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Jeremiah J. Bass, Department of Mathematics, Tarleton State University, Stephenville, Texas 76402, U.S.A. email: jbass@tarleton.edu

MYCIELSKI-REGULAR MEASURES

Abstract

Let μ be a Radon probability measure on the Euclidean space $\mathbb{R}^d$ for $d \geq 1$, and $f : \mathbb{R}^d \to \mathbb{R}$ a measurable function. Given a sequence in $(\mathbb{R}^d)^{\mathbb{N}}$, for any $x \in \mathbb{R}^d$ define $f_n(x) = f(x_k)$, where x_k is the first among $x_0, \ldots, x_{n-1}$ that minimizes the distance from x to x_k, $0 \leq k \leq n-1$. The measures for which the sequence $(f_n)_{n=1}^{\infty}$ converges in measure to f for almost every sequence $(x_0, x_1, \ldots)$ are called Mycielski-regular. We show that the self-similar measure generated by a finite family of contracting similitudes and which up to a constant is the Hausdorff measure in its dimension on an invariant set C is Mycielski-regular.

1 Introduction

Consider the measure space (X, Σ, ν), where $X = \mathbb{R}^d$, for $d \geq 1$, and Σ is the domain of ν. We say that ν is a *topological measure* if Σ contains the open sets. Note that, in this case, Σ contains also all the closed sets and all Borel sets. We say that ν is *locally finite* if every bounded set has finite outer measure. If ν is a topological measure, ν is *inner regular* with respect to the compact sets means

$$\nu(E) = \sup\{\nu(K) : K \subseteq E, K \text{ is compact}\} \tag{1}$$

for all $E \in \Sigma$. Finally, ν is called a *Radon measure* if it is a complete, locally finite topological measure that is inner regular with respect to the compact sets (a *complete* measure includes all the subsets of sets of measure 0). Of course, when we say that ν is a *probability* measure, we simply mean that $\nu(X) = 1$ [3].

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In his paper, *Learning Theorems* [4], Jan Mycielski poses the following scenario: Given a metric space M and a sequence of points $(x_k)_{k=0}^{\infty}$ in M and an unknown real-valued function $f : M \to \mathbb{R}$, for which we have learned its values for $x_0, x_1, \ldots, x_{n-1}$ (but perhaps not for x_n), we predict the value of $f(x_n)$ by the following algorithm. Let $f_n : M \to \mathbb{R}$ be the function $x \mapsto f(x_k)$, where x_k is the first term of the first n elements of the sequence that minimizes the distance from x to x_i, for $0 \le i \le n-1$. To make the dependence of f_n on the sequence $\vec{x} = (x_k)_{k=0}^{\infty}$ clear, we denote $f_n(x) = f_n(x; \vec{x}\restriction_{n-1})$. In his paper, Mycielski proves the following theorem:

Theorem 1.1. *Let ν be a Radon probability measure on the Euclidean space $\mathbb{R}^d$, and $P = \nu^{\mathbb{N}}$ the product measure in $(\mathbb{R}^d)^{\mathbb{N}}$. If $f : \mathbb{R}^d \to \mathbb{R}$ is ν-measurable, then*

$$\lim_{n\to\infty} P(|f_n(x_n; \vec{x}\restriction_{n-1}) - f(x_n)| < \epsilon) = 1 \tag{2}$$

for every $\epsilon > 0$.

In other words, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ and $\delta > 0$ such that for all $n \ge N$,

$$\nu^{\mathbb{N}}(E_{n,\epsilon}) = P(E_{n,\epsilon}) \ge 1 - \delta, \tag{3}$$

where $\vec{x} \in E_{n,\epsilon}$ if and only if $|f_n(x_n, \vec{x}\restriction_{n-1}) - f(x_n)| < \epsilon$. So given $\epsilon > 0$, the probability that you will choose a sequence in $(\mathbb{R}^d)^{\mathbb{N}}$ such that f_n differs from f by less than ϵ at the nth term of that sequence goes to 1 as $n \to \infty$.

