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A FEW REMARKS ON METRIC DENSITY AND SET POROSITY THAT ARE SIMPLE AND INTERESTING CONSEQUENCES OF WELL KNOWN RESULTS

Abstract

To every Lebesgue measurable subset of $\mathbb{R}$ is associated a certain subcollection of points where the given measurable set possesses a density. By virtue of Lebesgue's famous theorem on metric density, this associated set is a set of full measure in $\mathbb{R}$ and is hence measure-theoretically very large. But are these sets also topologically large? In Lebesgue's theorem, the set is kept fixed while the point is allowed to vary. If instead, we keep the point fixed a vary the set, then we may have corresponding to each point in $\mathbb{R}$ a certain subclass of measurable sets each member of which possesses a density at that point. How large is this subclass in the "topology of measurable subsets of $\mathbb{R}$"? In this paper, in an endeavour to seek out answers to the questions set above, we have arrived at certain interesting and significant conclusions. Somewhat similar conclusions have been derived over analogous questions relating to 'set-porosity'.

1 Introduction.

Apart from the ones introduced (when required) in the sequel, we will be using in general the following set of symbols

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(i) $\mathbb{N}$ for the set of naturals.

(ii) $\mu, \mathcal{M}$ for the Lebesgue measure, the class of Lebesgue measurable sets of finite measure, and $\mathcal{K}$ for the class of compact subsets of $\mathbb{R}$.

(iii) $A \setminus B$ (resp, $A \Delta B$) for the difference (resp, symmetric difference) of two sets A and B.

(iv) $C(X)$ for the class of real-valued continuous functions on any general topological space X.

(v) $E_{(x)}$ and $E^{(y)}$ for the x-section ($x \in X$) and the y-section ($y \in Y$) of any set $E \subseteq X \times Y$.

(vi) $f(x, .)$ and $f(., y)$ for the x-section ($x \in X$) and the y-section ($y \in Y$) of any function $f : X \times Y \to Z$.

And also the following definitions.

Definition 1. *A set $E \subseteq X$ is called meagre (or, of first category) with respect to some topology on X if we can express E as $E = \bigcup\limits_n E_n$ where each $E_n (n \in \mathbb{N})$ is a nowhere dense subset of X.*

Definition 2. *A set $E \subseteq X$ is called co-meager (or, residual) with respect to some topology on X if $X \setminus E$ is a meager subset of X.*

Definition 3. *In a topological space X, a subset E is said to have the "property of Baire" (see [8]) if $E = G \triangle P$, where G is open and P is meager in X.*

Definition 4. *(see [1]) Let (X, Σ) be a measurable space and Y, Z are topological spaces with $\mathcal{B}_Z$ as the σ-algebra of Borel subsets of Z. A function $f : X \times Y \to Z$ is called a "Carathéodory function" provided that*

(i) for each $x \in X$, the function $f(x, .) : Y \to Z$ is continuous and

(ii) for each $y \in Y$, the function $f(., y) : X \to (Z, \mathcal{B}_Z)$ is measurable.

Metric density (or, simply 'density') of a set (see [6]) in $\mathbb{R}$ is defined as follows. For each $E \in \mathcal{M}$ and $x \in \mathbb{R}$, we write

$$\overline{D}_x(E) = \limsup_{I \to x} \frac{\mu(E \cap I)}{|I|} = \sup_{\{I_k\}} \left\{ \limsup_{k \to \infty} \frac{\mu(E \cap I_k)}{|I_k|} : I_k \to x \right\}$$

and

$$\underline{D}_x(E) = \liminf_{I \to x} \frac{\mu(E \cap I)}{|I|} = \inf_{\{I_k\}} \left\{ \liminf_{k \to \infty} \frac{\mu(E \cap I_k)}{|I_k|} : I_k \to x \right\}$$

to denote respectively the 'upper' and 'lower' densities of E at x, where $I \to x$ means that I, I_k are non-degenerate intervals in $\mathbb{R}$ such that $x \in I_k (k \in \mathbb{N})$, $|I|$, $|I_k|$ stands for their Lebesgue measures and the $\text{diam}(I_k) \to 0$. The common value of the two limits (whenever it exists) is called the "Density of E at x" and is denoted by the symbol $D_x(E)$. Now let $\mathcal{D}(E) = \{x \in \mathbb{R} : D_x(E) \text{ exists}\}$ and $\mathcal{D}^*(E) = \{x \in \mathbb{R} : D_x(E) = 0 \text{ or } 1\}$. Evidently, $\mathcal{D}^*(E) \subseteq \mathcal{D}(E)$.

Lebesgue's famous theorem on metric density states that given a Lebesgue measurable subset E of $\mathbb{R}$, there is a subcollection of points of full measure in $\mathbb{R}$ at each of which the density of E is either 0 or 1. In a more concise form, this may be expressed as: for any $E \in \mathcal{M}$, $\mu(\mathbb{R} \setminus \mathcal{D}^*(E)) = 0$ and consequently, $\mu(\mathbb{R} \setminus \mathcal{D}(E)) = 0$ since $\mathcal{D}^*(E) \subseteq \mathcal{D}(E)$.

