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A LARGE GROUP OF ABSOLUTELY NONMEASURABLE ADDITIVE FUNCTIONS

Abstract

By assuming the Continuum Hypothesis, it is proved that there exists a subgroup of $\mathbf{R}^{\mathbf{R}}$ of cardinality strictly greater than the cardinality of the continuum, all nonzero members of which are absolutely nonmeasurable additive functions.

The existence of Lebesgue-nonmeasurable sets and functions on the real line $\mathbf{R}$ is a very important fact of analysis and plays a seminal role for the foundations of contemporary mathematics (because, e.g., this fact is closely connected with uncountable forms of the Axiom of Choice). In some questions of analysis nonmeasurable sets and functions turn out to be endowed with additional algebraic structure. For instance, one may be required to have a Lebesgue-nonmeasurable subgroup of the additive group $\mathbf{R}$ or a Lebesgue-nonmeasurable homomorphism of $\mathbf{R}$ into itself. As a rule, one needs more delicate constructions for proving the existence of such Lebesgue-nonmeasurable algebraic objects. The following example illustrates this circumstance.

Example 1. Denote by λ the standard Lebesgue measure on the real line $\mathbf{R}$. As is well known (see, e.g., [11, Chapter 1, p. 4]), $\mathbf{R}$ admits a partition $\{A, B\}$ such that the set A is of λ-measure zero and the set B is of first category in $\mathbf{R}$. This fact easily implies that there exists a subset B' of B which is not λ-measurable. Consequently, B' is of first category, but is not Lebesgue measurable. The analogous question can be posed for subgroups of $\mathbf{R}$. Namely, does there exist a subgroup of $\mathbf{R}$ which is of first category, but is not λ-measurable?

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The answer is positive under the Continuum Hypothesis (or under the much weaker Martin's Axiom), but the corresponding techniques need the method of transfinite induction. Moreover, assuming Martin's Axiom, there exists a subset S of $\mathbf{R}$ that is a generalized Sierpiński set and simultaneously a vector space over the field $\mathbf{Q}$ of all rational numbers. This set S is of first category in $\mathbf{R}$, but every subset of S of cardinality continuum is not λ-measurable. Extensive information about Sierpiński sets and generalized Sierpiński sets may be found in [9] (see also [11, Chapter 20]).

A number of works are devoted to algebraic properties of various families of nonmeasurable functions (see, e.g., [1, 2, 5, 10, 12]). In this paper we will construct, with the aid of the Continuum Hypothesis, a family $\mathcal{F} \subset \mathbf{R}^{\mathbf{R}}$ of functions such that $\mathrm{card}(\mathcal{F})$ is strictly greater than the cardinality of the continuum, $\mathcal{F}$ itself is a group with respect to the standard addition operation, and each nonzero member from $\mathcal{F}$ is an absolutely nonmeasurable additive function.

For our further purposes, we need some auxiliary notions and statements.

Let μ be a nonzero measure defined on a σ-algebra of subsets of a nonempty set E. As usual, we denote by the symbol $\mathrm{dom}(\mu)$ the domain of μ (i.e., the σ-algebra of all μ-measurable sets) and by the symbol $\mathcal{I}(\mu)$ the σ-ideal generated by the family of all μ-measure zero sets.

We say that μ is a *continuous* (or *diffused*) measure on E if $\{x\} \in \mathrm{dom}(\mu)$ and $\mu(\{x\}) = 0$ for all elements $x \in E$.

Let $\mathcal{M}$ be some class of measures on E (in general, their domains are various σ-algebras of subsets of E) and let $f : E \to \mathbf{R}$ be a function.

We shall say that f is *absolutely nonmeasurable* with respect to $\mathcal{M}$ if there exists no measure from $\mathcal{M}$ for which f is measurable.

Accordingly, we shall say that a set $X \subset E$ is absolutely nonmeasurable with respect to $\mathcal{M}$ if the characteristic function (i.e., indicator) of X is absolutely nonmeasurable with respect to $\mathcal{M}$.

