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PRODUCTS OF EXTRA STRONG ŚWIĄTKOWSKI FUNCTIONS

Abstract

In this paper we characterize products of four or more extra strong Świątkowski functions.

1 Preliminaries

We use mostly standard terminology and notation. The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The symbols $\mathrm{I}(a,b)$ and $\mathrm{I}[a,b]$ denote the open and the closed interval with endpoints a and b, respectively. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, and $\operatorname{bd} A$ to denote the interior, the closure, and the boundary of A, respectively. We say that a set $A \subset \mathbb{R}$ is *simply open* [1], if it can be written as the union of an open set and a nowhere dense set. The symbol $\operatorname{Ent}(x)$ denotes the greatest integer not larger than $x \in \mathbb{R}$.

Let $f\colon I \to \mathbb{R}$, where I is a nondegenerate interval. The symbol $\mathcal{C}(f)$ stands for the set of all points of continuity of f. We say that f is a *Darboux function* ($f \in \mathcal{D}$), if it maps connected sets onto connected sets. We say that f is *quasi-continuous* in the sense of Kempisty [2], if for all $x \in I$ and open sets $U \ni x$ and $V \ni f(x)$, the set $\operatorname{int}(U \cap f^{-1}(V))$ is nonempty. We say that f is *cliquish* [7], if the set of points of continuity of f is dense in I. We say that f is a *strong Świątkowski function* [3] ($f \in \acute{\mathcal{S}}_s$), if whenever $\alpha, \beta \in I$ and

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$y \in \mathrm{I}(f(\alpha), f(\beta))$, there is an $x_0 \in \mathrm{I}(\alpha, \beta) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. We say that f is an *extra strong Świątkowski function* [6] ($f \in \acute{\mathcal{S}}_{es}$), if whenever $\alpha, \beta \in I$, $\alpha \neq \beta$, and $y \in \mathrm{I}[f(\alpha), f(\beta)]$, there is an $x_0 \in \mathrm{I}[\alpha, \beta] \cap \mathcal{C}(f)$ such that $f(x_0) = y$. One can easily see that each strong Świątkowski function is both Darboux and quasi-continuous and each extra strong Świątkowski function is strong Świątkowski. The symbol $[f = a]$ stands for the set $\{x \in I : f(x) = a\}$. We say that a function f *changes its sign* in interval J, if there are points $x_1, x_2 \in J$ such that $\operatorname{sgn} f(x_1) \neq \operatorname{sgn} f(x_2)$. The symbol $\mathcal{M}$ denotes the class of all functions f such that f has a zero in each interval in which it takes on both positive and negative values.

2 Introduction

In 1996 A. Maliszewski proved the following theorem [4].

Theorem 2.1. *For each function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent*:

i) *f is a finite product of Darboux quasi-continuous functions,*

ii) *there are Darboux quasi-continuous functions g and h such that $f = gh$,*

iii) *$f \in \mathcal{M}$, f is cliquish, and the set $[f = 0]$ is simply open.*

He showed also that products of two and three strong Świątkowski functions are different, and asked for characterization of products of such functions. In 2006 I found the partial solution of this problem proving the following theorem [5].

Theorem 2.2. *For each function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent*:

i) *f is a finite product of strong Świątkowski functions,*

ii) *there are strong Świątkowski functions $g_1, \ldots, g_4$ such that $f = g_1 \ldots g_4$,*

iii) *the function f is cliquish, the set $[f = 0]$ is simply open, and there exist a G_δ-set $A \subset [f = 0]$ such that $I \cap A \neq \emptyset$ for every interval I in which f takes on both positive and negative values.*

Recently I found a bounded strong Świątkowski function which cannot be written as the finite product of extra strong Świątkowski functions [6, Proposition 4.2]. Moreover I presented a product of three extra strong Świątkowski functions that cannot be written as a product of two such functions and a product of four extra strong Świątkowski functions that cannot be expressed as a product of three functions of that kind [6, Propositions 4.4 and 4.5].

In this paper I characterize products of four or more extra strong Świątkowski functions. However, the following problem is still open.

Problem 2.3. Characterize the products of two extra strong Świątkowski functions and the products of three extra strong Świątkowski functions.

3 Auxiliary lemmas

The proof of Lemma 3.1 we can find in [6, Theorem 3.1].

Lemma 3.1. *For each function* $f\colon \mathbb{R} \to \mathbb{R}$ *the following conditions are equivalent*:

i) $f \in \acute{\mathcal{S}}_{es}$,

ii) $f \in \mathcal{D}$ *and* $f[I] = f[I \cap \mathcal{C}(f)]$ *for each nondegenerate interval* $I \subset \mathbb{R}$,

iii) $f \in \mathcal{D}$ *and* $f(x) \in f[\mathrm{I}[x,t] \cap \mathcal{C}(f)]$ *for each* $x \in \mathbb{R}$ *and each* $t \in \mathbb{R} \setminus \{x\}$.

The next lemma is interesting in itself.

Lemma 3.2. *Assume that* $I \subset \mathbb{R}$ *is an interval,* $g\colon I \to \mathbb{R}$*, and* $h\colon \mathbb{R} \to \mathbb{R}$*. If* $g, h \in \acute{\mathcal{S}}_{es}$*, then* $h \circ g \in \acute{\mathcal{S}}_{es}$.

Proof. Let $x \in I$ and $t \in I \setminus \{x\}$. If $g{\restriction}\mathrm{I}[x,t] \in \mathcal{C}onst$, then $(h \circ g){\restriction}\mathrm{I}[x,t] \in \mathcal{C}onst$ and

$$(h \circ g)(x) \in (h \circ g)[\mathrm{I}[x,t] \cap \mathcal{C}(h \circ g)].$$

In the other case, since $g \in \acute{\mathcal{S}}_{es} \subset \mathcal{D}$, then $g[\mathrm{I}[x,t]]$ is a nondegenerate interval. Since $h \in \acute{\mathcal{S}}_{es}$, by Lemma 3.1 we have

$$\begin{aligned}(h \circ g)(x) \in h\big[g[\mathrm{I}[x,t]]\big] &= h\big[g[\mathrm{I}[x,t]] \cap \mathcal{C}(h)\big] = h\big[g[\mathrm{I}[x,t] \cap \mathcal{C}(g)] \cap \mathcal{C}(h)\big] \\ &\subset h\big[g[\mathrm{I}[x,t] \cap \mathcal{C}(h \circ g)]\big] = (h \circ g)[\mathrm{I}[x,t] \cap \mathcal{C}(h \circ g)].\end{aligned}$$

Clearly $h \circ g \in \mathcal{D}$. By Lemma 3.1 we obtain that $h \circ g \in \acute{\mathcal{S}}_{es}$. □

Lemma 3.3 is due to A. Maliszewski [4, Lemma III.1.1].