Thus according to the above theorem, "the set of all points at which E has a density" is measure-theoretically very large in $\mathbb{R}$ in the sense that its complement in $\mathbb{R}$ is a set of measure zero (or in other words, measure-theoretically small). But are these sets also topologically large?

Goffman (see [3]) showed that for Lebesgue measurable subsets of $\mathbb{R}$, the set $\mathcal{D}(E) \setminus \mathcal{D}^*(E) = \{x : 0 < D_x(E) < 1\}$ is meager. But the fact that this set is also of measure zero is an immediate consequence of Lebesgue's theorem. Thus Goffman's result exhibits the topological smallness of a set which is also measure-theoretically small.

Again, Martin (see [7]) proved that given any F_σ subset Z of $\mathbb{R}$ with $\mu(Z) = 0$ and a real number $\gamma (0 < \gamma < 1)$, there exists a set $E \in \mathcal{M}$ such that $D_x(E) = \gamma$ for each $x \in Z$. But being an F_σ set of measure zero, Z is also meager (by Baire's theorem). Also by Goffman's result stated above, it is a subset of the set $\mathcal{D}(E) \setminus \mathcal{D}^*(E)$. But unfortunately, these results of Goffman and Martin are not enough to produce a proper answer to the questions raised above.

None the less it was shown in [5] that we can always construct sets E for which $\mathcal{D}(E)$ is meager. What has been shown there amounts to the fact that corresponding to every $Z \in \mathcal{M}$ in $\mathbb{R}$ with $\mu(Z) = 0$, a set E can be constructed such that $\overline{D}_x(E) = 1$ and $\underline{D}_x(E) = 0$ for every $x \in Z$. But as Z can be so chosen that it is a also co-meager (owing to an important decomposition theorem stated in Chapter 1 (see [8]) which shows that $\mathbb{R}$ can be decomposed into a meager set and a set of measure zero), the set E so constructed is such that $D_x(E)$ exists for at most a collection of points x which constitutes a meager subset of $\mathbb{R}$.

The above example shows that there are sets E in $\mathcal{M}$ for which $\mathcal{D}(E)$ is meager and therefore topologically small in $\mathbb{R}$. More explicitly, given any co-meager set of measure zero such a set can always be constructed. Thus such sets may even exist in uncountably infinite numbers. But the major question still remains unanswered. How large is this collection in the topology of $\mathcal{M}$? Where by the topology of $\mathcal{M}$ is meant the topology that is induced by the metric τ (on $\mathcal{M}$) defined by
$\tau(E,F) = \mu(E\Delta F)$ for $E, F \in \mathcal{M}$ (see [8])

In the density theorem of Lebesgue, the set is kept fixed while the point is made to vary. If instead, we keep the point fixed and vary the set, then there exists corresponding to each $x \in \mathbb{R}$, a certain subclass $\{E \in \mathcal{M} : D_x(E) \text{ exists}\}$ which we denote by the symbol $\mathcal{D}_x$. One may note here that the class $\mathcal{D}_x$ and the set $\mathcal{D}(E)$ are dual of each other, as one may be obtained from the other by simply interchanging the roles played by the 'set' and the 'point'. It is already known by Lebesgue's theorem that $\mathcal{D}(E)$ is measure theoretically very large in $\mathbb{R}$. Is $\mathcal{D}_x$ also topologically large in $\mathcal{M}$.

The concept of density of a set was originally introduced by Lebesgue. Lebesgue found that points in whose immediate neighbourhood a measurable set is either "highly concentrated" or "highly rarefied" occur in measure-theoretic abundance and this was made precise in a remarkable theorem by him mention of which is already made at the beginning of this article. A somewhat different notion of "set-porosity", on the other hand, was introduced by Denjoy whose aim was to obtain a classification for perfect sets in $\mathbb{R}$ in terms of the relative sizes of the complementary intervals. Delzhenko introduced the notion of σ-porous (countable union of porous sets) sets which forms a subclass of both the classes of "measure zero sets" and "meager sets". However since its inception, the concept of set-porosity has always played a significant role in answering numerous question in real analysis.

The general definition for "set-porosity" (see [4]) is as follows. Let (X, ρ) be a metric space. If $\mathcal{U}(x, r)$ stands for the open ball centred at x and of radius $r > 0$, $x \in X, M \subseteq X, R > 0$ and we set $\lambda(M, x, R) = \sup(\{r > 0 : \text{there is an } y \in X \text{ such that } \mathcal{U}(y, r) \subseteq \mathcal{U}(x, R) \setminus M\} \cup \{0\})$, then the number $\Pi(M, x) = \limsup\limits_{R\to 0+} \frac{\lambda(M,x,R)}{R}$ defines what is called the "porosity of M at x".

