases, $\lim_{x\to-\infty}f(x)=-\infty$.*

Proof. It is enough to apply Theorem 5 for the function $(0,\infty)\ni x\longmapsto f(-x)$. □

From Theorem 6 we obtain the following

Corollary 8. *Suppose that a bounded from below function $f:(0,\infty)\to\mathbb{R}$ satisfies inequality (4) and condition (5). If f is one-to-one then f is continuous, strictly increasing and $f(0+)=0$.*

Taking here $\alpha=a=1$ and a nonnegative function f, we obtain the result on subadditive functions presented in [11] (cf. also [12]) that is useful in the theory of converses of Minkowski and Hölder inequalities (cf. [3], [4], [6], [10]).

Remark 4. *A construction of a class of discontinuous subadditive bijections of* $(0,\infty)$ *which are bounded in the intervals* $(0,r)$ *for all* $r\in(0,\infty)$ *is given in [12]. Thus assumption (5) in Theorems 6, 7 and Corollary 1 is essential.*

Remark 5. *In Corollary 1, the assumption that* f *is bounded from below is essential. Indeed, the function* $f:(0,\infty)\to\mathbb{R}$ *defined by*

$$f(x)=\begin{cases} -x & \text{for } x\in(0,1] \\ -2x & \text{for } x\in(1,\infty)\end{cases}$$

is subadditive, one-to-one, unbounded from below and discontinuous.

More generally, if $\varphi:(0,\infty)\to(0,\infty)$ *is increasing and discontinuous, then the function* $f:(0,\infty)\to(-\infty,0)$ *defined by* $f(x)=-x\varphi(x)$ *is subadditive, strictly decreasing (so one-to-one), unbounded from below and discontinuous.*

Remark 6. *In Theorems 6 and 7 we assume that* f *is one-to-one. This assumption is essential as the function* $f:=|\sin|$ *is periodic, subadditive, continuous and* $f(0+)=f(0)$.

5 Periodicity and monotonicity of generalized subadditive functions in $\mathbb{R}$

In this section we deal with periodic generalized subadditive functions (cf. [8] where the periodic subadditive functions were considered). A key role is played by the following

Lemma 2. *Let* $\alpha>0$ *and* $a>0$. *If* $f:\mathbb{R}\to\mathbb{R}$ *satisfies the inequality*

$$f(\alpha x+y)\le af(x)+f(y), \qquad x,y\in\mathbb{R},$$

and there $q,r\in\mathbb{R}$,

$$q<0<r,$$

such that

$$f(q)\le 0 \quad \text{and} \quad f(r)\le 0,$$

then

$$f(\alpha(mq+nr)+x)\le f(x)\le f(x-\alpha(kq+lr)), \qquad k,l,m,n\in\mathbb{N};\quad x\in\mathbb{R}.$$

Proof. Applying inequality of Remark 2 with $n := m$, $x_1, \dots, x_n := q$, $y := x$, we get

$$f(\alpha mq + x) \le amf(q) + f(x), \qquad m \in \mathbb{N}; \ \ x \in \mathbb{R},$$

whence, as $f(q) \le 0$,

$$f(\alpha mq + x) \le f(x), \qquad m \in \mathbb{N}; \ \ x \in \mathbb{R}.$$

Similarly we get

$$f(\alpha nr + x) \le f(x), \qquad n \in \mathbb{N}; \ \ x \in \mathbb{R}.$$

From the last two inequalities we obtain

$$f(\alpha(mq + nr) + x) = f(\alpha mq + (\alpha nr + x)) \le f(\alpha nr + x) \le f(x),$$

that is

$$f(\alpha(mq + nr) + x) \le f(x), \qquad m, n \in \mathbb{N}; \ \ x \in \mathbb{R}.$$

Replacing x by $x - \alpha(kq + lr)$ with $m := k, n := l$, $k, l \in \mathbb{N}$ and $x \in \mathbb{R}$, we get

$$f(x) \le f(x - \alpha(kq + lr)), \qquad k, l, m, n \in \mathbb{N}; \ \ x \in \mathbb{R}.$$

This completes the proof. □

The main result of this section reads as follows.