Lemma 3.3. *Let* $A \subset \mathbb{R}$ *be nowhere dense and closed and* $\mathcal{I}$ *be the family of all components of* $\mathbb{R} \setminus A$*. There are pairwise disjoint families* $\mathcal{I}_1, \dots, \mathcal{I}_4 \subset \mathcal{I}$ *such that for each* $j \in \{1, \dots, 4\}$ *and* $x \in A$ *if* x *is not isolated in* A *from the left (from the right), then there is a sequence* $(I_{j,n}) \subset \mathcal{I}_j$ *with* $\inf I_{j,n} \to x^-$ *(with* $\sup I_{j,n} \to x^+$*, respectively).*

The proof of Lemma 3.4 is similar to the proof of [5, Lemma 3.4].

Lemma 3.4. *Assume that $F \subset C$ are closed and $\mathcal{J}$ is a family of components of $\mathbb{R} \setminus C$ such that $C \subset \operatorname{cl} \bigcup \mathcal{J}$. There is a family $\mathcal{J}' \subset \mathcal{J}$ such that*

i) *for each $J \in \mathcal{J}$, if $F \cap \operatorname{bd} J \neq \emptyset$, then $J \in \mathcal{J}'$,*

ii) *for each $c \in F$, if c is a right-hand (left-hand) limit point of C, then c is a right-hand (respectively left-hand) limit point of the union $\bigcup \mathcal{J}'$,*

iii) $\operatorname{cl} \bigcup_{J \in \mathcal{J}'} \{\inf J\} \subset F \cup \bigcup_{J \in \mathcal{J}'} \{\inf J\}$ *and* $\operatorname{cl} \bigcup_{J \in \mathcal{J}'} \{\sup J\} \subset F \cup \bigcup_{J \in \mathcal{J}'} \{\sup J\}$.

Proof. Let $\mathcal{P}$ be the family of all components of $\mathbb{R} \setminus F$ and $P \in \mathcal{P}$. One can easily see that there is a family $\mathcal{J}_P \subset \mathcal{J}$ such that $\bigcup \mathcal{J}_P \subset P$ and the following conditions hold:

$$\text{if } P \cap C \neq \emptyset, \text{ then } \mathcal{J}_P \neq \emptyset, \tag{1}$$
$$\text{for each } J \in \mathcal{J}, \text{ if } J \subset P \text{ and } \operatorname{bd} P \cap \operatorname{bd} J \neq \emptyset, \text{ then } J \in \mathcal{J}_P, \tag{2}$$
$$\text{if } \inf P \in \operatorname{cl}(P \cap C), \text{ then } \inf P \in \operatorname{cl} \bigcup \mathcal{J}_P, \tag{3}$$
$$\text{if } \sup P \in \operatorname{cl}(P \cap C), \text{ then } \sup P \in \operatorname{cl} \bigcup \mathcal{J}_P, \tag{4}$$
$$\operatorname{cl} \textstyle\bigcup_{J \in \mathcal{J}_P} \{\inf J\} \subset \operatorname{bd} P \cup \bigcup_{J \in \mathcal{J}_P} \{\inf J\}, \tag{5}$$
$$\operatorname{cl} \textstyle\bigcup_{J \in \mathcal{J}_P} \{\sup J\} \subset \operatorname{bd} P \cup \bigcup_{J \in \mathcal{J}_P} \{\sup J\}. \tag{6}$$

Define $\mathcal{J}' = \bigcup_{P \in \mathcal{P}} \mathcal{J}_P$. Clearly $\mathcal{J}' \subset \mathcal{J}$. We will show that $\mathcal{J}'$ satisfies the conditions i)–iii) of the lemma.

Assume that $F \cap \operatorname{bd} J \neq \emptyset$ for some $J \in \mathcal{J}$. Since $F \subset C$, there is a $P \in \mathcal{P}$ with $J \subset P$. Then by (2), $J \in \mathcal{J}_P \subset \mathcal{J}'$. This proves condition i).

To prove condition ii) assume that $c \in F$ is a right-hand limit point of C. We consider two cases.

If there is a $P \in \mathcal{P}$ with $c = \inf P$, then by (3),

$$c \in \operatorname{cl} \bigcup \mathcal{J}_P \subset \operatorname{cl}\big((c, \infty) \cap \bigcup \mathcal{J}'\big).$$

In the opposite case fix a $d > c$. Since $C \subset \operatorname{cl} \bigcup \mathcal{J}$, we obtain $(c, d) \cap \bigcup \mathcal{J} \neq \emptyset$. By our assumption, there is a $J \in \mathcal{J}$ such that $J \subset (c, d)$ and $(\sup J, d) \cap F \neq \emptyset$. Choose $P \in \mathcal{P}$ with $J \subset P$. Clearly $P \subset (c, d)$.

If $P \cap C = \emptyset$, then $P = J \in \mathcal{J}$, and by (2), $P \in \mathcal{J}_P \subset \mathcal{J}'$. Consequently $(c, d) \cap \bigcup \mathcal{J}' \neq \emptyset$.

If $P \cap C \neq \emptyset$, then by (1), $\mathcal{J}_P \neq \emptyset$. Since $\bigcup \mathcal{J}_P \subset P$, we obtain that $(c, d) \cap \bigcup \mathcal{J}' \neq \emptyset$. This completes the proof of ii).