Example 2. Recall that a measure μ on the real line $\mathbf{R}$ is *translation quasi-invariant* if both $\mathrm{dom}(\mu)$ and $\mathcal{I}(\mu)$ are translation invariant classes of subsets of $\mathbf{R}$. If, in addition, the values $\mu(X)$, where $X \in \mathrm{dom}(\mu)$, are preserved under all translations of X (i.e., $\mu(X + h) = \mu(X)$ for all $h \in \mathbf{R}$), then μ is called a *translation invariant* measure on $\mathbf{R}$. Denote by $\mathcal{M}_1(\mathbf{R})$ (respectively, by $\mathcal{M}_2(\mathbf{R})$) the class of all those translation quasi-invariant (respectively, translation invariant) measures on $\mathbf{R}$ which extend λ. Clearly, we have the proper inclusion $\mathcal{M}_2(\mathbf{R}) \subset \mathcal{M}_1(\mathbf{R})$. Recall that any selector of the quotient-group $\mathbf{R}/\mathbf{Q}$ is called a Vitali set in $\mathbf{R}$. It is widely known that all Vitali sets are absolutely nonmeasurable with respect to the class $\mathcal{M}_2(\mathbf{R})$. Moreover, there exists a Vitali set that is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on $\mathbf{R}$. On the other hand, there exists a Vitali set that is not absolutely nonmeasurable with respect to the class $\mathcal{M}_1(\mathbf{R})$. (For more details on the above-mentioned facts, see [6].) Notice also that the class $\mathcal{M}_2(\mathbf{R})$ is quite large (cf. [3, 13]). Among its members one may encounter even nonseparable extensions of λ (see, for instance, [4] and [8]).

In the sequel, for a given nonempty set E, we shall denote by $\mathcal{M}(E)$ the class of all nonzero σ-finite continuous measures on E.

Of course, any function $f : E \to \mathbf{R}$ that is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$ can be regarded as an utterly nonmeasurable real-valued function on E. In order to describe such functions, we need the classical notion of a universal measure zero subset of $\mathbf{R}$.

Let $Z \subset \mathbf{R}$. We recall that Z has *universal measure zero* if for any σ-finite continuous Borel measure μ on $\mathbf{R}$, the equality $\mu^*(Z) = 0$ holds, where μ^* denotes the outer measure associated with μ. Equivalently, we may say that $Z \subset \mathbf{R}$ has universal measure zero if there exists no nonzero σ-finite continuous Borel measure on Z (where Z is assumed to be endowed with the induced topology).

Some important properties of universal measure zero sets are discussed in [9].

The following auxiliary proposition yields a characterization of absolutely nonmeasurable functions with respect to the class $\mathcal{M}(E)$.

Lemma 1. *For any function $f : E \to \mathbf{R}$, these two assertions are equivalent:*

(1) *f is absolutely nonmeasurable with respect to $\mathcal{M}(E)$;*

(2) *the range of f is a universal measure zero subset of $\mathbf{R}$ and, for each point $t \in \mathbf{R}$, the set $f^{-1}(t)$ is at most countable.*

The proof of this lemma is not difficult and may be found in [7].

Remark 1. Denote by $\mathbf{c}$ the cardinality of the continuum. It directly follows from Lemma 1 that if $\mathrm{card}(E) > \mathbf{c}$, then there exist no functions on E which are absolutely nonmeasurable with respect to $\mathcal{M}(E)$. More precisely, the existence on E of an absolutely nonmeasurable function with respect to $\mathcal{M}(E)$ is equivalent to the existence of a universal measure zero set $Z \subset \mathbf{R}$ with $\mathrm{card}(Z) = \mathrm{card}(E)$. Thus, the following two assertions are equivalent:

(a) there exists a function $f : \mathbf{R} \to \mathbf{R}$ absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$;

(b) there exists a universal measure zero set $Z \subset \mathbf{R}$ with $\mathrm{card}(Z) = \mathbf{c}$.

Every Luzin set on $\mathbf{R}$ has universal measure zero and, under Martin's Axiom, every generalized Luzin set on $\mathbf{R}$ also has universal measure zero.

