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DIMENSION SPECTRUM FOR A NONCONVENTIONAL ERGODIC AVERAGE

Abstract

We compute the dimension spectrum of certain nonconventional averages, namely, the Hausdorff dimension of the set of $0, 1$ sequences, for which the frequency of the pattern 11 in positions $k, 2k$ equals a given number $\theta \in [0, 1]$.

1   Introduction.

For a dynamical system (X, T) (say, a continuous self-map of a compact metric space), the dimension spectrum of ordinary Birkhoff averages is defined as the function

$$\theta \mapsto \dim_H \Big\{ x \in X : \ \lim_{n \to \infty} \frac{S_n f(x)}{n} = \theta \Big\}.$$

where $S_n f(x) = \sum_{k=1}^n f(T^k x)$ and f is a function on X. It has been widely investigated in Multifractal Analysis, see e.g. [2]. The most basic example of such analysis goes back to Besicovitch [4] and Eggleston [7] who proved that

$$\dim_H \Big\{ (x_k)_1^\infty \in \{0, 1\}^{\mathbb{N}} : \ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n x_k = \theta \Big\} = H(\theta), \quad \theta \in [0, 1], \qquad (1)$$

where $H(\theta) = -\theta \log_2 \theta - (1-\theta) \log_2(1-\theta)$ is the entropy function. Throughout the paper, $\{0,1\}^{\mathbb{N}} = \Sigma_2$ is the symbolic space, with the usual metric $\varrho((x_k),(y_k)) = 2^{-\min\{n\colon x_n \neq y_n\}}$. For dimension purposes, this is equivalent to $[0,1]$ with the standard metric, since for any set $A \subset \Sigma_2$, its image under the binary representation map has the same dimension as A, see [8, Section 2.4]. Furstenberg [10] was the first to consider *multiple Birkhoff averages*, and their study has become a very active area of research, see e.g. Bourgain [6], Host and Kra [11], and others. For a system (X,T) one considers

$$\frac{1}{n} S_n(f_1, \ldots, f_\ell)(x) := \frac{1}{n} \sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x)$$

for some bounded functions $f_1, \ldots, f_\ell$. Very recently, Yu. Kifer [16] and A.-H. Fan, L. Liao, J. Ma [9] initiated the study of the dimension spectrum for such averages (in [16] more general "nonconventional averages" are considered as well). Multifractal analysis of this kind appears to be very complicated, so it is natural to start with the simplest situation, namely, the shift map T on the symbolic space and the functions $f_1, \ldots, f_\ell$ depending only on the first digit x_1, for $\ell \geq 2$. Specializing even further, to $\ell = 2$ and $f_1(x) \equiv f_2(x) = x_1$ leads to the sets

$$A_\theta := \left\{ (x_k)_1^\infty \in \Sigma_2 : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} = \theta \right\}, \ \theta \in [0,1]. \tag{2}$$

The question about the dimension of A_θ was raised in [9]. Note that this directly generalizes the Besicovitch-Eggleston set-up from $\ell = 1$ to $\ell = 2$.

Motivated by this problem, A.-H. Fan, L. Liao, J. Ma, and J. Schmeling [private communication in August 2010] computed the Minkowski (box-counting) dimension of another set

$$X_G := \left\{ (x_k)_1^\infty \in \Sigma_2 : x_k x_{2k} = 0 \text{ for all } k \right\}$$

and asked what is its Hausdorff dimension. It is obvious that $X_G \subset A_0$, and in fact, it is easy to see that $\dim_H(X_G) = \dim_H(A_0)$.

In joint work with R. Kenyon, we computed the Hausdorff dimension of X_G and a large class of similarly defined sets, putting it into the context of *subshifts invariant under the semi-group of multiplicative integers* [14, 15]. Here we adapt the techniques of [14, 15] to compute the full dimension spectrum $\dim_H(A_\theta)$.