Finally we will show iii). Note that by (5),

$$\operatorname{cl}\bigcup_{J\in\mathcal{J}'}\{\inf J\} = \operatorname{cl}\bigcup_{P\in\mathcal{P}}\bigcup_{J\in\mathcal{J}_P}\{\inf J\} \subset \operatorname{cl}\bigcup_{P\in\mathcal{P}}\Big(\operatorname{bd} P\cup\bigcup_{J\in\mathcal{J}_P}\{\inf J\}\Big) \subset$$
$$\subset \operatorname{cl} F\cup\bigcup_{P\in\mathcal{P}}\Big(\bigcup_{J\in\mathcal{J}_P}\{\inf J\}\cup\operatorname{bd} P\Big) = F\cup\bigcup_{J\in\mathcal{J}'}\{\inf J\}.$$

Similarly, using condition (6) we can prove that $\operatorname{cl}\bigcup_{J\in\mathcal{J}'}\{\sup J\} \subset F \cup \bigcup_{J\in\mathcal{J}'}\{\sup J\}$. This completes the proof of the lemma. □

Lemma 3.5. *Let $I=(a,b)$ be an open interval and assume that $y_1, y_2 \in [0,1]$. There is an extra strong Świątkowski function $g\colon \operatorname{cl} I \to [0,1]$ such that*

i) $g(a) = y_1, \quad g(b) = y_2$,

ii) $g[I] = (0,1]$,

iii) $\operatorname{bd} I \subset \mathcal{C}(g)$,

iv) *if $y_1 \neq 0$, then $g\big[[a, a+\delta)\big] = \{y_1\}$ for some $\delta > 0$,*

v) *if $y_2 \neq 0$, then $g\big[(b-\delta, b]\big] = \{y_2\}$ for some $\delta > 0$.*

PROOF. Define the function $\bar{g}\colon \mathbb{R} \to (0,1]$ by

$$\bar{g}(x) = \begin{cases} \min\big\{1, \sin x^{-1} + |x| + 1\big\} & \text{if } x \neq 0, \\ \;\; 2^{-1} & \text{if } x = 0. \end{cases}$$

Then clearly $\bar{g} \in \acute{\mathcal{S}}_{es}$. Choose elements $a < x_1 < \cdots < x_7 < b$ and define continuous functions $\varphi, \psi\colon \operatorname{cl} I \to [0,1]$ as follows:

$$\varphi(x) = \begin{cases} y_1 & \text{if } x \in [a, x_1], \\ y_2 & \text{if } x \in [x_7, b], \\ 1 & \text{if } x \in [x_2, x_6], \\ \text{linear} & \text{in intervals } [x_1, x_2] \text{ and } [x_6, x_7], \end{cases}$$

$$\psi(x) = \begin{cases} y_1 & \text{if } x = a, \\ y_2 & \text{if } x = b, \\ 1 & \text{if } x \in [x_2, x_6], \\ \text{linear} & \text{in intervals } [a, x_2] \text{ and } [x_6, b]. \end{cases}$$

Now define the function $g\colon \operatorname{cl} I \to [0,1]$ by the formula:

$$g(x) = \begin{cases} \bar{g}(x - x_4) & \text{if } x \in [x_3, x_5], \\ \varphi(x) & \text{if } x \in [a, x_2] \text{ and } y_1 \neq 0 \text{ or } x \in [x_6, b] \text{ and } y_2 \neq 0, \\ \psi(x) & \text{if } x \in [a, x_2] \text{ and } y_1 = 0 \text{ or } x \in [x_6, b] \text{ and } y_2 = 0, \\ \text{linear} & \text{in intervals } [x_2, x_3] \text{ and } [x_5, x_6]. \end{cases}$$

Clearly $\mathcal{C}(g) = \operatorname{cl} I \setminus \{x_4\}$. So, condition iii) holds and $g \in \acute{\mathcal{S}}_{es}$. Now assume that $y_1 \neq 0$ and put $\delta = x_1 - a > 0$. Then obviously $g\big[[a, a+\delta)\big] = \{y_1\}$. Similarly we can show that condition v) is fulfilled. The other requirements of the lemma are evident. □

Lemma 3.6. *Let $E \subset \mathbb{R}$ be a compact interval and $f\colon E \to \mathbb{R}$. Assume that $I = (a,b) \subset E$ is an open interval such that $I \subset [f = 0]$. Moreover, suppose that $c, d \in [-1,1]$ such that $c \neq 0$ if $f(a) \neq 0$ and $d \neq 0$ if $f(b) \neq 0$. There are extra strong Świątkowski functions $g_1, g_2\colon \operatorname{cl} I \to [-1,1]$ such that* $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$ *and for $i \in \{1,2\}$, we have:*

i) $g_i(a) = c(\operatorname{sgn} f(a))^{i+1}, \quad g_i(b) = d(\operatorname{sgn} f(b))^{i+1}$,

ii) $g_i[I] = [-1,1]$,

iii) *if $f(a) \neq 0$, then $g_i[(a,z) \cap \mathcal{C}(g_i)] = [-1,1]$ for each $z \in (a,b)$,*

iv) *if $f(a) = 0$, then $[a, a+\delta) \subset [g_i = 0]$ for some $\delta > 0$,*

v) *if $f(b) \neq 0$, then $g_i[(z,b) \cap \mathcal{C}(g_i)] = [-1,1]$ for each $z \in (a,b)$,*

vi) *if $f(b) = 0$, then $(b-\delta, b] \subset [g_i = 0]$ for some $\delta > 0$.*

PROOF. Choose $t \in (a,b)$ and a strictly decreasing sequence $(x_n) \subset (a,t)$ such that $x_n \to a^+$. Put $x_0 = t$. Define the function $h\colon (a,t] \to [-1,1]$ by the formula:

$$h(x) = \begin{cases} 0 & \text{if } x = x_{n-1},\ n \in \mathbb{N}, \\ (-1)^{n-1} & \text{if } x = (x_{n-1} + x_n)/2,\ n \in \mathbb{N}, \\ \text{linear} & \text{in each interval of the form } [x_n, (x_{n-1} + x_n)/2] \\ & \quad \text{or } [(x_{n-1} + x_n)/2, x_{n-1}],\ n \in \mathbb{N}. \end{cases}$$

Then h is continuous on $(a,t]$ and $h\big[(a,t]\big] = [-1,1]$. Now fix an $i \in \{1,2\}$ and define two functions $\varphi_i, \psi_i\colon [a,t] \to [-1,1]$ as follows:

$$\varphi_i(x) = \begin{cases} h(x) & \text{if } x \in \bigcup_{n=1}^{\infty} [x_{4n-2i+2}, x_{4n-2i}], \\ 0 & \text{if } x \in \bigcup_{n=1}^{\infty} [x_{4n+2i-4}, x_{4n+2i-6}], \\ c(\operatorname{sgn} f(x))^{i+1} & \text{if } x = a, \end{cases}$$

$$\psi_i(x) = \begin{cases} h(x) & \text{if } x \in [x_{6-2i}, x_{4-2i}], \\ 0 & \text{if } x \in [x_{2i}, x_{2i-2}] \cup [a, x_4]. \end{cases}$$