In connection with the above definition, a subset M of X is called porous at x provided $\Pi(M, x) > 0$. It is called non-porous at x provided $\Pi(M, x) = 0$ and superporous if $\Pi(M, x) = 1$.

If X is the real line $\mathbb{R}$ with its usual metric, then $\lambda(M, x, R)$ takes the form $\lambda(M, x, R) = \sup(\{r > 0 : \text{ there is } y \in R \text{ such that } (y - r, y + r) \subseteq (x - R, x + R) \setminus M\} \cup \{0\})$ but instead of $\lambda(M, x, R)$, here we prefer using the symbol $\lambda(M, x, (x - R, x + R))$ which is same as writing $\lambda(M, x, I)$ where I is a non-degenerate interval having its center at x. This slight change of notation helps explain how the definition of linear porosity $P_x(E)$ given below depends on the previously made observations. There is an extra advantage to using a notation similar to that of linear density $D_x(E)$ given above. It is that by doing so we may frame the definition of porosity much in the same manner as that of density.

Thus we define $P_x(E)$ for any $E \subseteq \mathbb{R}$ at $x \in \mathbb{R}$ as follows

$$P_x(E) = \limsup_{I \to x} \frac{2\lambda(E, x, I)}{|I|} = \sup_{\{I_k\}} \left\{ \limsup_{k \to \infty} \frac{2\lambda(E, x, I_k)}{|I_k|}, I_k \to x \right\}$$

where I, I_k are nondegenerate intervals having their common centers at x such that $I_k \to x$ in the sense introduced earlier and that $\lambda(E, x, I_k)$ is interpreted as above.

If $P_x(E) > 0$, then the set E is said to be porous at x. In particular, if $P_x(E) = 1$, then E is called "strongly porous" at x. It is called "non-porous" at x provided $P_x(E) = 0$.

2 Results on Metric Density.

In this paper, some interesting and significant conclusions with regard to the questions raised above in connection with measurability and density have been drawn. In addition, to this, we have also deduced somewhat similar type of results on set-porosity using analogous procedures. In both cases, apart from the essential use of other technical devices, our proofs also rests heavily on an important theorem in topology known as the "Kuratowski-Ulam's theorem" and its partial converse (see Theorem 15.1, theorem 15.4, pg. 56-57, [8]).

KURATOWSKI-ULAM'S THEOREM

Let X and Y be topological spaces with Y having a countable base. If $E \subseteq X \times Y$ is a meager set, then the sets $E_{(x)} = \{y \in Y : (x, y) \in E\}$ are meager (in Y) for all x except possibly those lying in a meager subset of X. Similarly the sets $E^{(y)} = \{x \in X : (x, y) \in E\}$ are meager (in X) for all y except possibly those lying in a meager subset of Y.

The following result is a partial converse of the above theorem

If $E \subseteq X \times Y$ has the property of Baire and $E_{(x)} \subseteq Y$ is meager for all x except possibly those lying in a meager subset of X, then E is also meager in the product topology of $X \times Y$.

In the first part of this paper which deals with metric density, we prove the following two theorems.

Theorem 1. *For each $x \in \mathbb{R}$, the class $\mathcal{D}_x$ is meager in the topology of $\mathcal{M}$.*

Let $\Im \subseteq \mathbb{R} \times \mathcal{M}$ be defined by $\Im = \{(x, E) : D_x(E) \text{ exists }\}$. Then clearly $\Im_{(x)} = \mathcal{D}_x$ and $\Im^{(E)} = \mathcal{D}(E)$ are the two sections of $\Im$ in $\mathbb{R} \times \mathcal{M}$. Now for each $x \in \mathbb{R}$, both $\overline{D}_x$ and $\underline{D}_x$ are actually functions from $\mathcal{M}$ to $\mathbb{R}$. Upon writing $\mathcal{J}$ for the class of all non-degenerate intervals in $\mathbb{R}$ and for each $k \in \mathbb{N}$, $\mathcal{J}_x^{(k)} = \left\{ I \in \mathcal{J} : x \in I \text{ and } \frac{1}{k+1} \leq |I| < \frac{1}{k} \right\}$, and $\Phi^{(k)}(x, E) = \sup\left\{ \frac{\mu(E\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\}$, $\Psi^{(k)}(x, E) = \inf\left\{ \frac{\mu(E\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\}$ we may note that both $\Phi^{(k)}$ and $\Psi^{(k)}$ are functions from $\mathbb{R}\times\mathcal{M}$ to $\mathbb{R}$ such that the identities $\overline{D}_x = \limsup\limits_{k\to\infty} \Phi^{(k)}(x, .)$ and $\underline{D}_x = \liminf\limits_{k\to\infty} \Psi^{(k)}(x, .)$ hold true. Moreover, the following set of propositions follow