Theorem 1. *Let A_θ be given by (2). For $\theta \in (0, 1)$ we have*

$$\dim_H(A_\theta) = f(\theta) := -\log_2(1-p) - \frac{\theta}{2}\log_2\left[\frac{(1-q)(1-p)}{qp}\right], \qquad (3)$$

where

$$p^2 q = (1-p)^3, \ 0 < p < 1, \ 0 < q < 1, \qquad (4)$$

$$\theta = \frac{2p(1-q)}{1+p+q}. \qquad (5)$$

We have $\dim_H(A_0) = \lim_{\theta \to 0} f(\theta) = -\log_2(1-p)$, *with* $p^2 = (1-p)^3$, *and* $\dim_H(A_1) = \lim_{\theta \to 1} f(\theta) = 0$.

The meaning of p and q will be explained in the next section. Of course, it is easy to eliminate q from (3) and (5). For a given θ, we get an algebraic equation of degree 4 for p. Solving the equation numerically yields the graph in Figure 1.
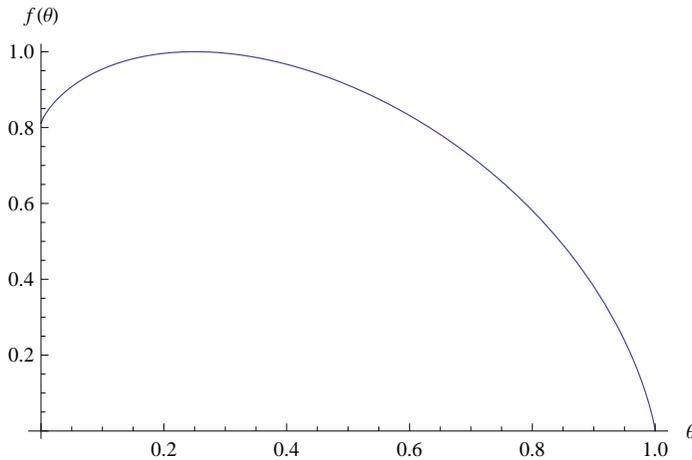

Figure 1: Dimension of A_θ

Remarks.

1. As already mentioned, the formula for $\dim_H(A_0)$ easily follows from [14]. Note that in [14, 15] notation was slightly different, so that p in those papers is $1 - p$ here.

2. It is immediate that A_1 is contained in the set of 0-1 sequences which have frequency of 1's equal to 1. Thus, $\dim_H(A_1) = 0$ by (1), and we assume $\theta < 1$ for the rest of the paper.

3. By the Strong Law of Large Numbers for weakly correlated random variables (see [18]), for a.e. sequence (x_k) with respect to the Bernoulli $(\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ measure, $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_j x_{2j} = 1/4$. This agrees with our result: for $\theta = 1/4$ we get $\dim_H(A_\theta) = 1$ and $p = q = 1/2$.

4. In [9] it is proved that $\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\left(\frac{1+\theta}{2}\right)$ for $\theta \in [-1, 1]$ and $\ell \geq 2$, where $B_\theta := \left\{ (x_k)_1^\infty \in \{-1, 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}$, using the techniques of Riesz products. It is further pointed out in [9] that the problem becomes drastically different if one takes the digits 0,1 (which reduces to A_θ for $\ell = 2$) instead of $-1, 1$.

5. Yu. Kifer [16] considered a slightly different question: he studied the Hausdorff dimension of sets defined by the frequencies of all ℓ-tuples of digits $i_1, \ldots, i_\ell$ in positions $k, 2k, \ldots, \ell k$. However, he was able to compute the dimensions only under the assumption that such frequencies are of the form $p_{i_1, \ldots, i_\ell} = p_{i_1} \cdots p_{i_\ell}$.

6. As pointed out in [15], there are some parallels between the multiplicative shifts of finite type and self-affine carpets [3, 19]; we should also add here self-affine sponges [13]. The present paper may similarly be compared to the work on multifractal self-affine carpets and sponges, see e.g. [17, 20, 1, 12]; however, we do not see any way to directly transfer the results.

2 Preliminaries and the scheme of the proof.

The dimension of A_θ is computed with the help of the following lemma which goes back to Billingsley [5]. We write $[u]$ for the cylinder set of sequences starting with a finite word u and $x_1^n := x_1 \ldots x_n$.