Since $h(x_n) = 0$ for each $n \in \mathbb{N} \cup \{0\}$, functions φ_i and ψ_i are well defined and ψ_i is continuous. Moreover the function $\varphi_i \in \acute{\mathcal{S}}_{es}$ and $t \in \mathcal{C}(\varphi_i) \cap \mathcal{C}(\psi_i)$. Proceeding similarly we construct functions $\bar{\varphi}_i, \bar{\psi}_i \colon [t, b] \to [-1, 1]$ having the same properties as φ_i and ψ_i, respectively. Define the function $g_i \colon \operatorname{cl} I \to [-1, 1]$ by the formula:

$$g_i(x) = \begin{cases} \varphi_i(x) & \text{if } x \in [a, t] \text{ and } f(a) \neq 0, \\ \psi_i(x) & \text{if } x \in [a, t] \text{ and } f(a) = 0, \\ \bar{\varphi}_i(x) & \text{if } x \in [t, b] \text{ and } f(b) \neq 0, \\ \bar{\psi}_i(x) & \text{if } x \in [t, b] \text{ and } f(b) = 0. \end{cases}$$

Since $\varphi_i(t) = \psi_i(t) = \bar{\varphi}_i(t) = \bar{\psi}_i(t) = 0$, the function g_i is well defined. Moreover, $t \in \mathcal{C}(g_i)$, whence $g_i \in \acute{\mathcal{S}}_{es}$. Now assume that $f(a) = 0$ and put $\delta = x_4 - a > 0$. Then

$$g_i\big[[a, a + \delta)\big] = \psi_i\big[[a, a + \delta)\big] = \{0\}.$$

So, $[a, a + \delta) \subset [g_i = 0]$. Similarly we can show that condition vi) holds.

Finally, $\operatorname{sgn} \circ (g_1 g_2) \restriction I = 0 = \operatorname{sgn} \circ f \restriction I$ and $\big(\operatorname{sgn} \circ (g_1 g_2)\big)(x) = (\operatorname{sgn} \circ f)(x)$ for each $x \in \operatorname{bd} I$. Consequently, $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f \restriction \operatorname{cl} I$. The other requirements of the lemma are evident. □

4 Main results

Theorem 4.1. *Assume that $E \subset \mathbb{R}$ is a compact interval, the function $f \colon E \to \mathbb{R}$ is cliquish, the set $[f = 0]$ is simply open, and there is a G_δ-set $A \subset [f = 0]$ such that $I \cap A \neq \emptyset$ for each interval I in which the function f changes its sign. Then there are functions $g_1, \dots, g_4 \in \acute{\mathcal{S}}_{es}$ such that $f = g_1 \dots g_4$.*

Proof. First we will show that

$$\text{there are functions } g_1, g_2 \in \acute{\mathcal{S}}_{es} \text{ with } \operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f. \tag{7}$$

Define $C \stackrel{\text{df}}{=} \operatorname{bd}[f = 0]$. Observe that the set C is closed and since $[f = 0]$ is simply open, C is nowhere dense. Let $\mathcal{I}$ be the family of all components of $\mathbb{R} \setminus C$. By Lemma 3.3 there are pairwise disjoint families $\mathcal{I}_1, \dots, \mathcal{I}_4 \subset \mathcal{I}$ such that for each $j \in \{1, \dots, 4\}$ and $x \in C$ if x is not isolated in C from the left (from the right), then there is a sequence $(I_{j,n}) \subset \mathcal{I}_j$ with $\inf I_{j,n} \to x^-$ (with $\sup I_{j,n} \to x^+$, respectively). Observe that, since $[f=0]$ is simply open, we have only $I \cap [f=0] = \emptyset$ or $I \subset [f=0]$ for each $I \in \mathcal{I}$. Now define

$$P \stackrel{\text{df}}{=} \{x \in C : x \in \operatorname{bd} I \cap \operatorname{bd} I' \text{ for some } I, I' \in \mathcal{I} \text{ such that } I' \neq I\}. \tag{8}$$

Clearly, P is the set of all points which are bilaterally isolated in C. Let

$$A_1 \stackrel{\text{df}}{=} (A \cap C) \cup P. \tag{9}$$

Since A is a G_δ-set, A_1 is a G_δ-set, too. Then $C \setminus A_1$ is an F_σ-set, whence there is an increasing sequence (F_n) consisting of closed sets such that

$$C \setminus A_1 = \bigcup_{n \in \mathbb{N}} F_n. \tag{10}$$

Let $C_1 \stackrel{\text{df}}{=} C \setminus P$. Then obviously C_1 is closed, nowhere dense, and $C \setminus A_1 = C_1 \setminus A_1$. So, $F_n \subset C_1$ for each $n \in \mathbb{N}$ and $C_1 \subset \operatorname{cl} \bigcup \mathcal{I}_j$ for each $j \in \{1, \dots, 4\}$.

Define $F'_0 = \emptyset$. For each $n \in \mathbb{N}$, use four times Lemma 3.4 to construct a sequence of sets (F'_n) and an increasing sequence of families of intervals $(\mathcal{J}'_n)$ such that

$$\mathcal{J}'_n = \bigcup_{j=1}^{4} \mathcal{J}'_{j,n}, \tag{11}$$

$$F'_n = F_n \cup \bigcup_{k<n} \Big(F'_k \cup \bigcup_{I \in \mathcal{J}'_k} (\operatorname{bd} I \setminus A_1) \Big), \tag{12}$$

and for $j \in \{1, \dots, 4\}$,

$$\mathcal{J}'_{j,n} \subset \mathcal{I}_j, \tag{13}$$

$$\text{for each } I \in \mathcal{I}_j, \text{ if } F'_n \cap \operatorname{bd} I \neq \emptyset, \text{ then } I \in \mathcal{J}'_{j,n}, \tag{14}$$

$$\begin{aligned}&\text{for each } c \in F'_n, \text{ if } c \text{ is a right-hand (left-hand) limit point of } C_1,\\ &\text{then } c \text{ is a right-hand (left-hand) limit point of the union } \textstyle\bigcup \mathcal{J}'_{j,n},\end{aligned} \tag{15}$$

$$\begin{aligned}&\operatorname{cl} \bigcup_{J \in \mathcal{J}'_{j,n}} \{\inf J\} \subset F'_n \cup \bigcup_{J \in \mathcal{J}'_{j,n}} \{\inf J\} \text{ and}\\ &\operatorname{cl} \bigcup_{J \in \mathcal{J}'_{j,n}} \{\sup J\} \subset F'_n \cup \bigcup_{J \in \mathcal{J}'_{j,n}} \{\sup J\}.\end{aligned} \tag{16}$$