Proposition 2. *For each $k \in \mathbb{N}$, both $\Phi^{(k)}(x, .), \Psi^{(k)}(x, .) \in C(\mathcal{M})$ and $\Phi^{(k)}(., E), \Psi^{(k)}(., E) \in C(\mathbb{R})$ and therefore Borel measurable.*

Proof. Since for any $E, F \in \mathcal{M}$,
$|\Phi^{(k)}(x, E) - \Phi^{(k)}(x, F)| = \left| \sup\left\{ \frac{\mu(E\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} - \sup\left\{ \frac{\mu(F\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} \right| \leq \sup\left\{ \frac{\mu(E\Delta F)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} \leq (k+1)\tau(E, F)$, the function $\Phi^{(k)}(x, .)$ is continuous on $\mathcal{M}$.
Similarly, as
$|\Psi^{(k)}(x, E) - \Psi^{(k)}(x, F)| = \left| \inf\left\{ \frac{\mu(E\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} - \inf\left\{ \frac{\mu(F\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} \right| \leq \sup\left\{ \frac{\mu(E\Delta F)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} \leq (k+1)\tau(E, F)$, the function $\Psi^{(k)}(x, .)$ is also continuous on $\mathcal{M}$. Thus both $\Phi^{(k)}(x, .)$, $\Psi^{(k)}(x, .) \in C(\mathcal{M})$.
Again for $x, y \in \mathbb{R}$,
$|\Phi^{(k)}(x, E) - \Phi^{(k)}(y, E)|$
$= \left| \sup\left\{ \frac{\mu(E\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} - \sup\left\{ \frac{\mu(E\cap I)}{|I|} : I \in \mathcal{J}_y^{(k)} \right\} \right|$
$= \left| \sup\left\{ \frac{\mu(E\cap I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} - \sup\left\{ \frac{\mu(E\cap (I+z))}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} \right|$ (where $z = y - x$)
$\leqq (k+1)|y - x|$
Likewise, $|\Psi^{(k)}(x, E) - \Psi^{(k)}(y, E)| \leqq (k+1)|x - y|$.
So both $\Phi^{(k)}(., E)$ and $\Psi^{(k)}(., E) \in C(\mathbb{R})$. Moreover, they are uniformly continuous. $\square$

As a consequence of the above result, we derive

Proposition 3. *For any $r \in \mathbb{R}$, the sets $\{E \in \mathcal{M} : \overline{D}_x(E) \geqq r\}$ and $\{E \in \mathcal{M} : \underline{D}_x(E) \leqq r\}$ are both G_δ in $\mathcal{M}$.*

Proof. Since $\overline{D}_x = \limsup_{k\to\infty} \Phi^{(k)}(x,.)$ and $\underline{D}_x = \liminf_{k\to\infty} \Psi^{(k)}(x,.)$, so $\{E \in \mathcal{M} : \overline{D}_x(E) \geq r\} = \bigcap_p \bigcap_q \bigcup_{k \geq q} \{E \in \mathcal{M} : \Phi^{(k)}(x,E) > r - \frac{1}{p}\}$ and also $\{E \in \mathcal{M} : \underline{D}_x(E) \leq r\} = \bigcap_p \bigcap_q \bigcup_{k \geq q} \{E \in \mathcal{M} : \Psi^{(k)}(x,E) < r + \frac{1}{p}\}$, the result therefore follows by using proposition 2. $\square$

Proof of Theorem 1. Now $\mathcal{D}_x = \mathcal{M} \setminus \{E \in \mathcal{M} : \underline{D}_x(E) < \overline{D}_x(E)\} = \mathcal{M} \setminus \bigcup_k \{E \in M : \underline{D}_x(E) < r_k < \overline{D}_x(E)\}$, where $\{r_k : k \in \mathbb{R}\}$ is the set of all rationals in $\mathbb{R}$.

But for any $k \in \mathbb{N}$ the class $\{E \in \mathcal{M} : \underline{D}_x(E) < r_k < \overline{D}_x(E)\} = \{E \in \mathcal{M} : \underline{D}_x(E) < r_k\} \bigcap \{E \in \mathcal{M} : \overline{D}_x(E) > r_k\} = \bigcup_p \bigcup_q [\{E \in \mathcal{M} : \underline{D}_x(E) \leq r_k - \frac{1}{p}\} \bigcap \{E \in \mathcal{M} : \overline{D}_x(E) \geq r_k + \frac{1}{q}\}]$ and is hence $G_{\delta\sigma}$ in $\mathcal{M}$ by proposition 3.
Thus $\mathcal{D}_x$ is $F_{\sigma\delta}$. A concrete representation of $\mathcal{D}_x$ as an $F_{\sigma\delta}$ set follows from the fact that $\mathcal{D}_x = \mathcal{M} \setminus \{E \in \mathcal{M} : \underline{D}_x(E) < \overline{D}_x(E)\} = \mathcal{M} \setminus \bigcup_{r,s} \{E \in \mathcal{M} : \underline{D}_x(E) \leqq r < s \leqq \overline{D}_x(E)\} = \bigcap_{r,s} (\{E \in \mathcal{M} : \underline{D}_x(E) > r\} \cap \{E \in \mathcal{M} : \overline{D}_x(E) < s\})$ (where r, s runs over the set of rationals in $\mathbb{R}$) and both the sets $\{E \in \mathcal{M} : \underline{D}_x(E) > r\}$ and $\{E \in \mathcal{M} : \overline{D}_x(E) < s\}$ are F_σ in $\mathcal{M}$ by virtue of proposition 3.