Lemma 2 (see Prop.4.9 in [8]). *Let E be a Borel set in Σ_2 and let ν be a finite Borel measure on Σ_2.*

(i) *If $\nu(E) > 0$ and $\liminf_{n\to\infty} \frac{-\log_2 \nu[x_1^n]}{n} \geq s$ for ν-a.e. $x \in E$, then $\dim_H(E) \geq s$.*

(ii) *If $\liminf_{n\to\infty} \frac{-\log_2 \nu[x_1^n]}{n} \leq s$ for all $x \in E$, then $\dim_H(E) \leq s$.*

Following [14, 15], for a probability measure μ on Σ_2, we define another measure $\mathbb{P}_\mu$ on Σ_2 by

$$\mathbb{P}_\mu[u] := \prod_{i \leq n,\, i \text{ odd}} \mu[u|_{J(i)}], \quad \text{where } J(i) = \{2^r i\}_{r=0}^\infty \tag{6}$$

and $u|_{J(i)}$ is the subsequence of u (viewed as a finite sequence) along the geometric progression $J(i)$. The new measure $\mathbb{P}_\mu$ is invariant under the action of the *multiplicative semigroup of odd positive numbers*:

$$(x_k)_{k=1}^\infty \mapsto (x_{ik})_{k=1}^\infty \quad \text{for odd } i.$$

We consider Markov measures $\mu_{\mathbf{p},P}$ on Σ_2, with the initial probability distribution $\mathbf{p} = (1-p, p)$ (so that p is the probability of initial 1), and the stochastic transition matrix $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$. Note that our Markov measures are not stationary; instead, their initial distribution coincides with the first row of the transition matrix.

Next we indicate the scheme of the proof of Theorem 2. Recall that $\theta \in [0, 1)$. In view of Lemma 2(i), the lower bound for $\dim_H(A_\theta)$ will be established once we prove the following.

Lemma 3. *Fix $p \in (0, 1), q \in [0, 1)$, and let $\mathbb{P}_\mu$, with $\mu = \mu_{\mathbf{p},P}$, be defined by (6).*

(i) *If p, q satisfy (5), then $\mathbb{P}_\mu(\Sigma_2 \setminus A_\theta) = 0$. For $\theta = 0$ we take $q = 1$.*

(ii) *For any p, q we have*

$$\lim_{n\to\infty} \frac{-\log_2 \mathbb{P}_\mu[x_1^n]}{n} = s(p, q) := \frac{(1+q)H(p) + pH(q)}{1 + p + q} \quad \text{for } \mathbb{P}_\mu\text{-a.e. } x.$$

(iii) *The maximum of $s(p, q)$, subject to (5), is achieved when $p^2 q = (1 - p)^3$, and it equals*

$$f(\theta) = -\log_2(1-p) - \frac{\theta}{2}\log_2\left[\frac{(1-q)(1-p)}{qp}\right]. \tag{7}$$

The upper bound in Theorem 1 will follow from Lemma 2(ii), once we prove the following:

Lemma 4. *Let $\mu = \mu_{\mathbf{p},P}$ be the Markov measure with initial probability vector $\mathbf{p} = (1-p, p)$ and transition matrix $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$, where $p^2 q = (1-p)^3$ and (5) holds, and let $\mathbb{P}_\mu$ be the corresponding multiplicative invariant measure. Then*

$$\liminf_{n\to\infty} \frac{-\log_2 \mathbb{P}_\mu[x_1^n]}{n} \le f(\theta) \quad \text{for all } x \in A_\theta. \tag{8}$$

3   Proofs.

The following elementary lemma will be useful. We provide the proof for completeness.

Lemma 5. *Suppose that $\{z_n\}$ is a bounded real sequence and there exists $c > 0$ such that*

$$|z_n - z_{n+m}| \leq \frac{cm}{n} \quad \text{for all } m, n \in \mathbb{N}. \tag{9}$$

If $z_{2^k n} \to \gamma$ as $k \to \infty$ for all $n \in \mathbb{N}$, then $z_n \to \gamma$.