Observe that for each $k<n$, the set $B_k \overset{\mathrm{df}}{=} F'_k \cup \bigcup_{I\in\mathcal{J}'_k}(\operatorname{bd} I\setminus A_1)$ is closed. Indeed, fix a $k<n$ and let $x\in \operatorname{cl} B_k$. Then there is a sequence $(x_m)\subset B_k$ such that $x_m\to x$. If $(x_m)\subset F'_k$, then $x\in \operatorname{cl} F'_k = F'_k\subset B_k$. In the opposite case, without loss of generality we can assume that $(x_m)\subset \bigcup_{I\in\mathcal{J}'_k}\{\inf I\}\setminus A_1$, whence $x\in \operatorname{cl}\bigcup_{I\in\mathcal{J}'_k}\{\inf I\}$. By (16), $x\in F'_k\cup\bigcup_{I\in\mathcal{J}'_k}\{\inf I\}$. If we would have $x\in A_1$, then since $(x_m)\subset\bigcup_{I\in\mathcal{J}'_k}\{\inf I\}$ and $x_m\to x$, there was a sequence $(y_m)\subset\bigcup_{I\in\mathcal{J}'_k}\{\sup I\}$ such that $y_m\to x$, which contradicts (16). Consequently $x\in B_k$, which proves that the set B_k is closed. So, by (12), the set F'_n is also closed and $F'_n\subset C_1\setminus A_1$.

Now let $a_I \overset{\mathrm{df}}{=} \inf I$, $b_I\overset{\mathrm{df}}{=}\sup I$ for each $I\in\mathcal{I}$, and

$$\mathcal{I}_5 \overset{\mathrm{df}}{=} \Big\{I\in\mathcal{I} : I\notin \bigcup_{j=1}^{4}\mathcal{I}_j\Big\}.$$

Fix an $I\in\mathcal{I}$ and put

$$n_I = \begin{cases}\min\{n\in\mathbb{N} : I\in\mathcal{J}'_n\} & \text{if } I\in\bigcup\mathcal{J}'_n,\\ \operatorname{Ent}(1/|I|)+1 & \text{if } I\in\mathcal{I}\setminus\bigcup\mathcal{J}'_n.\end{cases}$$

Note that if $a_I\in P$, then there is $I_R\in\mathcal{I}$ such that $a_I=b_{I_R}$. Similarly, if $b_I\in P$, then there is $I_L\in\mathcal{I}$ such that $b_I=a_{I_L}$. Define

$$r_{a_I} = \begin{cases} 0 & \text{if } a_I\in A,\\ 2^{-1}|\operatorname{sgn} f(a_I)| & \text{if } a_I=b_{I_R} \text{ and } n_I\ge n_{I_R},\\ 2^{n_I-n_{I_R}-1}|\operatorname{sgn} f(a_I)| & \text{if } a_I=b_{I_R} \text{ and } n_I< n_{I_R},\\ 2^{-n}|\operatorname{sgn} f(a_I)| & \text{if } a_I\in F'_n\setminus\bigcup_{k<n}F'_k,\ n\in\mathbb{N},\end{cases}$$

$$r_{b_I} = \begin{cases} 0 & \text{if } b_I\in A,\\ 2^{-1}|\operatorname{sgn} f(b_I)| & \text{if } b_I=a_{I_L} \text{ and } n_I\ge n_{I_L},\\ 2^{n_I-n_{I_L}-1}|\operatorname{sgn} f(b_I)| & \text{if } b_I=a_{I_L} \text{ and } n_I< n_{I_L},\\ 2^{-n}|\operatorname{sgn} f(b_I)| & \text{if } b_I\in F'_n\setminus\bigcup_{k<n}F'_k,\ n\in\mathbb{N}.\end{cases}$$

Observe that $r_{a_I}, r_{b_I}\in[0,1]$. Moreover we can easily see that if components $J,J'\in\mathcal{I}$ and $a_J=b_{J'}$, then

$$2^{-n_J}r_{a_J} = 2^{-n_{J'}}r_{b_{J'}}. \tag{17}$$

By (8), if $x\in\operatorname{bd} I\cap P$, there is $I'\in\mathcal{I}$ such that $I'\neq I$ and $x\in\operatorname{bd} I\cap\operatorname{bd} I'$.

For each $x \in \operatorname{bd} I$ define

$$s(x) = \begin{cases} |\operatorname{sgn} f(x)| & \text{if } x \notin P \text{ or } x \in P \text{ and } I' \subset [f=0], \\ (-1)^{j+1}|\operatorname{sgn} f(x)| & \text{if } x \in P,\ I' \in \mathcal{I}_j \text{ for } j \in \{1,2,5\}, \\ & \quad \text{and } I' \cap [f=0] = \emptyset, \\ (-1)^{j+1} \operatorname{sgn} f(x) & \text{if } x \in P,\ I' \in \mathcal{I}_j \text{ for } j \in \{3,4\}, \\ & \quad \text{and } I' \cap [f=0] = \emptyset. \end{cases}$$

If $I \cap [f=0] = \emptyset$, assuming that $y_1 = r_{a_I}$ and $y_2 = r_{b_I}$, we construct the function $g_I \colon \operatorname{cl} I \to [0,1]$ which fulfills the requirements of Lemma 3.5. And if $I \subset [f=0]$, assuming that $c = r_{a_I} s(a_I)$ and $d = r_{b_I} s(b_I)$, we construct functions $g_{1,I}, g_{2,I} \colon \operatorname{cl} I \to [-1,1]$ which fulfill the requirements of Lemma 3.6.