Consequently we can write $\mathcal{D}_x$ as $\mathcal{D}_x = \bigcap \mathcal{D}_x^{r,s}$, where $\mathcal{D}_x^{r,s} = \bigcap_{r,s} \{E \in \mathcal{M} : \underline{D}_x(E) > r\} \cap \{E \in \mathcal{M} : \overline{D}_x(E) < s\}$. We claim that each $\mathcal{D}_x^{r,s}$ is meager in the topology of $\mathcal{M}$. For otherwise, each $\mathcal{D}_x^{r,s}$ and hence each of the sets $\{E \in \mathcal{M} : \underline{D}_x(E) > r\}$ and $\{E \in \mathcal{M} : \overline{D}_x(E) < s\}$ will be a F_σ set of second category in $\mathcal{M}$. Consequently for any r (or, equally for any s) there should exist at least one $E_0 (\in \mathcal{M})$ and an $r_0 (> 0)$ such that $\{E \in \mathcal{M} : \tau(E, E_0) < r_0\} \subseteq \{E \in \mathcal{M} : \underline{D}_x(E) > r\}$. But this is impossible due to the reason that the value of $\underline{D}_x(E)$ can be made to alter drastically by either adding or deleting a set of sufficiently small measure containing x. Hence $\mathcal{D}_x$ is meager which finally proves Theorem 1. $\square$

Theorem 4. *The class of all those sets $E \in \mathcal{M}$ for which $D(E)$ is meager (in $\mathbb{R}$) is co-meager in the topology of $\mathcal{M}$.*

Proof. In order to prove theorem 4 which is stated below, our first attempt would be to show that the set $\Im$ is a set with the Baire-property in the product topology of $\mathbb{R} \times \mathcal{M}$ by showing that it is Borel.

Proposition 2 shows that $\Phi^{(k)}$ and $\Psi^{(k)}$ are both Carathéodory functions in the sense given by Definition 4 (in the introduction) and so are Borel measurable (see pg. 156, [1])

Certainly then both the functions $\Phi : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ and $\Psi : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ defined by $\Phi = \limsup_{k\to\infty} \Phi^{(k)}$ and $\Psi = \liminf_{k\to\infty} \Psi^{(k)}$ are Borel measurable and so also is the function $h : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ defined by $h = \Phi - \Psi$. But $\Im = \{(x, E) \in \mathbb{R} \times \mathcal{M} : h(x, E) = 0\}$. So $\Im$ is a Borel subset of $\mathbb{R} \times \mathcal{M}$ and hence has the property of Baire.

We have already established that for each $x \in \mathbb{R}$, the set $\mathcal{D}_x$ (or equivalently, the section $\Im_{(x)}$) is meager in $\mathcal{M}$. So by applying the converse of Kuratowski-Ulam's theorem (in the product topology of $\mathbb{R} \times \mathcal{M}$), it follows that $\Im$ is meager. Now upon applying Kuratowski-Ulam's theorem, it again follows that the sections $\Im^{(E)}$ are meager (in $\mathbb{R}$) for all E except possibly those which constitute a meager subset of $\mathcal{M}$. But $\Im^{(E)} = \mathcal{D}(E)$ and this finally proves the theorem.

□

The following theorem is in fact a corollary of Lebesgue's theorem and theorem 4. But judging by its importance in the present context, we present it here as a theorem.

Theorem 5. *For each member E belonging to a co-meager subclass of $\mathcal{M}$, there corresponds a decomposition of $\mathbb{R}$ into $\mathcal{D}(E)$ and its complement $\mathbb{R}\setminus\mathcal{D}(E)$ the first of which is meager while the second is of Lebesgue measure zero.*

The above theorem shows that there exists a subclass of measurable sets in $\mathbb{R}$ which is topologically very big (i.e., co-meager) in the class of measurable sets such that for each of its member E the corresponding sets $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ both exhibit a completely contrasting character in respect of topology and measure by being topologically small whereas at the same time also measure-theoretically very big. Thus every such set initiates a decomposition of $\mathbb{R}$ into mutually disjoint sets one of which is meager (i.e., small in the topological sense) while the other of Lebesgue measure zero (i.e., small in the measure-theoretic sense). Examples of such decompositions of $\mathbb{R}$ may be found in the first two chapters of the classic book of Oxtoby (see [8]). In particular, the example presented in Chapter 2 of this book shows that the set of Lioville numbers and its complement in $\mathbb{R}$ can induce such a decomposition. But the above theorem is far strong enough for it suggests that in the real line for each and every set in $\mathcal{M}$ which occur in topological abundance such decompositions are possible. Apart from this, theorem 4 also enriches Goffman's result (stated in the introduction) significantly by showing that not only $\mathcal{D}(E) \setminus \mathcal{D}^*(E)$ but even $\mathcal{D}(E)$ (which is evidently a much larger collection) is meager for every set E the class of which constitutes a co-meager collection in $\mathcal{M}$.