Extensive information about Luzin sets and generalized Luzin sets may be found in [9] (see also [11, Chapter 20]).

Remark 2. Several classical constructions (within **ZFC** theory) of uncountable universal measure zero subsets of $\mathbf{R}$ are known. Those constructions belong to Luzin, Hausdorff, Sierpiński, Marczewski, and others (see also a more recent paper [14] where a quite short argument is given). According to them, every nonempty perfect set $P \subset \mathbf{R}$ contains an uncountable universal measure zero subset. It was also shown that there exists a model of **ZFC** theory in which the Continuum Hypothesis fails to be true and every universal measure zero subset of $\mathbf{R}$ has cardinality less than or equal to ω_1, where ω_1 stands, as usual, for the least uncountable cardinal number (for more details, see [9] and the references therein).

Uncountable universal measure zero subsets of $\mathbf{R}$ can carry a certain algebraic structure. The following auxiliary proposition will be useful for our purposes.

Lemma 2. *There exists (within* **ZFC** *theory) an uncountable universal measure zero set* $Z \subset \mathbf{R}$ *which simultaneously is a vector space over* $\mathbf{Q}$.

This lemma is well known (see, e.g., [12] where a much deeper result is presented).

Remark 3. The assertion of Lemma 2 directly follows from one general statement of metamathematical character. We would like to formulate this statement as a metatheorem.

Let $S(X)$ be a property of a subset X of a Polish space. Suppose that the following conditions are satisfied:

(a) if $S(X)$ and $Y \subset X$, then $S(Y)$;

(b) if $\{X_i : i \in I\}$ is a countable family of subsets of a Polish space and $S(X_i)$ for all $i \in I$, then $S(\cup\{X_i : i \in I\})$;

(c) if $S(X)$ and $S(Y)$, then $S(X \times Y)$;

(d) if B_1 and B_2 are Borel subsets of Polish spaces, $h : B_1 \to B_2$ is an injective Borel mapping and X is a subset of B_1 with $S(X)$, then $S(h(X))$;

(e) there exists at least one Polish space containing an uncountable set X such that $S(X)$.

Then there exists an uncountable vector space $Z \subset \mathbf{R}$ (over $\mathbf{Q}$) such that $S(Z)$.

Let us sketch the proof. As is well known, there exists a nonempty perfect set $P \subset \mathbf{R}$ that is linearly independent over $\mathbf{Q}$. The conditions (d) and (e) imply that there exists an uncountable set $X \subset P$ satisfying $S(X)$. Clearly, this set X is also linearly independent over $\mathbf{Q}$. The conditions (b), (c) and (d) imply that for the set

$$Y = \{(q_1x_1, \dots, q_nx_n) : 0 < n < \omega, (q_1, \dots, q_n) \in (\mathbf{Q} \setminus \{0\})^n, (x_1, \dots, x_n) \in X^n\}$$

the property $S(Y)$ holds true. Consider an arbitrary element $(q_1x_1, \dots, q_nx_n)$ of Y, where $n > 0$ is a natural number, $(q_1, \dots, q_n) \in (\mathbf{Q} \setminus \{0\})^n$ and $(x_1, \dots, x_n) \in X^n$. We shall say that this element is admissible if $x_i \neq x_j$ for any two distinct natural indices $i \in [1, n]$ and $j \in [1, n]$. Let Y' denote the set of all admissible elements of Y. In view of condition (a), we have $S(Y')$. Now, for every natural number $n > 0$, define the set T_n by the equality

$$T_n = \{(x_1, \dots, x_n) \in P^n : (\forall i \in [1, n])(\forall j \in [1, n])(i \neq j \Rightarrow x_i \neq x_j)\}$$

and also define the set

$$T'_n = \{(q_1x_1, \dots, q_nx_n) : (x_1, \dots, x_n) \in T_n,\ (q_1, \dots, q_n) \in (\mathbf{Q} \setminus \{0\})^n\}.$$