PROOF. For $\varepsilon > 0$ let $i_0 \in \mathbb{N}$ be such that $2^{-i_0} < \varepsilon$. By the assumption, we can find $k_0 \in \mathbb{N}$ such that for all $\ell \leq 2^{i_0}$ and all $k \geq k_0 - i_0$ we have $|z_{\ell \cdot 2^k} - \gamma| < \varepsilon$. For $n > 2^{k_0 + i_0}$, with $2^{k-1} \leq n < 2^k$, let $\ell = \lceil n \cdot 2^{i_0 - k} \rceil \leq 2^{i_0}$. Then $0 \leq \ell \cdot 2^{k - i_0} - n < 2^{k - i_0}$, hence

$$|z_n - \gamma| \leq |z_n - z_{\ell \cdot 2^{k-i_0}}| + |z_{\ell \cdot 2^{k-i_0}} - \gamma| \leq \frac{c2^{k-i_0}}{n} + \varepsilon < \varepsilon(1 + 2c),$$

completing the proof.   □

For positive integers $m < n$, $i, j \in \{0, 1\}$, and $x \in \Sigma_2$ let

$$N_i(x_m^n) = \# \left\{ k \in [m, n] \cap \mathbb{N} : \ x_k = i \right\},$$

$$N_{ij}(x_m^n) = \# \left\{ k \in [m, n] \cap \mathbb{N} : \ x_k = i, \ x_{2k} = j \right\}.$$

Further, for $x \in \Sigma_2$ and even n denote

$$\alpha_n(x) = \frac{N_1(x_{n/2+1}^n)}{n/2} . \tag{10}$$

Observe that $\alpha_n(x) \in [0, 1]$, and it is easy to see that

$$|\alpha_n - \alpha_{n+m}| \leq \frac{2m}{n} \quad \text{for } n, m \text{ even.} \tag{11}$$

Lemma 6. *Let μ be a Markov measure on Σ_2, with initial probability (row) vector $\mathbf{p} = (1 - p, p)$ and transition matrix $P = \begin{pmatrix} 1 - p & p \\ q & 1 - q \end{pmatrix}$. Let $\mathbb{P}_\mu$ be the measure on Σ_2 defined by (6). Then for $\mathbb{P}_\mu$-a.e. $x \in \Sigma_2$ we have*

$$\lim_{n \to \infty} \alpha_n(x) = \frac{2p}{1 + p + q} =: \xi,$$

where $\alpha_n(x)$ is defined in (10).

PROOF. We assume n to be even. Consider $n\alpha_{2n}(x)$, the number of 1's in x_i's for $i \in (n, 2n]$. There are $n/2$ odd numbers i in $(n, 2n]$. By the definition of the measure $\mathbb{P}_\mu$, the corresponding x_i are chosen to be 1 independently with probability p. Thus, by the Law of Large Numbers, the number of 1's among them is $np/2 + o(n)$. Among the even numbers $2i$ in $(n/2, n]$, there is $n\alpha_n(x)/2$ numbers for which $x_i = 1$, and there will be approximately $n\alpha_n(x)(1-q)/2$ 1's there, again using the Law of Large Numbers, since $1-q$ is the probability of the transition $1 \to 1$ according to the Markov measure μ. Similarly, there is $n(1-\alpha_n(x))/2$ numbers $2i \in (n, 2n]$ for which $x_i = 0$, and approximately $n(1-\alpha_n(x))p/2$ 1's among those. Hence we have

$$
\begin{aligned}
\alpha_{2n}(x) &= \frac{p}{2} + \frac{\alpha_n(x)(1-q)}{2} + \frac{\alpha_n(x)(1-\alpha_n(x))p}{2} + \varepsilon_n(x) \\
&= p + \frac{\alpha_n(x)(1-q-p)}{2} + \varepsilon_n(x),
\end{aligned}
$$

where $\varepsilon_n(x) \to 0$ for $\mathbb{P}_\mu$-a.e. x. Since $|(1-q-p)/2| < 1$, it follows that $\alpha_{2^k n}(x) \to \xi$ for $\mathbb{P}_\mu$-a.e. x, where $\xi = p + \xi(1-q-p)/2$, i.e. $\xi = \frac{2p}{1+p+q}$. Then for $\mathbb{P}_\mu$-a.e. x we have $\alpha_{2^k n}(x) \to \xi$ for all $n \in \mathbb{N}$. It remains to recall (11) and apply Lemma 5. □