Mycielski then noted that it would be interesting to estimate the rate of convergence in Theorem 1.1; and it seems that *$f_n \to f$ in ν measure for P-almost every sequence* $(x_0, x_1, \ldots)$. That is, it seems in the context of Theorem 1.1 that for all $\epsilon > 0$

$$\lim_{n\to\infty} \nu(\{x \in \mathbb{R}^d : |f_n(x; \vec{x}\restriction_{n-1}) - f(x)| < \epsilon\}) = 1 \tag{4}$$

for P-a.e. sequence in $(\mathbb{R}^d)^{\mathbb{N}}$.

In other words, for P-a.e. $\vec{x} \in (\mathbb{R}^d)^{\mathbb{N}}$, $\nu(G(n, \vec{x})) \to 1$, where $x \in G(n, \vec{x})$ if and only if $|f_n(x; \vec{x}\restriction_{n-1}) - f(x)| < \epsilon$. We think that both (2) and (4) hold for Radon probability measures on $\mathbb{R}^d$. Theorem 1.1 certainly shows that for all Radon probability measures on $\mathbb{R}^d$, (2) holds. We do not know if it can be shown that (2) implies (4). If it could, then we would have (4) immediately by virtue of Theorem 1.1. Instead, we have used some results in a note by David H. Fremlin.

Fremlin has called measures that satisfy the condition that $f_n \to f$ in ν-measure for P-almost every sequence $(x_0, x_1, \ldots)$, *Mycielski-regular*. He has proved that for the the unit cube with the Euclidean metric, Lebesgue measure is Mycielski-regular [2]. The purpose of this paper is to extend Fremlin's result to other measures on $\mathbb{R}^d$ with the Euclidean metric. In particular, we show in Theorem 3.2 that the self-similar measure generated by a finite family of contracting similitudes and which up to a constant is the Hausdorff measure in its dimension on the invariant set C is Mycielski-regular. We begin by defining the central idea behind our proof. That idea is what Fremlin calls *moderated Voronoi tessellations*. He proves in [2] that a measure is Mycielski-regular if it has moderated Voronoi tessellations. It is this implication which provides the basic foundation for our method. In particular, we apply this result to certain self-similar measures. Since the self-similar measures we study have all their mass on an invariant set which is constructed via similitudes that obey the open set condition, we then spend some time reminding the reader of the theory in general of such self-similar measures and the open set condition.

Finally, we present our results in Theorem 3.2, in which we show that if a self-similar measure μ concentrating its mass on the invariant set C which is constructed via similitudes which obey the open set condition, then μ is Mycielski-regular. We note here that our proof includes the case where the contraction ratios of the similitudes vary. The proof for the case where the contraction ratios are the same is almost identical, although somewhat cleaner in the sense that at each level the images of the similitudes have the same size. This, of course, is no longer true if you allow the contraction ratios to vary.

To begin, however, we provide the proper setting and definitions, and give a couple of examples of measures - one which is *not* Mycielski-regular, and one which is. The first example shows that this is a non-trivial question: there are measures which do not converge in measure for these sequences of functions.

2 Foundations

We begin with notation. When referring to sets, the absolute value notation will refer to the diameter of a set (otherwise, it carries its usual meaning). So, if A is a subset of some metric space (X, ρ), then $|A| = \sup\{\rho(x, y) : x, y \in A\}$. When referring to the number of elements in a set - its cardinality - we will simply write $\mathrm{Card}(A)$. The interior of a set A will be denoted by $\mathrm{int}\, A$.

The following definitions are from David Fremlin [2]:

Definition 2.1. *Let (X, ρ) be a metric space. Let $\omega = (x_k)_{k=0}^{\infty}$ be an infinite sequence in $X^{\mathbb{N}}$ and let $\omega[n] = \{x_0, \ldots, x_{n-1}\}$. Suppose that $z \in \omega[n]$. Define the Voronoi tile $V(\omega\restriction n, z)$ by*

$$V(\omega\restriction n, z) = \{x \in X : \rho(x, z) = \rho(x, \omega[n]) \text{ and if } i < j < n \text{ and } z = x_j \neq x_i, \text{ then } \rho(x, z) < \rho(x, x_i)\}. \tag{5}$$

We call the collection of such $V(\omega\restriction n, z)$ the Voronoi tessellation defined by $\omega[n]$.