Theorem 9. *Let $\alpha > 0$ and $a > 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies (4):*

$$f(\alpha x + y) \le af(x) + f(y), \qquad x, y \in \mathbb{R};$$

there exists for a set $A \subset (0, \infty)$ such that $\delta_0^+(A) > 0$,

$$\limsup_{x \to 0+} f|_A(x) \le 0;$$

and put

$$C = \{x \in \mathbb{R} \backslash \{0\} : f(x) \le 0\} \ne \emptyset.$$

(i) *If there exist $q, r \in C$ such that*

$$q < 0 < r \quad \text{and} \quad \frac{q}{r} \quad \text{is rational,}$$

then f is periodic, nonnegative, bounded, for every $x \in \mathbb{R}$ there exist $f(x-)$, $f(x+)$, and

$$f(x+) \le f(x) \le f(x-).$$

(ii) *If there exist* $q, r \in C$ *such that*

$$q < 0 < r \quad \text{and} \quad \frac{q}{r} \quad \text{is irrational,}$$

then $f(x) = 0$ *for all* $x \in \mathbb{R}$.

(iii) *If* $\sup C = 0$ *then* f *is increasing and continuous.*

(iv) *If* $\inf C = 0$ *then* f *is decreasing.*

Proof. By Theorem 3 and Remark 3, for every for every $x \in \mathbb{R}$ there exist $f(x-)$, $f(x+)$ and

$$f(x+) \leq f(x) \leq f(x-), \qquad x \in \mathbb{R}. \tag{7}$$

Proof of (i). Choosing $k, l, m, n \in \mathbb{N}$ such that

$$\frac{l+n}{k+m} = -\frac{q}{r} \quad \text{and} \quad \frac{n}{m} \neq -\frac{q}{r},$$

we have

$$l = -(m+k)\frac{q}{r} - n,$$

whence

$$kq + lr = kq - \left[(m+k)\frac{q}{r} + n\right] r = -(mq + nr).$$

Therefore

$$p := \alpha(mq + nr) \neq 0$$

and applying Lemma 2, with such chosen $k, l, m, n \in \mathbb{N}$, we obtain

$$f(p+x) = f(\alpha(mq+nr) + x) \leq f(x) \leq f(x - \alpha(kq + lr)) = f(x+p),$$

whence

$$f(p+x) = f(x), \qquad x \in \mathbb{R}.$$

which proves that p is a period of f.

By Theorem 6, the function f is bounded on every finite subinterval of $(0, \infty)$. The periodicity of f implies that f is bounded. To show that f is nonnegative, assume, on the contrary, that $f(x_0) < 0$ for some $x_0 \in \mathbb{R}$. Then, for arbitrary $M < 0$ we could choose $n \in \mathbb{N}$ such that $naf(x_0) < M - f(0)$. By Remark 2, we would have

$$f\left(\alpha \sum_{j=1}^{n} x_0\right) = f\left(\alpha \sum_{j=1}^{n} x_0 + 0\right) \leq naf(x_0) + f(0) < M,$$

that contradicts the boundedness of f. This completes the proof of (i).

Proof of (ii). Since the number $\frac{q}{r}$ is irrational and $q < 0 < p$, the set

$$D = \{\alpha\,(mq + nr) : m, n \in \mathbb{N}\}$$

is dense in $\mathbb{R}$ (cf. [3]). From Lemma 2 we have

$$f\,(t + x) \le f\,(x) \le f\,(x - s)\,; \qquad s, t \in D,$$

for arbitrarily fixed $x \in \mathbb{R}$. Letting here t tend to $-x$ from the right and s tend to x from the left, we obtain $f\,(x) = f\,(0+)$ for all $x \in \mathbb{R}$. Now it is obvious that $f\,(x) = 0$ for all $x \in \mathbb{R}$.

Proof of (iii). By the assumption there exists an increasing sequence $z_k < 0, \in \mathbb{N}$, such that

$$f\,(z_k) \le 0 \quad \text{for} \quad k \in \mathbb{N}, \qquad \text{and} \qquad \lim_{k\to\infty} z_k = 0.$$

Taking $x_1 = \ldots = x_n = y = z_k$ in Lemma 2, we get

$$f\,((n\alpha + 1)\,z_k) \le (na + 1)\,f\,(z_k) \le 0, \qquad k, n \in \mathbb{N},$$

whence, setting

$$B = \{(n\alpha + 1)\,z_k : k, n \in \mathbb{N}\}\,,$$

we obtain $f\,(t) \le 0$ for $t \in B$. Since the set B is dense in $(-\infty, 0)$, inequalities (7) imply that $f(x) \le 0$ for all $x < 0$. Hence, taking arbitrary $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, and making use of the inequality (4), we obtain

$$f\,(x_1) = f\left(\alpha\frac{x_1 - x_2}{\alpha} + x_2\right) \le af\left(\frac{x_1 - x_2}{\alpha}\right) + f\,(x_2) \le f\,(x_2)\,,$$

which proves that f is increasing. It follows that

$$f(x-) \le f(x) \le f(x+), \qquad x \in \mathbb{R}.$$

Taking into account inequalities (7) we obtain the continuity of f.