PROOF OF LEMMA 3(i). We claim that

$$
\lim_{n \to \infty} \frac{N_{11}(x_1^n)}{n} = \xi(1-q) = \frac{2p(1-q)}{1+p+q} = \theta
$$

for a $\mathbb{P}_\mu$-typical point $x \in \Sigma_2$. Indeed, the initial part of the sequence does not impact the limit, and for n big enough we see 1's with frequency ξ and thus 11's (in positions $i, 2i$) with frequency $\xi(1-q)$, by the definition of $\mathbb{P}_\mu$. This means that $\mathbb{P}_\mu$-a.e. x is in A_θ, as desired. □

Fix an even integer n. Denote

$$
N_{1,\text{odd}} = N_{1,\text{odd}}(x_1^n) := \#\{k \le n : k \text{ odd}, x_k = 1\}.
$$

By the definition of $\mu = \mu_{\mathbf{p}, P}$ and $\mathbb{P}_\mu$ we have, for any x and even n:

$$
\mathbb{P}_\mu[x_1^n] = p^{N_{1,\text{odd}}}(1-p)^{n/2 - N_{1,\text{odd}}}(1-p)^{N_{00}} p^{N_{01}} q^{N_{10}} (1-q)^{N_{11}} \tag{12}
$$

where $N_{ij} = N_{ij}(x_1^{n/2})$.

For the rest of the proof we write log for $\log_2$ to simplify notation.

Proof of Lemma 3(ii). For $\mathbb{P}_\mu$-a.e. $x \in \Sigma_2$, we see 1's with frequency p in odd places, and similarly to the proof of Lemma 3(i), the frequency of 00's, 01's, 10's, and 11's is $(1-\xi)(1-p)$, $(1-\xi)p$, ξq, and $\xi(1-q)$ respectively. Therefore, for $\mathbb{P}_\mu$-a.e. $x \in \Sigma_2$ the formula (12) yields

$$
\begin{aligned}
\lim_{n\to\infty} \frac{-\log \mathbb{P}_\mu[x_1^n]}{n} &= -(1/2)\big[(2-\xi)p\log p + (2-\xi)(1-p)\log(1-p) \\
&\quad + \xi q \log q + \xi(1-q)\log(1-q)\big] \\
&= \frac{(1+q)H(p) + pH(q)}{1+p+q},
\end{aligned}
$$

as desired.   □

In view of Lemma 3(i),(ii) and Lemma 2(i), we have that

$$
\dim_H(A_\theta) \geq s(p,q) = \frac{(1+q)H(p) + pH(q)}{1+p+q}, \quad \text{where} \quad \theta = \frac{2p(1-q)}{1+p+q}. \quad (13)
$$

Thus, we should find the constrained maximum of $s(p,q)$ on $[0,1]^2$. This is a straightforward exercise, but we include it for the record and in order to explain where the formula (4) comes from. It is actually not needed for the proof of the main result, since we could just produce the answer for the optimization problem and refer to Lemma 4. We also include the verification of (7).

Proof of Lemma 3(iii). We use the method of Lagrange multipliers. Differentiating $s(p,q)$ yields

$$
\begin{aligned}
(1+p+q)^2 \frac{\partial s(p,q)}{\partial p} &= (1+q)[(1+p+q)\log(\tfrac{1-p}{p}) - H(p) + H(q)] \\
&= (1+q)[(2+q)\log(1-p) - (1+q)\log p + H(q)],
\end{aligned}
$$

$$
\begin{aligned}
(1+p+q)^2 \frac{\partial s(p,q)}{\partial q} &= p[(1+p+q)\log(\tfrac{1-q}{q}) + H(p) - H(q)] \\
&= p[(2+p)\log(1-q) - (1+p)\log q + H(p)].
\end{aligned}
$$