Fix an $i \in \{1,2\}$. Define the function $g_i \colon E \to [-1,1]$ by the formula:

$$g_i(x) = \begin{cases} 0 & \text{if } x \in A \cap C, \\ 2^{-n_I} g_{i,I}(x) & \text{if } x \in \operatorname{cl} I \text{ and } I \subset [f=0], \\ (-1)^{j+1} 2^{-n_I} (\operatorname{sgn} f(x))^{i+1} g_I(x) & \text{if } x \in \operatorname{cl} I,\ I \cap [f=0] = \emptyset, \\ & \quad \text{and } I \in \mathcal{I}_j \text{ for } j \in \{1,2,5\}, \\ (-1)^{j+1} 2^{-n_I} (\operatorname{sgn} f(x))^{i} g_I(x) & \text{if } x \in \operatorname{cl} I,\ I \cap [f=0] = \emptyset, \\ & \quad \text{and } I \in \mathcal{I}_j \text{ for } j \in \{3,4\}, \\ 2^{-n} (\operatorname{sgn} f(x))^{i+1} & \text{if } x \in F'_n \setminus \left(\bigcup_{I \in \mathcal{I}} \operatorname{bd} I \cup F'_{n-1}\right), \\ & \quad n \in \mathbb{N}. \end{cases}$$

First we will show that the function g_i is well defined. If $x \in \operatorname{bd} I \cap A$ for some $I \in \mathcal{I}$, then $g_i(x) = 0$. Now let $x \notin A$ and $x \in \operatorname{bd} I \cap \operatorname{bd} I'$ for some $I, I' \in \mathcal{I}$ such that $I' \neq I$. Then obviously $x \in P$. Note that if $x \in [f=0] \cap P$, then $g_i(x) = 0$. So, let $(\operatorname{sgn} \circ f)(x) \neq 0$. Without loss of generality we can assume that $x = a_I = b_{I'}$. We consider the following cases.

<u>*Case* 1.</u> $I \subset [f=0]$ and $I' \subset [f=0]$.
Then, by (17) and since $s(a_I) = 1 = s(b_{I'})$, we have

$$\begin{aligned} g_i(a_I) &= 2^{-n_I} g_{i,I}(a_I) = 2^{-n_I} r_{a_I} s(a_I) (\operatorname{sgn} f(a_I))^{i+1} = \\ &= 2^{-n_{I'}} r_{b_{I'}} s(b_{I'}) (\operatorname{sgn} f(b_{I'}))^{i+1} = 2^{-n_{I'}} g_{i,I'}(b_{I'}) = g_i(b_{I'}). \end{aligned}$$

<u>*Case* 2.</u> $I \cap [f=0] = \emptyset$, $I' \subset [f=0]$, and $I \in \mathcal{I}_j$ for $j \in \{1,2,5\}$.
Since $g_I(a_I) = r_{a_I}$ and $s(b_{I'}) = (-1)^{j+1}$, then by (17)

$$\begin{aligned} g_i(a_I) &= (-1)^{j+1} 2^{-n_I} (\operatorname{sgn} f(a_I))^{i+1} g_I(a_I) = \\ &= 2^{-n_{I'}} r_{b_{I'}} s(b_{I'}) (\operatorname{sgn} f(b_{I'}))^{i+1} = 2^{-n_{I'}} g_{i,I'}(b_{I'}) = g_i(b_{I'}). \end{aligned}$$

Case 3. $I \cap [f = 0] = \emptyset$, $I' \subset [f = 0]$, and $I \in \mathcal{I}_j$ for $j \in \{3, 4\}$.
Since $g_I(a_I) = r_{a_I}$, $(\operatorname{sgn} f(a_I))^i = (\operatorname{sgn} f(b_{I'}))^{i+2}$, and

$$s(b_{I'}) = (-1)^{j+1} \operatorname{sgn} f(b_{I'}),$$

then by (17)

$$g_i(a_I) = (-1)^{j+1} 2^{-n_I} (\operatorname{sgn} f(a_I))^i g_I(a_I) = \\ = 2^{-n_{I'}} r_{b_{I'}} s(b_{I'}) (\operatorname{sgn} f(b_{I'}))^{i+1} = 2^{-n_{I'}} g_{i,I'}(b_{I'}) = g_i(b_{I'}).$$

So, the function g_i is well defined. Moreover, we can easily see that $\operatorname{sgn} \circ (g_1 g_2) = \operatorname{sgn} \circ f$.

Now we will show that

$$A_1' \stackrel{\text{df}}{=} A \cap C \subset \mathcal{C}(g_i). \tag{18}$$

Take an $x_0 \in A_1'$. Observe that $A_1' \subset [f = 0] \cap C$. If there is an $I \in \mathcal{I}$ such that $x_0 = b_I$, then by condition iii) of Lemma 3.5 or condition vi) of Lemma 3.6, respectively, the function g_i is continuous from the left at x_0.

In the opposite case take an $x_0 \in A_1' \setminus \{b_I : I \in \mathcal{I}\}$ and let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \varepsilon$ and define the set F as follows:

$$F \stackrel{\text{df}}{=} \begin{cases} \left(\operatorname{cl} \bigcup \mathcal{J}_{n_0}'\right) \setminus \left(I \cup \{x_0\}\right) & \text{if there is an } I \in \mathcal{I} \text{ such that } x_0 \in \operatorname{cl} I, \\ \operatorname{cl} \bigcup \mathcal{J}_{n_0}' & \text{otherwise.} \end{cases}$$

Observe that, by (16), the set F is closed. Put $\delta = \min\{2^{-n_0}, \operatorname{dist}(F, x_0)\}$. (If $C \setminus A_1 = \emptyset$, then $\delta = 2^{-n_0}$.) Since $x_0 \notin F$, we have $\operatorname{dist}(F, x_0) > 0$. Consequently $\delta > 0$. Choose a $\delta' \in (0, \delta)$ such that $x_0 - \delta' \notin \bigcup \mathcal{I}$. (Recall that x_0 is not isolated in C from the left.) Observe that if $I \in \mathcal{I}_5$ and $I \subset (x_0 - \delta', x_0)$, then $|I| < 2^{-n_0}$ and $n_I > n_0$. For every $x \in (x_0 - \delta', x_0)$, we have $x \notin F$, which shows that $x \notin \bigcup_{I \in \mathcal{J}_{n_0}'} \operatorname{cl} I$. Condition (15) yields $F_{n_0}' \subset \operatorname{cl} \bigcup \mathcal{J}_{n_0}'$, whence $F_{n_0}' \subset F \cup \{x_0\}$ and in particular $x \notin F_{n_0}'$. Thus

$$|g_i(x) - g_i(x_0)| = |g_i(x)| \le 2^{-n_0} < \varepsilon.$$

So, in this case the function g_i is continuous from the left at x_0, too. Similarly we can prove that the function g_i is continuous from the right at each point $x_0 \in A_1'$. Consequently $A_1' \subset \mathcal{C}(g_i)$.