That is, $x \in V(\omega\restriction n, z)$ either if the distance from x to z is smaller than its distance to any other element of $\omega[n]$, or if x is equidistant from two such points x_i and x_j, then x belongs to the $V(\omega\restriction n, z)$ such that z is equal to the first of the two elements x_i and x_j.

It is useful to note the fact that the Voronoi tessellation induces a partition on the space X. (If a point is repeated in the first n entries of the sequence ω, say $\omega(i) = \omega(j)$ for $i < j$, then $V(\omega\restriction n, \omega(j)) = \varnothing$.) It is also easily seen that in a Banach space, the Voronoi tiles $V(\omega\restriction n, z)$ are convex sets and that $\varnothing \neq \operatorname{int} V(\omega\restriction n, z) \subseteq V(\omega\restriction n, z) \subseteq \overline{\operatorname{int} V(\omega\restriction n, z)}$. The next definition is another way to define the function f_n above.

Definition 2.2. *Let $f : X \to \mathbb{R}$, and $\omega[n]$ as above, and write $x_i = x(i)$. Let $k(\omega[n], x)$ be the least i such that $\rho(x, \omega[n]) = \rho(x, x(i))$, so that $x \in V(\omega\restriction n, x(k(\omega[n], x)))$. Define $F(\omega\restriction n, f)(x) = f(x(k(\omega[n], x)))$.*

Definition 2.3. *Let (X, Σ, μ) be a measure space with μ a topological probability measure. Let λ be the product measure $\mu^{\mathbb{N}}$ on $\Omega = X^{\mathbb{N}}$. We say that μ is Mycielski-regular provided for every measurable $f : X \to \mathbb{R}$, the sequence $(F(\omega\restriction n, f))_{n=1}^{\infty}$ converges in measure to f for λ-almost every ω.*

Example. An easy example of a measure which is not Mycielski-regular is given by Fremlin [2]. Assume that there exists a countably additive extension μ of Lebesgue measure to all subsets of $X = [0, 1]$. Let $A = [0, 1/2)$ and $B = [1/2, 1]$, and let μ_A and μ_B be conditional measure induced by μ on A and B, respectively.

Let ρ be the zero-one metric on X; that is,

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{otherwise.} \end{cases}$$

Given a sequence $\omega = (x_i)_{i=0}^{\infty}$, then for any n, $V(\omega\restriction n, x_0) = X \setminus \omega[n] \cup \{x_0\}$, and has measure 1, while $\mu\{z\} = 0$ for $z \in \omega[n]$. Hence, for almost every $x \in X$,

$$F(\omega\restriction n, f)(x) = f(x_0).$$

In particular, if $f = \chi_A$ (where χ_A is the characteristic function on the set A), then

$$F(\omega\restriction n, f)(x) = \begin{cases} 1, & \text{if } x_0 \in A, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\mu\{x \in X : |F(\omega\restriction n, f)(x) - f(x)| \geq \epsilon\} \geq 1 > 1/2$ for every sequence and for every n, and so $F(\omega\restriction n, f)$ never converges in measure to f, and hence is not Mycielski-regular.

We are aware that it is not certain that such a measure exists; however, in [2] Fremlin gives another example of a measure on a compact metric space which is not Mycielski-regular. This example does not depend on the questionable assumption we asked our readers to make in the one above. (The reason we give it instead of the other is that the above example is more straightforward.)