Differentiating the constraint $g(p,q) = \theta(1+p+q) - 2p(1-q) = 0$ yields

$$
\nabla g(p,q) = (\theta - 2(1-q), \theta + 2p) = \left(\frac{-2(1-q)(1+q)}{1+p+q}, \frac{2p(2+p)}{1+p+q}\right).
$$

At the point of constrained maximum we have $\nabla s(p,q) = \lambda \nabla g(p,q)$, which reduces to

$$(2+p)[(2+q)\log(1-p) - (1+q)\log p + H(q)]$$
$$= -(1-q)[(2+p)\log(1-q) - (1+p)\log q + H(p)].$$

The latter becomes, after collecting the terms:

$$3(1+p+q)\log(1-p) = 2(1+p+q)\log p + (1+p+q)\log q,$$

so $p^2q = (1-p)^3$, as claimed.

It remains to verify the formula (7). We have

$$
\begin{aligned}
f(\theta) &= -\log(1-p) - \frac{\theta}{2}\log\left[\frac{(1-q)(1-p)}{qp}\right] \\
&= -\log(1-p) - \frac{p(1-q)}{1+p+q}\log\left[\frac{(1-q)(1-p)}{qp}\right].
\end{aligned}
$$

Comparing the latter with

$$s(p,q) = \frac{\left(p\log\frac{1-p}{p} - \log(1-p)\right)(1+q) + p\left(q\log\frac{1-q}{q} - \log(1-q)\right)}{1+p+q}$$

results in $(1+p+q)(f(\theta) - s(p,q)) = -p[2\log\frac{1-p}{p} + \log\frac{1-q}{q} + \log(1-p) - \log(1-q)] = 0$, whenever $p^2q = (1-p)^3$, as desired. $\qquad\square$

PROOF OF LEMMA 4. Fix any x and an even $n \in \mathbb{N}$. We continue to use the notation $N_{ij} = N_{ij}(x_1^{n/2})$. Since $N_{00} + N_{01} + N_{10} + N_{11} = n/2$, we have from (12):

$$
\begin{aligned}
\mathbb{P}_\mu[x_1^n] &= p^{N_{1,\text{odd}}+N_{01}}(1-p)^{n-N_{1,\text{odd}}-N_{01}-N_{10}-N_{11}}q^{N_{10}}(1-q)^{N_{11}} \\
&= (1-p)^n\left(\frac{p}{1-p}\right)^{N_{1,\text{odd}}+N_{01}}\left(\frac{q}{1-q}\right)^{N_{10}}\left(\frac{1-q}{1-p}\right)^{N_{10}+N_{11}}.
\end{aligned}
$$

Observe that

$$N_{10} + N_{11} = N_1(x_1^{n/2}) \quad \text{and} \quad N_{1,\text{odd}} = N_1(x_1^n) - N_{01} - N_{11}.$$

The equation $p^2q = (1-p)^3$ can be rewritten as $\frac{1-q}{1-p} = \frac{1-q}{q}\left(\frac{1-p}{p}\right)^2$. Combining this with the last several equalities yields

$$
\begin{aligned}
\mathbb{P}_\mu[x_1^n] &= (1-p)^n\left(\frac{p}{1-p}\right)^{N_1(x_1^n)-N_{11}-2N_1(x_1^{n/2})}\left(\frac{1-q}{q}\right)^{N_{11}} \\
&= (1-p)^n\left(\frac{(1-q)(1-p)}{qp}\right)^{N_{11}}\left(\frac{p}{1-p}\right)^{N_1(x_1^n)-2N_1(x_1^{n/2})}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{-\log \mathbb{P}_\mu[x_1^n]}{n} &= -\log(1-p) - \frac{N_{11}(x_1^{n/2})}{n} \log\Big(\frac{(1-q)(1-p)}{qp}\Big) \\
&\quad + \Big(\frac{N_1(x_1^n)}{n} - \frac{N_1(x_1^{n/2})}{n/2}\Big) \log\Big(\frac{p}{1-p}\Big) \\
&= f(\theta) + \Big(\theta - \frac{N_{11}(x_1^{n/2})}{n}\Big) \log\Big(\frac{(1-q)(1-p)}{qp}\Big) \\
&\quad + \Big(\frac{N_1(x_1^n)}{n} - \frac{N_1(x_1^{n/2})}{n/2}\Big) \log\Big(\frac{p}{1-p}\Big), \qquad (14)
\end{aligned}
$$