Now we will prove that

$$\mathop{\forall}_{n \in \mathbb{N}} \mathop{\forall}_{\delta > 0} \Big(x \in F_n' \setminus \{b_I : I \in \mathcal{I}\} \Rightarrow g_i[(x - \delta, x) \cap \mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}] \Big). \tag{19}$$

Let $n \in \mathbb{N}$, $\delta > 0$ and $x \in F_n' \setminus \{b_I : I \in \mathcal{I}\}$. Then $x \notin P$, whence for $j \in \{1, \dots, 4\}$, by (15), there is an $I_j \in \mathcal{J}_{j,n}'$ with $I_j \subset (x - \delta, x)$. Notice that $\max\{n_{I_j} : j \in \{1, \dots, 4\}\} \le n$. So,

$$g_i[(x-\delta,x)\cap\mathcal{C}(g_i)] \supset \bigcup_{j=1}^{4} g_i[I_j \cap \mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}] \setminus \{0\}.$$

If there is an $I \in \mathcal{I}$ such that $I \subset [f = 0]$ and $(x - \delta, x) \cap I \neq \emptyset$, then since $g_{i,I} \in \acute{\mathcal{S}}_{es}$ and conditions v) and vi) of Lemma 3.6 hold, we have

$$g_i[(x-\delta,x)\cap\mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}].$$

In the opposite case, since $x \notin P$, the function f changes its sign in each left-hand neighborhood of x. Hence, by assumption

$$\emptyset \neq (x-\delta,x)\cap A_1' \subset (x-\delta,x)\cap\mathcal{C}(g_i)\cap[g_i = 0]$$

and finally

$$g_i[(x-\delta,x)\cap\mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}].$$

Similarly we can prove that

$$\forall_{n\in\mathbb{N}} \forall_{\delta>0} \Big(x \in F_n' \setminus \{a_I : I \in \mathcal{I}\} \Rightarrow g_i[(x, x+\delta)\cap\mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}]\Big).$$

Further we will show that

for each $I \in \mathcal{I}$, if $x \in \operatorname{bd} I$, then $g_i(x) \in g_i\big[\mathrm{I}[x,t] \cap \mathcal{C}(g_i)\big]$ for each $t \neq x$ such that $\mathrm{I}[x,t] \subset E$. (20)

Let $I \in \mathcal{I}$, $x \in \operatorname{bd} I$, and $t \neq x$. Without loss of generality we can assume that $x \notin \mathcal{C}(g_i)$. Hence by (18), $x \notin A_1'$. Let $t < x$. (If $t > x$ we proceed analogously.)

First assume that $x = b_I$. We consider two cases.

Case 1. $I \subset [f = 0]$.

Then $g_i = 2^{-n_I} g_{i,I}$ on $\operatorname{cl} I$. If $g_{i,I}(b_I) \neq 0$, then by condition v) of Lemma 3.6 we obtain that

$$g_{i,I}[(t_0, b_I)\cap\mathcal{C}(g_{i,I})] = [-1, 1],$$

where $t_0 = \sup\{a_I, t\}$. Hence and by the definition of g_i we have

$$g_i[[t,x]\cap\mathcal{C}(g_i)] \supset g_i[(t_0,b_I)\cap\mathcal{C}(g_i)] = [-2^{-n_I}, 2^{-n_I}] \ni g_i(b_I) = g_i(x).$$

If $g_{i,I}(b_I) = 0$, then by condition vi) of Lemma 3.6 there is a $\delta > 0$ such that

$$(b_I - \delta, b_I] \subset [g_{i,I} = 0] \subset [g_i = 0],$$

whence condition (20) holds.

Case 2. $I \cap [f = 0] = \emptyset$.

Then $g_i = 2^{-n_I} g_I$ or $g_i = -2^{-n_I} g_I$. If $g_i(b_I) = 0$, then $b_I \in A \cap C = A_1' \subset \mathcal{C}(g_i)$, which is impossible, or $b_I \in P$. But if $b_I \in P$, then by condition iii) of Lemma 3.5 and by condition iv) of Lemma 3.6, we would also have $x = b_I \in \mathcal{C}(g_i)$, a contradiction. So, $g_i(b_I) \neq 0$. By condition v) of Lemma 3.5 there is a $z \in (t, b_I) \cap \mathcal{C}(g_i)$ such that $g_i(z) = g_i(b_I) = g_i(x)$, whence condition (20) holds.

Now let $x = a_I$. We can assume that $x \notin P$. Since $x \notin A_1'$, then $x \notin A_1$. Consequently $x \in C \setminus A_1 = \bigcup_{n\in\mathbb{N}} F_n$, whence $x \in F_n' \setminus \bigcup_{k<n} F_k'$ for some $n \in \mathbb{N}$. Since $g_i(a_I) = 0$, $g_i(a_I) = 2^{-n-n_I}$, or $g_i(a_I) = -2^{-n-n_I}$, we have $|g_i(a_I)| < 2^{-n}$. Therefore, by (19),

$$g_i[[t, x] \cap \mathcal{C}(g_i)] \supset [-2^{-n}, 2^{-n}] \ni g_i(a_I) = g_i(x),$$

which completes the proof of (20).

To complete the proof of (7) we must show that $g_i \in \acute{\mathcal{S}}_{es}$. Let $\alpha, \beta \in E$, $\alpha < \beta$, and $y \in \mathrm{I}[g_i(\alpha), g_i(\beta)]$. Assume that $g_i(\alpha) \le g_i(\beta)$. (The other case is similar.) If $\alpha, \beta \in \mathrm{cl}\, I$ for some $I \in \mathcal{I}$, then by (20) and since $g_{1,I}, g_{2,I}, g_I \in \acute{\mathcal{S}}_{es}$, there is an $x_0 \in [\alpha, \beta] \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$. So, assume that the opposite case holds.

Assume that $y \ge 0$. (The case $y < 0$ is analogous.) If $\beta \in A_1'$, then $y = g_i(\beta) = 0$ and by (18), $\beta \in \mathcal{C}(g_i)$. So, let $\beta \notin A_1'$. We consider two cases.

Case 1. $\beta \notin \bigcup_{n\in\mathbb{N}} F_n'$ or $\beta \in \{b_I : I \in \mathcal{I}\}$.

Then there is an $I \in \mathcal{I}$ such that $\beta \in \mathrm{cl}\, I$, $\alpha \notin \mathrm{cl}\, I$ and $\beta \neq a_I$. If $y \in \mathrm{I}[g_i(a_I), g_i(\beta)]$, then by (20) and since $g_{1,I}, g_{2,I}, g_I \in \acute{\mathcal{S}}_{es}$, there is an $x_0 \in [a_I, \beta] \cap \mathcal{C}(g_i) \subset [\alpha, \beta] \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$.