Proof of (iv). Applying (iii) for the function $g(x) := f(-x)$, $x \in \mathbb{R}$, we obtain part (iv). □

Of course, in part (ii) the function $f = 0$ is also periodic (even microperiodic).

In a similar way we can prove the following

Theorem 10. *Let $\alpha > 0$ and $a > 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies inequality (4):*

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in \mathbb{R};$$

for a set $A \subset (-\infty, 0)$ such that $\delta_0^-(A) > 0$,

$$\limsup_{x \to 0-} f|_A(x) \leq 0;$$

and put

$$C = \{x \in \mathbb{R} \backslash \{0\} : f(x) \leq 0\} \neq \emptyset.$$

(i) *If there exist $q, r \in C$ such that*

$$q < 0 < r \quad \text{ and } \quad \frac{q}{r} \text{ is rational,}$$

then f is periodic, nonnegative, bounded, for every $x \in \mathbb{R}$ there exist $f(x-)$, $f(x+)$, and

$$f(x-) \leq f(x) \leq f(x+).$$

(ii) *If there exist $q, r \in C$ such that*

$$q < 0 < r \quad \text{ and } \quad \frac{q}{r} \text{ is irrational,}$$

then $f(x) = 0$ for all $x \in \mathbb{R}$.

(iii) *If $\sup C = 0$ then f is increasing.*

(iv) *If $\inf C = 0$ then f is decreasing and continuous.*

6 Application to one-to-one generalized subadditive functions in $\mathbb{R}$

Theorem 11. *Let $\alpha > 0$ and $a > 0$. Suppose that a one-to-one function $f : \mathbb{R} \to \mathbb{R}$ satisfies inequality (4):*

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in \mathbb{R},$$

and there exists a set $A \subset (0, \infty)$ such that $\delta_0^+(A) > 0$ and

$$\limsup_{x \to 0+} f|_A(x) \leq 0.$$

Then

$$f(0+) = 0$$

and one of the following cases occurs

(i) *f is continuous and strictly increasing;*

(ii) *f is strictly decreasing;*

(iii) *f is continuous, strictly increasing (positive) and bounded in $(0,\infty)$,*

$$\sup f\left((0,\infty)\right) \leq f(x+) \leq f(x) \leq f(x-), \qquad x \in (-\infty, 0),$$

and either $f(0) = 0$ or $f(0) \geq \sup f\left((0,\infty)\right)$;

(iv) *there is $c \in (0,\infty)$ such that f is continuous, strictly increasing positive either in $(0,c)$ or $(0,c]$, f is strictly decreasing and negative, respectively, in $[c,\infty)$ or (c,∞), and, in both cases, $\lim_{x\to\infty} f(x) = -\infty$,*

$$\sup f\left((0,\infty)\right) \leq f(x+) \leq f(x) \leq f(x-), \qquad x \in (-\infty, 0),$$

and either $f(0) = 0$ or $f(0) \geq \sup f\left((0,\infty)\right)$.

Proof. In view of Theorem 3 and Lemma 1 we have $f(0+) = 0$ and f is pointwise decreasing. Setting $x = y = 0$ in (4) we obtain $f(0) \geq 0$.

In view of Theorem 5, for the function $f\big|_{(0,\infty)}$ one of the following cases occurs:

$\mathbf{1}^0$ $f\big|_{(0,\infty)}$ is continuous, strictly increasing (and positive);

$\mathbf{2}^0$ $f\big|_{(0,\infty)}$ is strictly decreasing (negative) and $\lim_{x\to\infty} f\big|_{(0,\infty)}(x) = -\infty$;

$\mathbf{3}^0$ there is $c \in (0,\infty)$ such that $f\big|_{(0,\infty)}$ is continuous, strictly increasing (positive) either in $(0,c)$ or $(0,c]$, and $f\big|_{(0,\infty)}$ is strictly decreasing (and negative), respectively, in $[c,\infty)$ or (c,∞), and, in both cases, $\lim_{x\to\infty} f\big|_{(0,\infty)}(x) = -\infty$.

Put $C := \{x \in \mathbb{R} : f(x) < 0\}$.

Now we consider these three cases.

Case $\mathbf{1}^0$.