Obviously, both T_n and T'_n are Borel subsets of the Euclidean space $\mathbf{R}^n$. Since all spaces $\mathbf{R}^n$ $(n < \omega)$ are canonically embedded in the space $\mathbf{R}^\omega$, the set $\cup\{T'_n : 0 < n < \omega\}$ is a Borel subset of $\mathbf{R}^\omega$, and $Y' \subset \cup\{T'_n : 0 < n < \omega\}$. Further, consider the Borel mapping

$$g : \cup\{T'_n : 0 < n < \omega\} \to \mathbf{R}$$

given by the formula

$$g(y_1, \dots, y_n) = y_1 + \cdots + y_n, \qquad (0 < n < \omega,\ (y_1, \dots, y_n) \in T'_n).$$

This mapping g is such that, for any $t \in \mathbf{R}$, the set $g^{-1}(t)$ is finite. According to a classical theorem of descriptive set theory, g admits a representation in the form $g = \cup\{g_k : k < \omega\}$, where all g_k are injective Borel mappings and are defined on pairwise disjoint Borel sets. It can easily be checked that the set

$$Z = g(Y') \cup \{0\} = (\cup\{g_k(Y') : k < \omega\}) \cup \{0\}$$

is an uncountable vector space over $\mathbf{Q}$ and the relation $S(Z)$ holds true.

The next auxiliary proposition belongs to infinite combinatorics and states the existence of a quite large almost disjoint family of subsets of a given infinite set. This proposition is crucial for obtaining the main result of the paper.

Lemma 3. *Let Ξ be an arbitrary infinite set. There exists a family $\{\Xi_j : j \in J\}$ of subsets of Ξ such that:*

(1) $\mathrm{card}(J) > \mathrm{card}(\Xi)$*;*

(2) $\mathrm{card}(\Xi_j) = \mathrm{card}(\Xi)$ *for each index $j \in J$;*

(3) $\mathrm{card}(\Xi_j \cap \Xi_{j'}) < \mathrm{card}(\Xi)$ *for any two distinct indices $j \in J$ and $j' \in J$.*

We omit the proof of this lemma. It is based on a rather standard argument by transfinite recursion, and we refer the reader to the classical monograph by Sierpiński [15, pp. 451–452], where Lemma 3 is proved in detail.

Let $(V,+)$ be a vector space (over some field of scalars) and let $\{V_j : j \in J\}$ be a family of vector subspaces of V.

We shall say that this family is *admissible* if, for any finite sequence j_0, j_1, j_2, ..., j_k of distinct indices from J, the relation

$$\mathrm{card}(V_{j_0} \cap (V_{j_1} + V_{j_2} + \cdots + V_{j_k})) < \mathrm{card}(V)$$

holds true.

The next proposition guarantees the existence of a large admissible family of vector subspaces of an uncountable vector space (over $\mathbf{Q}$).

Lemma 4. *Let V be an uncountable vector space over $\mathbf{Q}$. There exists an admissible family $\{V_j : j \in J\}$ of vector subspaces of V such that:*

(1) $\mathrm{card}(J) > \mathrm{card}(V)$*;*

(2) $\mathrm{card}(V_j) = \mathrm{card}(V)$ *for all* $j \in J$.

PROOF. Let Ξ be a basis of V. Clearly, we have the equality

$$\mathrm{card}(\Xi) = \mathrm{card}(V).$$

Let $\{\Xi_j : j \in J\}$ be a family of subsets of Ξ satisfying the relations (1) - (3) of Lemma 3. For each index $j \in J$, let us put

$$V_j = \mathrm{span}_{\mathbf{Q}}(\Xi_j).$$

It is not difficult to check that the obtained family $\{V_j : j \in J\}$ of vector subspaces of V is as required. □

Now, we are ready to establish the following statement.

Theorem 1. *Under the Continuum Hypothesis, there exists a family $\mathcal{F} \subset \mathbf{R}^{\mathbf{R}}$ satisfying the following relations:*

(1) $\mathrm{card}(\mathcal{F}) > \mathbf{c}$*;*

(2) *$\mathcal{F}$ is a vector space over $\mathbf{Q}$;*

(3) *all functions from $\mathcal{F}$ are homomorphisms of the additive group $\mathbf{R}$ into itself;*

(4) *all nonzero functions from $\mathcal{F}$ are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.*

PROOF. Let V be an uncountable universal measure zero set in $\mathbf{R}$ which simultaneously is a vector space over $\mathbf{Q}$. As was already mentioned, such a V does exist (see Lemma 2).