Example. An example of a measure that is Mycielski-regular - though it is rather uninteresting! - is one that concentrates all its mass on a single point. For example, let (X, Σ, μ) be a measure space and let $x_0 \in X$ such that $\mu\{x_0\} = 1$, and $\mu(X \setminus \{x_0\}) = 0$. If $P = \mu^{\mathbb{N}}$, then P-almost every sequence is the constant sequence $(x_0)_{i=0}^{\infty}$. Hence, using Mycielski's notation, $f_n(x_0) = f(x_0)$ and so $\mu(x \in X : |f_n(x) - f(x)| < \epsilon) = \mu\{x_0\} = 1$ for P-almost every sequence in $X^{\mathbb{N}}$. Indeed, Fremlin has shown that any probability measure such that $supp(\mu)$ is countable is Mycielski-regular [2].

2.1 The Condition for a Measure to be Mycielski-Regular

The proof that Fremlin gives for Lebesgue measure on the unit cube as well as the proof we give for self-similar measures is based on the idea of moderated Voronoi tessellations and the relation between them and Mycielski-regular measures. Following Fremlin, we now define what moderated Voronoi tessellations are, and then state the theorem which will provide the fundamental tool we use to get our result. Theorem 2.5 is proved by Fremlin in [2].

Definition 2.4. *Let $X, \rho, \mu, \Omega,$ and λ be as defined above. We say that μ has moderated Voronoi tessellations if for every $\epsilon > 0$ there exists $M \geq 0$ such that*

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V^{'}(\omega\restriction n, z) : z \in \omega[n], \mu(V^{'}(\omega\restriction n, z)) \geq M/n\}\right) \geq \epsilon\} < \infty, \tag{6}$$

where each $V^{'}(\omega\restriction n, z)$ is the punctured Voronoi tile $V(\omega\restriction n, z) \setminus \{z\}$.

Note that if μ has moderated Voronoi tessellations for M then μ has moderated Voronoi tessellations for all $M^{'} \geq M$.

Theorem 2.5. *Let (X, ρ) be a separable metric space, μ a topological probability measure on X which has moderated Voronoi tessellations. Then μ is Mycielski-regular.*

2.2 Self-Similar Measures and the Open Set Condition

Here we give a quick review of self-similar sets and measures. As stated, the goal is to extend Fremlin's results to other measures besides Lebesgue measure. Our setting is as follows; we begin with some definitions.

A *similitude* is a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists a constant $c > 0$ for which $|\phi(x) - \phi(y)| = c|x - y|$ for all $x, y \in \mathbb{R}^n$. If $0 < c < 1$, then we call ϕ a *contracting* similitude. Let X be a subset of $\mathbb{R}^d$, ρ the Euclidean metric, and let $\phi_i : X \to X$ for $1 \leq i \leq l$, be similitudes with contraction ratios $r_1, r_2, \ldots, r_l$, and for which the open set condition holds (to be explained shortly). Let

$$\phi(F) = \bigcup_{i=1}^{l} \phi_i(F). \tag{7}$$

A set C is called an *invariant set* if $C = \bigcup_{i=1}^{l} \phi_i(C)$. Moreover, if $\mathcal{H}^s(C) > 0$ for some $s > 0$ and $\mathcal{H}^s(\phi_i(C) \cap \phi_j(C)) = 0$ (where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure), then we call C a *self similar set.* We will be looking at measures that concentrate their mass on an invariant subset of X which is compact, and which is constructed via these similitudes. It is shown in [1], for example, that such a set exists.

The following theorem gives us the existence of the self-similar measure we are interested in, and is proved in [1].

Theorem 2.6. *There exists a Borel measure μ with support contained in C, such that $\mu(\mathbb{R}^d) = 1$ and such that for any measurable set F,*

$$\mu(F) = \sum_{j=1}^{l} r_j^s \mu(\phi_j^{-1}(F)). \tag{8}$$

In the sequel, we will assume that the open set condition is satisfied for a finite family of similitudes $\{\phi_i\}_{i=1}^{l}$. The *open set condition* means that there exists a bounded open set V such that

$$\phi(V) = \bigcup_{j=1}^{l} \phi_j(V) \subseteq V, \tag{9}$$

and this union is disjoint. Let $\phi_{j_1} \circ \cdots \circ \phi_{j_k} = \phi_{j_1 \ldots j_k}$, and $V_{j_1 \ldots j_k} = \phi_{j_1} \circ \cdots \circ \phi_{j_k}(V)$. By applying $\phi_{j_1 \ldots j_k}$, we can see that the sets $V_{j_1 \ldots j_k}$ form a net in the sense that each one is disjoint from the other, contains or is contained in the other. We also have that