Assume first that the set $C := \{x \in \mathbb{R} : f(x) < 0\}$ is nonempty. In this case, of course, $x_0 := \sup C \leq 0$. We shall show that $x_0 = 0$.

Assume, on the contrary, that $x_0 < 0$. From the definition of C and Theorem 3 we get $f(x_0+) \leq f(x_0) \leq f(x_0-) \leq 0$, whence $f(x_0+) \leq 0$. Since f is one-to-one and $(0,r) \subset f\big|_{(0,\infty)}(0,\infty)$ for some $r > 0$, it follows that $f(x) < 0$ for some $x \in (x_0, 0)$ which contradicts the definition of x_0 and proves the claim.

Now Theorem 8 (iii) implies that f is increasing and continuous. Since f is one-to-one, it is strictly increasing.

If C is empty then, obviously, $f\left|_{(0,\infty)}\right.$ must be bounded (in the opposite case the function would not be one-to-one), and f is of the form (iii).

Case $\mathbf{2}^0$**.**

Take arbitrary $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. In this case from the assumed inequality we have

$$\begin{aligned} f(x_2) &= f\left(\alpha\frac{x_2-x_1}{\alpha}+x_1\right) \le af\left(\frac{x_2-x_1}{\alpha}\right)+f(x_1) \\ &= af\left|_{(0,\infty)}\right.\left(\frac{x_2-x_1}{\alpha}\right)+f(x_1) < f(x_1), \end{aligned}$$

so f is strictly decreasing.

Case $\mathbf{3}^0$.

In this case, for $x > c$, the function f takes negative values. It follows that f must be positive in $(-\infty, 0)$, as in the opposite case, in view of Theorem 8, f would be either periodic or a zero function, contradicting injectivity. For the same reason the function $f\left|_{(0,\infty)}\right.$ must be bounded from above. It follows that in this case f must be of the form (iv).

This completes the proof. □

In a similar way we can prove

Theorem 12. *Let $\alpha > 0$ and $a > 0$. Suppose that a one-to-one function $f : \mathbb{R} \to \mathbb{R}$ satisfies inequality (4):*

$$f(\alpha x + y) \le af(x) + f(y), \qquad x, y \in \mathbb{R},$$

and there exists a set $A \subset (-\infty, 0)$ such that $\delta_0^-(A) > 0$ and

$$\limsup_{x\to 0-} f\left|_A\right.(x) \le 0.$$

Then

$$f(0-) = 0$$

and one of the following cases occurs

(i) *f is continuous and strictly decreasing;*

(ii) *f is strictly increasing;*

(iii) *f is continuous, strictly decreasing (positive) and bounded in $(-\infty, 0)$,*

$$\sup f((-\infty,0)) \le \ f(x+) \le f(x) \le f(x-), \qquad x \in (0,\infty),$$

and either $f(0) = 0$ or $f(0) \ge \sup f((-\infty, 0))$;

(iv) *there is* $c \in (-\infty, 0)$ *such that* f *is continuous, strictly decreasing (positive) either in* $(c, 0)$ *or* $[-c, 0)$, f *is strictly increasing (and negative), respectively, in* $(-\infty, c]$ *or* $(-\infty, c)$, *and, in both cases,* $\lim_{x\to-\infty} f(x) = -\infty$,

$$\sup f((-\infty, 0)) \leq \ f(x+) \leq f(x) \leq f(x-), \qquad x \in (0, \infty),$$

and either $f(0) = 0$ *or* $f(0) \geq \sup f((-\infty, 0))$.

From Theorem 10 and Theorem 11 we obtain the following

Theorem 13. *Let* $\alpha > 0$ *and* $a > 0$. *Suppose that a one-to-one function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$f(ax + y) \leq \alpha f(x) + f(y), \qquad x, y \in \mathbb{R}.$$

If there exists a set $A \subset \mathbb{R}$ *such that* $\delta_0^+(A) > 0$, $\delta_0^-(A) > 0$ *and*

$$\limsup_{x\to 0} f|_A(x) \leq 0,$$

then f *is continuous, strictly monotonic and* $f(0) = 0$.

Corollary 14. *Let* $\alpha > 0$ *and* $a > 0$. *Suppose that a one-to-one function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$f(\alpha x + y) \leq af(x) + f(y), \qquad x, y \in \mathbb{R}.$$

If f *is continuous at* 0 *and* $f(0) = 0$ *then* f *is continuous and strictly monotonic.*

7 Remarks on generalized subadditive functions and open problems

The results of the previous section, showing that the functions satisfying inequality (4) have similar properties as the classical subadditive functions, partially justify their name ”generalized subadditive functions”. In this section, to answer some questions concerning the relations between these two classes of functions, we present a few remarks, examples and we pose some open problems.