where $f(\theta)$ is from (7). Observe that $\lim_{n\to\infty} \frac{N_{11}(x_1^{n/2})}{n} = \theta/2$ for all $x \in A_\theta$. Now replace n by 2^ℓ for $\ell = 1, \ldots, L$, and take the average over ℓ. The expression in the last line of (14) "telescopes," so we obtain

$$
\begin{aligned}
\frac{1}{L}\sum_{\ell=1}^{L}\Big(\frac{-\log \mathbb{P}_\mu[x_1^{2^\ell}]}{2^\ell} - f(\theta)\Big) &= \frac{1}{L}\sum_{\ell=1}^{L}\Big(\frac{\theta}{2} - \frac{N_{11}(x_1^{2^\ell})}{2^\ell}\Big) \log\Big(\frac{(1-q)(1-p)}{qp}\Big) \\
&\quad + \frac{1}{L}\log\Big(\frac{p}{1-p}\Big)\Big(\frac{N_1(x_1^{2^L})}{2^L} - \frac{N_1(x_1^2)}{2}\Big),
\end{aligned}
$$

which tends to zero, as $L \to \infty$. It follows that

$$
\liminf_{\ell\to\infty} \frac{-\log \mathbb{P}_\mu[x_1^{2^\ell}]}{2^\ell} \le f(\theta) \quad \text{for all } x \in A_\theta,
$$

and the proof of (8) is complete.   □

4   Concluding remarks.

1. It is not hard to verify that, under the conditions (4) and (5) we have $\frac{(1-q)(1-p)}{qp} < 1$ if and only if $\theta < 1/4$. Therefore, by the argument in the last section, it immediately follows that

$$
\dim_H(A_\theta^+) = f(\theta) \ \text{ for } \theta \in (0, 1/4), \qquad \dim_H(A_\theta^-) = f(\theta) \ \text{ for } \theta \in (1/4, 1),
$$

where

$$
A_\theta^+ := \Big\{(x_k)_1^\infty \in \Sigma_2 : \ \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k x_{2k} \le \theta\Big\},
$$

$$A_\theta^- := \Big\{ (x_k)_1^\infty \in \Sigma_2 : \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^n x_k x_{2k} \geq \theta \Big\}.$$

2. We extended the result of Theorem 1 to the case of arbitrary functions f_1, f_2 on the shift Σ_2 depending on the first digit x_1. The method is the same, but the calculations are more involved, so we only state the result.

Theorem 7. *For $\beta, \gamma \in \mathbb{R}$, let*

$$A_\theta(\beta, \gamma) := \Big\{ (x_k)_1^\infty \in \Sigma_2 : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n (x_k + \beta)(x_{2k} + \gamma) = \theta \Big\}.$$

We have

$$\dim_H (A_\theta(\beta, \gamma)) = -\frac{1}{2} \log_2[p_0(1-p)] - \frac{\theta}{2} \log_2 \Big[\frac{(1-q)(1-p)}{qp} \Big],$$

where

$$\frac{1-p}{1-q} = \Big(\frac{q}{1-q} \Big)^{1+2\beta+\gamma} \Big(\frac{p}{1-p} \Big)^{2+2\beta+\gamma}, \qquad \frac{1-p_0}{p_0} = \Big(\frac{q}{1-q} \Big)^\beta \Big(\frac{p}{1-p} \Big)^{1+\beta},$$

and

$$\theta = \beta\gamma + \frac{(1+\beta+\gamma-q)(1+p-p_0) + \beta(p_0(p+q)-q)}{1+p+q}.$$

The appropriate measure is $\mathbb{P}_\mu$, with μ Markov, having the initial distribution $(p_0, 1-p_0)$ and the transition matrix $\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$.

3. After this work was essentially completed, we were informed that A.-H. Fan, J. Schmeling, and M. Wu have computed the dimension of A_θ (in a different, but equivalent form) and other sets of this type, independently, but also building on [15].

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