$$C = \bigcap_{k=0}^{\infty} \phi^k(\overline{V}). \tag{10}$$

It turns out (see [1]) that if you have the open set condition for the similitudes $\{\phi_i\}$ with corresponding reduction ratios r_i such that $\sum_{i=1}^{l} r_i^s = 1$, then the set C is an s-set; that is, $0 < \mathcal{H}^s(C) < \infty$. In this case, we say that the self-similar measure μ is associated with the s-dimensional Hausdorff measure. Moreover, if the open set condition holds for the similitudes, then it follows that C is self-similar, and as μ concentrates its mass on a self-similar set, we call μ a self-similar measure.

3 Self-Similar Measures and Mycielski-Regularity

In this section, X is a convex and bounded subset of $\mathbb{R}^d$, ρ is the Euclidean metric, and μ is the self-similar measure which up to a constant is Hausdorff measure on the invariant set C. The maps ϕ_i are similitudes and $r_i < 1$ are the corresponding contraction ratios, where $1 \le i \le l$. We assume that the open set condition is satisfied (taking V as the interior of X), so that the results in the previous paragraphs apply. First, we give an important geometric lemma, the proof of which is given in [1].

Lemma 3.1. *Let $\{V_i\}$ be a collection of disjoint open subsets of $\mathbb{R}^d$ such that V_i contains a ball of radius $c_1\zeta$ and is contained in a ball of radius $c_2\zeta$. Then any ball B of radius ζ intersects no more than $\Gamma = (1+2c_2)^d c_1^{-d}$ of the sets $\overline{V}_i$.*

We now present our contribution to the question posed by Mycielski. This theorem, our most general result, will show that any self-similar measure which is Hausdorff measure (up to a constant) on an invariant set is Mycielski-regular.

Theorem 3.2. *Let (X, ρ, μ), ϕ_i and r_i be as defined above. Then μ has moderated Voronoi tessellations, and thus is Mycielski-regular.*

Proof. Suppose that the open set V given by the OSC contains a ball of radius c_1 and is contained in a ball of radius c_2. Let $\zeta = 1/l^{t/s}$, let $\beta = \min r_j$, and $\mathcal{S}_t$ be the set of finite sequences obtained in the following way: for each infinite sequence $\{j_1, j_2, \ldots\}$, $1 \le j_i \le l$, truncate the sequence at the first $k \ge 1$ for which $\beta\zeta \le r_{j_1} r_{j_2} \cdots r_{j_k} \le \zeta$. It follows from the net property of the open sets that $\{V_{j_1 \ldots j_k} : j_1 \ldots j_k \in \mathcal{S}_t\}$ is a disjoint collection. Each such $V_{j_1 \ldots j_k}$ contains a ball of radius $c_1 r_{j_1} \cdots r_{j_k}$ and hence a ball of radius $c_1\beta\zeta$ and similarly is contained in a ball of radius $c_2\zeta$. By Lemma 3.1, any ball of radius ζ intersects, at most, $(1+2c_2)^d c_1^{-d}(\min r_j)^{-d}$ sets of the collection $\mathcal{V}_t = \{\overline{V}_{j_1\ldots j_k} : j_1\ldots j_k \in \mathcal{S}_t\}$. Also, for large $n \in \mathbb{N}$, choose $t(n) \in \mathbb{N}$ such that

$$\log_l n - \log_l M \geq t(n) \geq \log_l n - \log_l M - 1. \tag{11}$$

Note that as $n \to \infty$, so does $t(n)$. Also note that usually we denote $t(n)$ by t, unless there is a reason to specifically highlight its dependence on n.