Remark 7. *If $f:(0,\infty)\to\mathbb{R}$ is subadditive, that is if*

$$f(x+y)\leq f(x)+f(y), \qquad x,y>0,$$

then, obviously,

$$f(2x+y)\leq 2f(x)+f(y), \qquad x,y>0, \tag{8}$$

i.e. f satisfies inequality (4) with $a=\alpha=2$. In this connection a question arises whether there exists a non-subadditive function satisfying inequality (8). The affirmative answer gives the following

Example 2. *Define $f:\mathbb{R}\to\mathbb{R}$ by*

$$f(x):=\begin{cases} 1 & x\in\mathbb{R}\backslash\mathbb{Q} \\ 3 & x\in\mathbb{Q}\end{cases}.$$

(Here $\mathbb{Q}$ stands for the set of all rational numbers.) It is easy to verify that f satisfies inequality (8). To see that f is not subadditive take $x=2-\sqrt{2}$, $y=2\sqrt{2}$, and note that, by the definition of f,

$$f(x+y)>f(x)+f(y).$$

Note that the function f in this example does not satisfy the condition $f(0+)=0$ (as well as a the formally weaker condition (5)). In this connection it is natural to ask wether there exists a continuous at the point 0 function $f:\mathbb{R}\to\mathbb{R}$ with $f(0)=0$ and such that f is not subadditive.

Trying to answer this question we shall consider some criteria for $f:(0,\infty)\to\mathbb{R}$ to satisfy inequality (4) with $\alpha=a$. Recall the following

Theorem 15. *([10]) Let $a,b>0$ be such that $\min(a,b)<1<a+b$. If $f:(0,\infty)\to\mathbb{R}$ satisfies the inequality*

$$f(ax+by)\leq af(x)+bf(y), \qquad x,y>0,$$

and

$$\limsup_{x\to 0+} f(x)\leq 0,$$

then $f(x)=f(1)x$ for all $x>0$.

(Cf. also [4] where f is assumed to be nonpositive, and M. Pycia [14] where a nontrivial generalization for inequality (4), involving power functions, is proved.)

Note the following consequence for a subfamily of generalized subadditive functions.

Corollary 16. *Let $a \in (0,1)$. If $f : (0,\infty) \to \mathbb{R}$ satisfies the inequality*

$$f(ax+y) \le af(x) + f(y), \qquad x, y > 0,$$

and $\limsup_{x\to 0+} f(x) \le 0$, *then* $f(x) = f(1)x$ *for all* $x > 0$.

This shows that, under the conditions of this corollary, the family of generalized subadditive functions in $(0,\infty)$ reduces to the class of linear functions. Therefore, in the sequel, we exclude this case from our considerations.

Remark 8. *Let* $\min(a,b) \ge 1$.

(i) *If $f : (0,\infty) \to \mathbb{R}$ is such that the function*

$$x \longmapsto \frac{f(x)}{x} \quad \text{is decreasing,} \tag{9}$$

then f satisfies the inequality:

$$f(ax+by) \le af(x) + bf(y), \qquad x, y > 0. \tag{10}$$

(ii) *If $f : (0,\infty) \to \mathbb{R}$ is subadditive and a- and b-subhomogeneous, that is*

$$f(ax) \le af(x), \qquad f(bx) \le bf(x), \qquad x > 0,$$

then f satisfies (10).

Proof. For all $x, y > 0$, by the assumptions, we have

$$\begin{aligned} f(ax+by) &= \frac{f(ax+by)}{ax+by}(ax+by) = \frac{f(ax+by)}{ax+by}ax + \frac{f(ax+by)}{ax+by}by \\ &\le \frac{f(ax)}{ax}ax + \frac{f(by)}{by}by \le \frac{f(x)}{x}ax + \frac{f(y)}{y}by = af(x) + bf(y), \end{aligned}$$

which proves part (i). Part (ii) is obvious. $\square$

Note that for $b = 1$ inequality (10) becomes a special case of inequality (4). For $a = b = 1$ we hence get a well-known criterion of subadditivity (cf. Hille-Phillips [1], p. 239, Theorem 7.2.4 (i)). Therefore Remark 8 is not helpful in answering the question. Note also that, by the monotonicity condition (9), this remark offers only quite regular solutions of inequality (10).