Theorem 1 on the otherhand shows that $\mathcal{D}_x$ (which is the dual of $\mathcal{D}(E)$) is topologically a very small set (in $\mathcal{M}$) for each $x \in \mathbb{R}$, something which stands in contradistinction to the measure-theoretic largeness of the set $\mathcal{D}(E)$. Martin (see [6]) observed that in the context of the real line $\mathbb{R}$, $\overline{D}_x$ is a "finitely subadditive outer measure function" on $\mathcal{M}$ with range in $[0, 1]$ and its restriction D_x on the class $\mathcal{D}_x$ is a finitely additive, subtractive, monotone, non-negative set function which is also onto. But it is not a finitely additive measure for although $\mathcal{D}_x$ is closed under the formation of complements, proper difference and disjoint union, it fails to be so with respect to intersection (see [6]). He however established in (see [6]) that the class of $\overline{D}_x$-measurable sets is precisely the collection $\mathcal{M}_x = \{E \in M : D_x(E) = 0 \text{ or } 1\}$. So from what we have deduced above, it follows that not only the class $\mathcal{M}_x$ of all $\overline{D}_x$-measurable sets but even a much larger collection such as $\mathcal{D}_x$ is topologically small in $\mathcal{M}$.

3 Results on Set Porosity.

In this part we show that results somewhat analogous to that of theorem 1 and theorem 4 can be formulated in the case of set-porosity. We begin by setting

For each $x \in \mathbb{R}, \mathcal{P}_x^{(0)} = \{E \in \mathcal{K} : P_x(E) = 0\}$ and $\mathcal{P}_x^{(1)} = \{E \in \mathcal{K} : P_x(E) = 1\}$. Also, for each $E \in \mathcal{K}$, let us write

$\mathcal{P}^{(0)}(E) = \{x \in \mathbb{R} : P_x(E) = 0\}$ and $\mathcal{P}^{(1)}(E) = \{x \in \mathbb{R} : P_x(E) = 1\}$.

Thus $\mathcal{P}_x^{(0)} = \{E \in \mathcal{K} : E \text{ is non-porous at } x\}$ and

$\mathcal{P}_x^{(1)} = \{E \in \mathcal{K} : E \text{ is strongly porous at } x\}$; and also for any $E \in \mathcal{K}, \mathcal{P}^{(0)}(E)$ (resp, $\mathcal{P}^{(1)}(E)$) is the set of all points in $\mathbb{R}$ at which E is non-porous (resp, strongly-porous).

As in the measure-theoretic case, here also we like to know: are the classes $\mathcal{P}_x^{(0)}$ and $\mathcal{P}_x^{(1)}$ small in the topology of $\mathcal{K}$? and also what can be said regarding the topological size of the collection comprising of those sets E (in the topology of $\mathcal{K}$) for which the corresponding sets $\mathcal{P}^{(0)}(E)$ and $\mathcal{P}^{(1)}(E)$ are meager in $\mathbb{R}$. Here by the topology of $\mathcal{K}$ is meant the topology that is induced by the "Hausdorff metric" h defined by

For any $A, B \in \mathcal{K}$,

$h(A, B) = \inf\{\delta > 0 : A \subseteq B_\delta \text{ and } B \subseteq A_\delta\}$, where A_δ (resp, B_δ) is the union of closed intervals of length 2δ centred at the points of A (resp, B).

The following two theorems answers these questions much in the same manner in which the corresponding measure-theoretic situation is dealt with.

Theorem 6. *For each $x \in \mathbb{R}$, the class $\mathcal{P}_x^{(0)}$ (resp, $\mathcal{P}_x^{(1)}$) is meager in the topology of $\mathcal{K}$.*

and

Theorem 7. *The class of sets E (in $\mathcal{K}$) for which $\mathcal{P}^{(0)}(E)$ (resp, $\mathcal{P}^{(1)}(E)$) is meager (in $\mathbb{R}$) is co-meager in the topology of $\mathcal{K}$.*

Let $\mathcal{H} \subseteq \mathbb{R} \times \mathcal{K}$ and $\mathcal{G} \subseteq \mathbb{R} \times \mathcal{K}$ be defined by

$$\mathcal{H} = \{(x, E) : E \in \mathcal{P}_x^{(0)} \text{ exists } \} \text{ and } \mathcal{G} = \{(x, E) : E \in \mathcal{P}_x^{(1)} \text{ exists } \}.$$