Let $\{V_j : j \in J\}$ be an admissible family of vector subspaces of V satisfying relations (1) and (2) of Lemma 4.

Consider $\mathbf{R}$ as a vector space over $\mathbf{Q}$. In view of the supposed equality $\mathbf{c} = \omega_1$, the vector space $\mathbf{R}$ is isomorphic to each vector space from the family $\{V_j : j \in J\}$.

For any $j \in J$, denote by $f_j : \mathbf{R} \to V_j$ some isomorphism between $\mathbf{R}$ and V_j.

Notice that the family of functions $\{f_j : j \in J\}$ is linearly independent over $\mathbf{Q}$. Indeed, consider any linear (over $\mathbf{Q}$) combination

$$q_0 f_{j_0} + q_1 f_{j_1} + \cdots + q_k f_{j_k},$$

where $k \geq 1$, all coefficients $q_0, q_1, \ldots, q_k$ are nonzero, and $j_0, j_1, \ldots, j_k$ are distinct indices from J. Since we have

$$\mathrm{card}(V_{j_0}) = \mathbf{c}, \quad \mathrm{card}(V_{j_0} \cap (V_{j_1} + \cdots + V_{j_k})) \leq \omega,$$

there exists $y \in V_{j_0} \setminus (V_{j_1} + \cdots + V_{j_k})$. Since $f_{j_0}(\mathbf{R}) = V_{j_0}$, we can find $x \in \mathbf{R}$ such that $y = f_{j_0}(x)$. Now, it readily follows that

$$(q_0 f_{j_0} + q_1 f_{j_1} + \cdots + q_k f_{j_k})(x) \neq 0,$$

so $q_0 f_{j_0} + q_1 f_{j_1} + \cdots + q_k f_{j_k}$ is not identically equal to zero.

Further, we put

$$\mathcal{F} = \mathrm{span}_{\mathbf{Q}}\{f_j : j \in J\}$$

and we claim that $\mathcal{F}$ is the required family of functions.

Notice that the relations (1), (2), and (3) of the theorem trivially hold by virtue of the definition of $\mathcal{F}$. So it remains to verify the validity of relation (4).

In view of Lemma 1, we must check that if f is an arbitrary function from $\mathcal{F} \setminus \{0\}$, then the set $\mathrm{ran}(f)$ is universal measure zero and the set $f^{-1}(t)$ is at most countable for every point $t \in \mathbf{R}$.

First, observe that if $f \in \mathcal{F}$, then $\mathrm{ran}(f) \subset V$, hence $\mathrm{ran}(f)$ is indeed universal measure zero.

Further, if $f \in \mathcal{F} \setminus \{0\}$, then f admits a unique representation in the form

$$f = q_0 f_{j_0} + q_1 f_{j_1} + \cdots + q_k f_{j_k},$$

where k is a natural number, $j_0, j_1, j_2, \ldots, j_k$ are distinct indices from J, and $q_0, q_1, \ldots, q_k$ are nonzero rational numbers.

It suffices to demonstrate that, for any $t \in V$, the set $f^{-1}(t)$ is at most countable. We will show this by induction on k.

If $k = 0$, then $f = q_0 f_{j_0}$, where $q_0 \neq 0$. In this case f is an isomorphism between $\mathbf{R}$ and V_{j_0}, and the set $f^{-1}(t)$ either is empty or is a singleton.