To determine a lager class of solutions of inequality (10) consider the following

Remark 9. *Let $a, b > 0$ and let $h : (0, \infty) \to \mathbb{R}$ be a convex function such that*

$$h(x) \le (a+b)\, h\left(\frac{x}{a+b}\right), \qquad x > 0. \tag{11}$$

If $f : (0, \infty) \to \mathbb{R}$ is an arbitrary function such that

$$h(x) \le f(x) \le (a+b)\, h\left(\frac{x}{a+b}\right), \qquad x > 0, \tag{12}$$

then f satisfies inequality (10):

$$f(ax+by) \le af(x) + bf(y), \qquad x, y > 0.$$

PROOF. Applying the second of inequalities (12), the convexity of h, and the first of inequalities (12), we obtain, for all $x, y > 0$,

$$\begin{aligned} f(ax+by) &\le (a+b)\, h\left(\frac{a}{a+b}x + \frac{b}{a+b}y\right) \\ &\le (a+b)\left[\frac{a}{a+b}h(x) + \frac{b}{a+b}h(y)\right] \\ &= ah(x) + bh(y) \le af(x) + bf(y), \end{aligned}$$

which completes the proof. □

Note that condition (11) guarantees that the set of functions f satisfying (12) is nonempty. If moreover inequality (11) is sharp i.e., if

$$h(x) < (a+b)\, h\left(\frac{x}{a+b}\right), \qquad x > 0, \tag{13}$$

then this remark gives a lot of discontinuous functions f satisfying inequality (10).

Moreover, taking in this remark $a = 2$, $b = 1$ and a convex function $h : (0, \infty) \to \mathbb{R}$ such that $h(0+) = 0$, one could expect to get the answer to the question. Unfortunately, we have the following

Remark 10. *If $a, b > 0$ are such that $a + b > 1$, then there is no convex function $h : (0, \infty) \to \mathbb{R}$ satisfying inequality (13) and such that $h(0+) = 0$.*

Indeed, if h is convex and $h(0+) = 0$, then the function

$$(0, \infty) \ni x \longmapsto \frac{h(x)}{x} = \frac{h(x) - h(0+)}{x - 0} \quad \text{is nondecreasing in } (0, \infty).$$

Since $a+b>1$, it follows that

$$\frac{h(x)}{x} \le \frac{h((a+b)x)}{(a+b)x}, \qquad x>0,$$

whence

$$(a+b)\,h\left(\frac{x}{a+b}\right) \le h(x), \qquad x>0.$$

Thus the aforementioned question is open. In this connection we pose the following more general

Problem. *Let the real numbers a_1, a_2 such $1 \le a_1 < a_2$ be fixed. Does there exist a function $f:(0,\infty)\to R$ with $f(0+)=0$ satisfying the inequality*

$$f(a_2x+y) \le a_2 f(x) + f(y), \qquad x,y>0,$$

and not satisfying the inequality

$$f(a_1x+y) \le a_1 f(x) + f(y), \qquad x,y>0?$$

Remark 11. *If $a+b<1$ then Remark 9, for every convex function $h:(0,\infty)\to\mathbb{R}$ such that $h(0+)=0$, gives a large family of functions $f:(0,\infty)\to\mathbb{R}$ with $f(0+)=0$ satisfying inequality (10).*

Replacing in the presented results the function f by $-f$ we obtain, in particular, the theory of the functional inequality

$$f(\alpha x+y) \ge af(x) + f(y), \qquad x,y>0,$$

(the generalized superadditive functions) satisfying the condition

$$\liminf_{x\to 0} f(x) \ge 0.$$

In reference to Example 2 and the general linear inequality considered by Pycia [13] we consider the following

Remark 12. *Let $a,b,\alpha,\beta>0$ be fixed and let $J=\mathbb{R}$ or $J=(0,\infty)$.*

(i) *Suppose that $a+b>1$. Then, for any $c>0$, the function $f:J\to\mathbb{R}$ such that*

$$c \le f(x) \le (a+b)\,c, \qquad x\in J,$$

satisfies inequality

$$f(\alpha x+\beta y) \le af(x) + bf(y), \qquad x\in J. \tag{14}$$

(ii) *Suppose that $a+b<1$. Then, for any $c<0$, the function $f:(0,\infty)\to\mathbb{R}$ such that*

$$c\le f(x)\le c\,(a+b)\,,\qquad x\in J,$$

satisfies inequality (14).