But this means that $\mathcal{H}_{(x)} = \mathcal{P}_x^{(0)}$ and $\mathcal{H}^{(E)} = \mathcal{P}^{(0)}(E)$ are the two sections of $\mathcal{H}$ in $\mathbb{R} \times \mathcal{K}$. Similarly, $\mathcal{G}_{(x)} = \mathcal{P}_x^{(1)}$ and $\mathcal{G}^{(E)} = \mathcal{P}^{(1)}(E)$ are the two sections of $\mathcal{G}$ in $\mathbb{R} \times \mathcal{K}$.

Now for a fixed $x \in \mathbb{R}$, P_x is a set function from $\mathcal{K}$ to $\mathbb{R}$. Upon writing $\eta^{(k)}(x, E) = \sup \left\{ \frac{2\lambda(E,x,I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\}$ (for each $k \in \mathbb{N}$) we note that $\eta^{(k)}$ is a function from $\mathbb{R} \times \mathcal{K}$ to $\mathbb{R}$ such that $P_x = \limsup\limits_{k\to\infty} \eta^{(k)}(x, .)$ (which may be easily checked). Moreover

Proposition 8. *For each $k \in \mathbb{N}$, $\eta^{(k)}(x, .) \in C(\mathcal{K})$ and $\eta^{(k)}(., E) \in C(\mathbb{R})$ and therefore Borel measurable.*

Proof. For any $E, F \in \mathcal{K}$,
$|\eta^{(k)}(x, E) - \eta^{(k)}(x, F)| = \left| \sup \left\{ \frac{2\lambda(E,x,I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} - \sup \left\{ \frac{2\lambda(F,x,I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} \right| \leq 2(k + 1)h(E, F)$
Hence $\eta^{(k)}(x, .) \in C(\mathcal{K})$.
Since for $x, y \in \mathbb{R}$,
$|\eta^{(k)}(x, E) - \eta^{(k)}(y, E)|$
$= \left| \sup \left\{ \frac{2\lambda(E,x,I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} - \sup \left\{ \frac{2\lambda(E,y,I)}{|I|} : I \in \mathcal{J}_y^{(k)} \right\} \right|$
$= \left| \sup \left\{ \frac{2\lambda(E,x,I)}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} - \sup \left\{ \frac{2\lambda(E,x,(I+z))}{|I|} : I \in \mathcal{J}_x^{(k)} \right\} \right|$ (where$z = y - x$)
$\leqq 2(k + 1)|y - x|$, so $\eta^{(k)}(., E) \in C(\mathbb{R})$. Even more it is uniformly continuous. □

Therefore

Proposition 9. *For any $r \in \mathbb{R}$, $\{E \in \mathcal{K} : P_x(E) \geqq r\}$ is G_δ in $\mathcal{K}$.*

PROOF. Since $P_x = \limsup_{k\to\infty} \eta^{(k)}(x,.)$, it follows that
$\{E \in \mathcal{K} : P_x(E) \geq r\} = \bigcap_p \bigcap_q \bigcup_{k\geq q} \{E \in \mathcal{K} : \eta^{(k)}(x,E) > r - \frac{1}{p}\}$
whereupon the result is obtained by using proposition 8. $\square$

PROOF OF THEOREM 6. Now, $\mathcal{P}_x^{(0)} = \mathcal{K} \setminus \{E \in \mathcal{K} : P_x(E) > 0\} = \mathcal{K} \setminus \bigcup_k \{E \in \mathcal{K} : P_x(E) \geq r_k > 0\}$, where $r_k > 0$ are rational in $\mathbb{R}$. Hence by proposition 9, $\mathcal{P}_x^{(0)}$ is $F_{\sigma\delta}$ in the topology of $\mathcal{K}$. Again as $\mathcal{P}_x^{(1)} = \{E \in \mathcal{K} : P_x(E) = 1\} = \{E \in \mathcal{K} : P_x(E) \geqq 1\}$, it is also $F_{\sigma\delta}$ by virtue of the same proposition (since in any metric space every G_δ is also $F_{\sigma\delta}$).

A more concrete description of $\mathcal{P}_x^{(0)}$ as an $F_{\sigma\delta}$ set also follows from the fact that $\mathcal{P}_x^{(0)} = \bigcap_k \{E \in \mathcal{K} : P_x(E) < r_k\}$, where each $\{E \in \mathcal{K} : P_x(E) < r_k\}$ is F_σ in $\mathcal{K}$ by proposition 9. In a similar manner, $\mathcal{P}_x^{(1)} = \bigcap_p \{E \in \mathcal{K} : P_x(E) > 1 - \frac{1}{p}\}$ is also $F_{\sigma\delta}$ in $\mathcal{K}$ where each $\{E \in \mathcal{K} : P_x(E) > 1 - \frac{1}{p}\}$ is F_σ in $\mathcal{K}$.