Assume that our assertion has already been proved for natural numbers strictly smaller than k and suppose to the contrary that, for f represented in the above-mentioned form, there exists $\tau \in V$ such that

$$\mathrm{card}(f^{-1}(\tau)) \geq \omega_1.$$

It is not difficult to see that, in this case, the set

$$\{x \in \mathbf{R} : q_0 f_{j_0}(x) = (-q_1 f_{j_1} - q_2 f_{j_2} - \cdots - q_k f_{j_k})(x)\}$$

is uncountable. Since the Continuum Hypothesis is assumed and the family of vector spaces $\{V_j : j \in J\}$ is admissible, the vector space

$$V_{j_0} \cap (V_{j_1} + V_{j_2} + \cdots + V_{j_k})$$

must be at most countable. But we obviously have

$$\mathrm{ran}(q_0 f_{j_0}) \subset V_{j_0},$$

$$\mathrm{ran}(-q_1 f_{j_1} - q_2 f_{j_2} - \cdots - q_k f_{j_k}) \subset V_{j_1} + V_{j_2} + \cdots + V_{j_k}.$$

The latter two inclusions readily imply that there exists a point

$$t' \in V_{j_0} \cap (V_{j_1} + V_{j_2} + \cdots + V_{j_k})$$

such that

$$\mathrm{card}((-q_1 f_{j_1} - q_2 f_{j_2} - \cdots - q_k f_{j_k})^{-1}(t')) \geq \omega_1,$$

which contradicts the inductive assumption on $q_1 f_{j_1} + q_2 f_{j_2} + \cdots + q_k f_{j_k}$.

The obtained contradiction finishes the proof of Theorem 1. $\square$

Remark 4. Denote by $\mathbf{c}^+$ the least cardinal number strictly greater than $\mathbf{c}$. As a straightforward consequence of Theorem 1 we obtain that if $2^{\mathbf{c}} = \mathbf{c}^+$, then $\mathrm{card}(\mathcal{F}) = \mathrm{card}(\mathbf{R}^{\mathbf{R}})$. We do not know whether the assertion of Theorem 1 remains valid under Martin's Axiom (instead of the Continuum Hypothesis).

Remark 5. Let K be an uncountable subfield of $\mathbf{R}$ and consider $\mathbf{R}$ as a vector space over K. Then there does not exist a vector space $\mathcal{G} \subset \mathbf{R}^{\mathbf{R}}$ over K such that:

(1) $\mathrm{card}(\mathcal{G}) > \mathbf{c}$;

(2) all $g \in \mathcal{G}$ are K-linear homomorphisms of $\mathbf{R}$ into itself;

(3) all $g \in \mathcal{G} \setminus \{0\}$ are absolutely nonmeasurable functions with respect to the class $\mathcal{M}(\mathbf{R})$.

Indeed, suppose to the contrary that such a $\mathcal{G}$ does exist and choose a point $x \in \mathbf{R} \setminus \{0\}$. Obviously, there are two distinct elements $g \in \mathcal{G}$ and $h \in \mathcal{G}$, for which we have $g(x) = h(x)$ and, consequently, $(g - h)(x) = 0$. This fact directly implies that

$$(g - h)(yx) = y((g - h)(x)) = 0$$

for any $y \in K$, i.e. the set $(g - h)^{-1}(0)$ is uncountable, which contradicts the absolute nonmeasurability of $g - h$ (see Lemma 1).

As was mentioned earlier, the existence of at least one function from $\mathbf{R}^{\mathbf{R}}$ that is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$ necessarily needs additional set-theoretical hypotheses (see Remarks 1 and 2). For the class $\mathcal{M}_0(\mathbf{R})$ of all nonzero σ-finite translation quasi-invariant measures on $\mathbf{R}$, a certain analogue of Theorem 1 can be established within **ZFC** theory.

Theorem 2. *There exists a family $\mathcal{F} \subset \mathbf{R}^{\mathbf{R}}$ satisfying the following relations:*

(1) $\mathrm{card}(\mathcal{F}) > \omega_1$*;*

(2) *$\mathcal{F}$ is a vector space over $\mathbf{Q}$;*

(3) *all functions from $\mathcal{F}$ are homomorphisms of the additive group $\mathbf{R}$ into itself;*

(4) *all nonzero functions from $\mathcal{F}$ are absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$.*

The proof of Theorem 2 is very similar to the proof of Theorem 1, so we omit it here.

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