PROOF. Indeed, in both cases, for all $x,y\in J$,

$$f\left(\alpha x+\beta y\right)\le (a+b)\,c=ac+bc\le af\left(x\right)+bf\left(y\right).$$

□

In the case when $a+b=1$, the convexity case, this remark gives only constant solutions of (14).

Remark 12 shows also that if $a+b\ne 1$, then there are a lot of solutions f of (14). But all these functions are bounded and uniformly separated from the zero function. The situation changes completely if close to zero, the function f has values close to zero (cf. for instance Theorem 13).

Note also the following

Remark 13. *Let $0<d\le\infty$ be fixed (so we admit $d=\infty$). If $f:[0,d)\to\mathbb{R}$ is convex and $f(0)=0$, then for any $a,b>0$ such that $a+b<1$ we have*

$$f(ax+by)\le af(x)+bf(y),\qquad x,y\in[0,d).$$

PROOF. Indeed, by the convexity of f, for all $x,y\in[0,d)$,

$$\begin{aligned} f(ax+by) &= f(ax+by+(1-a-b)0)\le af(x)+bf(y)+(1-a-b)f(0)\\ &= af(x)+bf(y). \end{aligned}$$

□

To show that the converse implication is not true consider the following

Example 3. *The function $f:[0,4)\to\mathbb{R}$ defined by*

$$f\left(x\right)=\begin{cases} x & \text{for } x\in[0,3] \\ 4 & \text{for } x\in[3,4) \end{cases}$$

satisfies the inequality

$$f\left(\frac{1}{3}x+\frac{1}{3}y\right)\le\frac{1}{3}f\left(x\right)+\frac{1}{3}f\left(y\right),\qquad x,y\in[0,4),$$

$f(0)=0$, but f is not convex.

Clearly, f is not convex and $f(0)=0$. Obviously this inequality is satisfied for all $x,y\in[0,3]$. If $x\in[0,3]$ and $y\in[3,4)$, then $\frac{1}{3}(x+y)<\frac{7}{3}<3,$ whence

$$f\left(\frac{1}{3}x+\frac{1}{3}y\right)=\frac{1}{3}x+\frac{1}{3}y=\frac{1}{3}f(x)+\frac{1}{3}y<\frac{1}{3}f(x)+\frac{1}{3}\cdot 4=\frac{1}{3}f(x)+\frac{1}{3}f(y).$$

By the symmetry, the inequality is satisfied for all $x\in[3,4)$ and $y\in[0,3]$. If $x,y\in(3,4)$ then $\frac{1}{3}(x+y)<\frac{1}{3}(4+4)=\frac{8}{3}<3$, and we have

$$f\left(\frac{1}{3}x+\frac{1}{3}y\right)=\frac{1}{3}x+\frac{1}{3}y<\frac{1}{3}\cdot 4+\frac{1}{3}\cdot 4=\frac{1}{3}f(x)+\frac{1}{3}f(y).$$

Remark 14. *Let $a,b>0$ be such that $a+b<1$. If $f:(0,\infty)\to\mathbb{R}$ satisfies (9) then*

$$kx\le f(x)\le mx,\qquad x>0,$$

where

$$k:=\lim\inf_{x\to 0+}\frac{f(x)}{x},\qquad m:=\lim\sup_{x\to\infty}\frac{f(x)}{x}.$$

Proof. Taking $y=x$ in (10) we get

$$f((a+b)x)\le(a+b)f(x),\qquad x>0,$$

whence, by induction

$$f((a+b)^n x)\le(a+b)^n f(x),\qquad x>0,\ n\in\mathbb{N}.$$

Hence we easily get

$$\frac{f((a+b)^n x)}{(a+b)^n x}\le\frac{f(x)}{x}\le\frac{f\left((a+b)^{-n}x\right)}{(a+b)^{-n}x},\qquad x>0,\ n\in\mathbb{N}.$$

Letting $n\to\infty$ we conclude that

$$k\le\frac{f(x)}{x}\le m,\qquad x>0.$$

□

Now consider the following

Example 4. *The function* $f:(0,\infty)\to\mathbb{R}$ *defined by*

$$f(x)=\begin{cases}\sqrt{x} & \text{for } 0<x\le 2\\ \frac{3}{2} & \text{for } x>2\end{cases}$$

satisfies the condition $f(0+)=0$*, the inequality*

$$f(2x+2y)\le 2f(x)+2f(y),\qquad x,y>0,$$

and it is not pointwise decreasing in $(0,\infty)$*; consequently,* f *is not subadditive.*