Each of the classes $\mathcal{P}_x^{(0)}$ and $\mathcal{P}_x^{(1)}$ are also meager in the topology of $\mathcal{K}$. For on the contrary, there should exists $E_0, E_1 \in \mathcal{K}, t_0, t_1 > 0$ such that $\{E \in \mathcal{K} : h(E, E_0) < t_0\} \subseteq \{E \in \mathcal{K} : P_x(E) < r_k\}$ and also $\{E \in \mathcal{K} : h(E, E_1) < t_1\} \subseteq \{E \in \mathcal{K} : P_x(E) > 1 - \frac{1}{p}\}$. But as the value of $P_x(E)$ can be made to alter drastically either by adding or by deleting a small non-degenerate interval centered at x, such inclusions as indicated above are impossible. This proves theorem 6. $\square$

PROOF OF THEOREM 7. We now start proving theorem 7 by showing that the set $\mathcal{H}$ is with Baire-property in the product topology of $\mathbb{R}\times\mathcal{K}$ by showing that it is Borel.

Proposition 8 shows that, $\eta^{(k)}$ is a Carathéodory function in the sense given by definition 4 (introduction) and is hence Borel measurable (see pg. 156, [1])

Certainly then the function $\eta : \mathbb{R}\times\mathcal{K} \to \mathbb{R}$ defined by $\eta = \limsup_{k\to\infty} \eta^{(k)}$ is also Borel measurable. As $\{\mathcal{H} = \{(x,E) \in \mathbb{R}\times\mathcal{K} : \eta(x,E) = 0\}$, $\mathcal{H}$ is a Borel subset of $\mathbb{R}\times\mathcal{K}$ and consequently has the property of Baire in the product topology.

We have already established (by proving theorem 6) that for each $x \in \mathbb{R}$, the set $\mathcal{P}_x^{(0)}$ (or, equivalently the set $\mathcal{H}_{(x)}$) is meager, in $\mathcal{H}$. So by applying the converse of Kuratowski-Ulam's theorem (in the product topology of $\mathbb{R}\times\mathcal{K}$), it follows that $\mathcal{H}$ is meager. But then (by Kuratowski-Ulam's theorem again), it follows that the sections $\mathcal{H}^{(E)}$ are meager in $\mathbb{R}$ for all E except those which constitute a meager subset of $\mathcal{K}$. As $\mathcal{H}^{(E)} = \mathcal{P}^{(0)}(E)$, we finally derive that the class of sets (in $\mathcal{K}$) for which $\mathcal{P}^{(0)}(E)$ is meager (in $\mathbb{R}$) is co-meager in the topology of $\mathcal{K}$. In a similar manner, we may also prove that the class of sets (in $\mathcal{K}$) for which $\mathcal{P}^{(1)}(E)$ is meager (in $\mathbb{R}$) is co-meager in the topology of $\mathcal{K}$. This proves theorem 7. $\square$

4 Some Final Remarks.

Since $\eta(.,E) = \limsup_{k\to\infty} \eta^{(k)}(.,E)$, so from proposition 8 it follows that the function $\eta(.,E)$ is lower semi-Borel function of class 2. Such conclusion holds irrespective of whether E is compact or not, for it may be noted from the proof of proposition 8 that it makes no use of the fact that E is a member of the class $\mathcal{K}$. Infact proposition 8 helps illustrating the lower semi-Borel character of $\eta(.,E)$ in a much simpler fashion than given by lemma 3.6 (see [2]).

Just as $\mathcal{D}_x$ and $\mathcal{D}(E)(E \in \mathcal{M})$, so are $\mathcal{P}_x^{(0)}$ and $\mathcal{P}^{(0)}(E)(E \in \mathcal{K})$ and likewise $\mathcal{P}_x^{(1)}$ and $\mathcal{P}^{(1)}(E)(E \in \mathcal{K})$ the dual of each other as one may be obtained from the other by simply interchanging the roles played by the 'set' and the 'point'. Moreover both $\mathcal{D}(E)$ and $\mathcal{D}_x$ are $F_{\sigma\delta}$ subsets of their respective spaces. The first one is a consequence of the identity $\mathcal{D}(E) = \{x \in \mathbb{R} : \eta(x,E) = 0\}$ and proposition 2, whereas the second one is a direct outcome of the proof of theorem 1. Likewise, both $\mathcal{P}^{(0)}(E)$ and $\mathcal{P}_x^{(0)}$ are $F_{\sigma\delta}$ subsets of their respective spaces. The first one is a consequence of the identity $\mathcal{P}^{(0)}(E) = \{x \in \mathbb{R} : \eta(x,E) = 0\}$ and proposition 8, whereas the second one is a direct outcome of the proof of theorem 6.

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