Now let $y \in [0, g_i(a_I))$. Then $g_i(a_I) > 0$, whence $a_I \notin A$.

If $a_I \in C \setminus A_1 = \bigcup_{n\in\mathbb{N}} F_n$, then $a_I \in F_n' \setminus \bigcup_{k<n} F_k'$ for some $n \in \mathbb{N}$ and $g_i(a_I) = 2^{-n-n_I} < 2^{-n}$. By (19),

$$y \in [0, g_i(a_I)) \subset [0, 2^{-n}] \subset g_i[(\alpha, a_I) \cap \mathcal{C}(g_i)].$$

So, there is an $x_0 \in (\alpha, a_I) \cap \mathcal{C}(g_i) \subset [\alpha, \beta] \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$.

If $a_I \in P$, then there is an $I' \in \mathcal{I}$ such that $a_I = b_{I'}$. Since $g_i(a_I) > 0$, we have $I \subset [f = 0]$ or $I' \subset [f = 0]$. Assume that the first inclusion holds. (If

$I' \subset [f = 0]$, then we proceed similarly.) By condition iii) of Lemma 3.6 and since $g_i = 2^{-n_I} g_{i,I}$ on $\operatorname{cl} I$ we obtain that

$$y \in [0, g_i(a_I)] \subset [0, 2^{-n_I}] \subset g_i[(a_I, \beta) \cap \mathcal{C}(g_i)].$$

Hence there is an $x_0 \in (a_I, \beta) \cap \mathcal{C}(g_i) \subset [\alpha, \beta] \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$.

Case 2. $\beta \in \bigcup_{n\in\mathbb{N}} F'_n \setminus \{b_I : I \in \mathcal{I}\}$.
Then $\beta \in F'_n \setminus F'_{n-1}$ for some $n \in \mathbb{N}$. By (19),

$$y \in [0, g_i(\beta)] \subset [0, 2^{-n}] \subset g_i[(\alpha, \beta) \cap \mathcal{C}(g_i)].$$

Consequently, there is an $x_0 \in (\alpha, \beta) \cap \mathcal{C}(g_i) \subset [\alpha, \beta] \cap \mathcal{C}(g_i)$ with $g_i(x_0) = y$. This completes the proof of condition (7).

Now define the function $\tilde{f} \colon E \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} \dfrac{f}{g_1 g_2}(x) & \text{if } f(x) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Notice that $\tilde{f}$ is cliquish. Indeed, it is obvious that

$$\mathcal{C}(\tilde{f}) \supset \mathcal{C}(f) \cap \mathcal{C}(g_1) \cap \mathcal{C}(g_2) \cap U,$$

where

$$U \overset{\text{df}}{=} \operatorname{int}[f = 0] \cup \operatorname{int}[f \neq 0].$$

Observe that $E \setminus U = \operatorname{bd}[f = 0] = C$ is nowhere dense, whence U is residual. Since the sets $\mathcal{C}(f)$, $\mathcal{C}(g_1)$, and $\mathcal{C}(g_2)$ are also residual, the set $\mathcal{C}(\tilde{f})$ is dense. Hence the function $\tilde{f}$ is cliquish.

Clearly $\tilde{f} > 0$ on E. So, the function $\ln \circ \tilde{f} \colon E \to \mathbb{R}$ is cliquish. By [6, Corollary 3.4], there are functions $h_1, h_2 \in \acute{\mathcal{S}}_{es}$ such that $\ln \circ \tilde{f} = h_1 + h_2$.

Define $g_3 \overset{\text{df}}{=} \exp \circ h_1$ and $g_4 \overset{\text{df}}{=} \exp \circ h_2$. By Lemma 3.2, $g_3, g_4 \in \acute{\mathcal{S}}_{es}$. Clearly

$$f = g_1 g_2 \tilde{f} = g_1 g_2 (\exp \circ h_1)(\exp \circ h_2) = g_1 \dots g_4,$$

which completes the proof. □

Theorem 4.2. *Let $f \colon \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent*:

i) *there are functions $g_1, \dots, g_4 \in \acute{\mathcal{S}}_{es}$ such that $f = g_1 \dots g_4$,*

ii) *there is a $k \in \mathbb{N}$ and functions $g_1, \dots, g_k \in \acute{\mathcal{S}}_{es}$ such that $f = g_1 \dots g_k$,*

iii) *the function f is cliquish, the set $[f = 0]$ is simply open, and there is a G_δ-set $A \subset [f = 0]$ such that $I \cap A \neq \emptyset$ for every interval I in which f changes its sign.*

PROOF. The implication i) $\Rightarrow$ ii) is evident, while the implication ii) $\Rightarrow$ iii) follows by [6, Theorem 4.1].

iii) $\Rightarrow$ i). Put $E = [-\pi/2, \pi/2]$. Define the function $\tilde{f} \colon E \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} (f \circ \tan)(x) & \text{if } x \in (-\pi/2, \pi/2), \\ 0 & \text{if } x \in \{-\pi/2, \pi/2\}. \end{cases}$$

Then clearly $\tilde{A} \overset{\text{df}}{=} \arctan[A] \cup \{-\pi/2, \pi/2\} \subset [\tilde{f} = 0]$ is a G_δ-set, the function $\tilde{f}$ is cliquish, and by [1], the set $[\tilde{f} = 0] = \arctan\big[[f = 0]\big] \cup \{-\pi/2, \pi/2\}$ is simply open. Moreover for each interval $I \subset E$, if the function $\tilde{f}$ changes its sign in I, then $I \cap \tilde{A} \neq \emptyset$. So, by Theorem 4.1, there are functions $\tilde{g}_1, \ldots, \tilde{g}_4 \in \acute{\mathcal{S}}_{es}$ such that $\tilde{f} = \tilde{g}_1 \ldots \tilde{g}_4$. For $i \in \{1, \ldots, 4\}$ define $g_i = \tilde{g}_i \circ \arctan$ and notice that by Lemma 3.2, $g_i \in \acute{\mathcal{S}}_{es}$. Clearly

$$f = \tilde{f} \circ \arctan = (\tilde{g}_1 \circ \arctan) \ldots (\tilde{g}_4 \circ \arctan) = g_1 \ldots g_4,$$

which completes the proof. □

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