The inequality is obvious for all $x,y>0$ such that $x+y\le 1$. If $x+y>1$ then $f(2x+2y)=\frac{3}{2}$. Assume that for some $x,y>0$, $x+y>1$, the inequality is not satisfied. Then we would have $2f(x)+2f(y)<\frac{3}{2}$. It follows that $x,y\in(0,2)$ and, by the definition of f, $\sqrt{x}+\sqrt{y}<\frac{3}{4}$, whence $x+y<\frac{9}{16}-2\sqrt{xy}<1$, that is a contradiction. Since $f(2)=\sqrt{2}<\frac{3}{2}=f(2+)$, the function f is not decreasing at the point $x=2$. Consequently, f cannot be subadditive.

We end this paper with the following open

Problem 1. *Let* $a,b>0$ *and let* $\varphi:(0,\infty)\to(0,\infty)$ *be a bijective function. Suppose that, for all* $x_1,x_2,y_1,y_2>0$,

$$\varphi^{-1}\left(a\varphi(x_1+y_1)+b\varphi(x_2+y_2)\right)\le\varphi^{-1}\left(a\varphi(x_1)+b\varphi(x_2)\right)+\varphi^{-1}\left(a\varphi(y_1)+b\varphi(y_2)\right),$$

that is the function

$$(0,\infty)^2\ni(x_1,x_2)\longmapsto\varphi^{-1}\left(a\varphi(x_1)+b\varphi(x_2)\right)$$

is subadditive in $(0,\infty)^2$.

Is then $\varphi^{-1}(0+)=0$*?*

Remark 15. *The answer is yes if* $a+b=1$ *(cf. [5]) or if* $a=\frac{1}{n}$ *and* $b=n$ *or* $a=n$ *and* $b=\frac{1}{n}$ *for some positive integer* $n\ge 2$ *(cf. [7]). In particular the problem is open in the case when* $a=b=1$.

It is not difficult to see that the function $f:=\varphi^{-1}$*, the inverse of* φ*, is subadditive in* $(0,\infty)$.

Acknowledgement. Many thanks are due to Miroslav Zelený and the anonymous referees for their valuable remarks.

References

[1] E. Hille, S. P. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc., Colloquium Publ., **31**, Providence, Rhode Island, 1957.

[2] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, PWN, Uniwersytet Ślkąski, Warszawa - Kraków - Katowice, 1985.

[3] J. Matkowski, *The converse of the Minkowski's inequality theorem and its generalizations*, Proc. Amer. Math. Soc., **109** (1990), 663–675.

[4] -, *L^p-like paranorms,* in Selected Topics in Functional Equations and Iteration Theory, Proceedings of the Austrian-Polish Seminar, Universität Graz, October 24-26, 1991, edited by D. Gronau and L. Reich, Grazer Math. Ber., **316** (1992), 103–139.

[5] -, *On α-Wright convexity and the converse of Minkowski's inequality*, Aequationes Math., **43** (1992), 106–112.

[6] -, *Subadditive functions and a relaxation of the homogeneity condition of seminorm*, Proc. Amer. Math. Soc., **117**(1993), 991–1001.

[7] -, *On a characterization of L^p-norm and a converse of Minkowski's inequality*, Hiroshima Math. J., **26**(1996), 277–287.

[8] -, *Subadditive periodic functions*, Opuscula Math., **31** (2011), 75–96.

[9] -, *Results of Tadeusz Świątkowski on algebraic sum of sets in the theory of subadditive functions*, accepted to J. Appl. Anal.

[10] J. Matkowski, M. Pycia, *Convex-like inequality, homogeneity, subadditivity, and a characterization of L^p-norm,* Ann. Polon. Math., **60** (1995), 221–230.

[11] J. Matkowski, T. Świątkowski, *Quasi-monotonicity, subadditive bijections of R_+, and a characterization of L^p-norm*, J. Math. Anal. Appl., **154** (1991), 493–506.

[12] J. Matkowski, T. Świątkowski, *On subadditive functions*, Proc. Amer. Math. Soc., **119** (1993), 187–197.

[13] M. Pycia, *Linear functional inequalities - a general theory and new special cases*, Dissertationes Math. (Rozprawy Mat.), **438** (2006).

[14] D. A. Raikov, *On the addition of point-sets in the sense of Schnirelmann*, Mat. Sb. **5** (1939), 425–440 (in Russian).

[15] A. Rosenbaum, *Subadditive functions*, Duke Math. J., **17** (1950), 227–242.