Define $\mu_{j_1\ldots j_k}(F) = \mu((\phi_{j_1} \circ \cdots \circ \phi_{j_k})^{-1}(F)) = \mu(\phi_{j_k}^{-1} \circ \cdots \circ \phi_{j_1}^{-1}(F))$. Then the measure $\mu_{j_1\ldots j_k}$ is supported on $C_{j_1\ldots j_k}$ and

$$\mu_{j_1\ldots j_k} = \sum_j r_j^s \mu_{j_1\ldots j_k j}. \tag{12}$$

By iterating (12) where appropriate, we get that

$$\mu = \sum_{j_1\ldots j_k \in \mathcal{S}_t} (r_{j_1\ldots j_k})^s \mu_{j_1\ldots j_k}. \tag{13}$$

Now let $z \in C$ and let $E = B(z, 2c_2\zeta)$. Then E intersects at most $\Gamma = (6c_2)^d c_1^{-d}(\min r_j)^{-d}$ members of $\mathcal{V}_t$. Let $\mathcal{K} = \{\overline{V}_\sigma \in \mathcal{V}_t : \overline{V}_\sigma \cap E \neq \varnothing\}$ (where $\sigma \in \{1,2,\ldots,l\}^k$ for some k). So $\mathrm{Card}(\mathcal{K}) \leq \Gamma$.

Let $W \subseteq X$ be convex and let $y \in W \setminus \cup\mathcal{K}$; suppose that $y \in \overline{V}_\sigma$, $\sigma \in \mathcal{S}_t$ for some $\overline{V}_\sigma \notin \mathcal{K}$. Then $\overline{V}_\sigma \subseteq \operatorname{int} B(y, \rho(y,z))$, since $\rho(y,z) > 2c_2\zeta$. So $y \in V_z = W \cap \cup_{y\in W}\{\overline{V}_\sigma : \overline{V}_\sigma \subseteq \operatorname{int} B(y,\rho(y,z))\}$. So $W \setminus V_z \subseteq \cup\mathcal{K}$.

Claim 3.3. *$\cup\mathcal{K}$ has measure at most $\Gamma\zeta^s = \Gamma/l^t$ (and hence so does $W \setminus V_z$).*

Proof. To see this, let $\overline{V}_\sigma \in \mathcal{K}$ and let $\sigma \in \mathcal{S}_t$ be such that $\sigma = (a_1, a_2, \ldots, a_m)$, and so

$$\begin{aligned}
\mu(V_\sigma) &= \sum_{j_1\ldots j_k \in \mathcal{S}_t} (r_{j_1}\cdots r_{j_k})^s \mu_{j_1\ldots j_k}(V_\sigma) && (14)\\
&= \sum_{j_1\ldots j_k \in \mathcal{S}_t} (r_{j_1}\cdots r_{j_k})^s \mu(\phi_{j_k}^{-1}\circ\cdots\circ\phi_{j_1}^{-1}(\phi_{a_1}\circ\cdots\circ\phi_{a_m}(V))) && (15)\\
&= (r_{a_1}\cdots r_{a_m})^s && (16)\\
&\leq \zeta^s, && (17)
\end{aligned}$$

since $(a_1, a_2, \ldots, a_m) \in \mathcal{S}_t$ and since $\mu(\phi_{a_m}^{-1}\circ\cdots\circ\phi_{a_1}^{-1}(\phi_{a_1}\circ\cdots\circ\phi_{a_m}(V))) = 1$, whereas every other term in the sum has measure zero, since $\phi_i^{-1}\circ\phi_j(F) = \varnothing$ when $i \neq j$ and for any $F \subseteq X$. This shows the claim. □

Let $\omega \in \Omega$. Define $H_n(\omega)$ as before and let

$$\mathcal{K}_\omega = \{V_\sigma \in \mathcal{S}_t : V_\sigma \cap \omega[n] = \varnothing\}. \tag{18}$$

Let $\epsilon > 0$. Suppose that $\mu(V(\omega \restriction n, z)) \geq 2\Gamma l^{-t}$. It follows that if $y \in V(\omega \restriction n, z)$, $(y \neq z)$, then $\operatorname{int} B(y, \rho(y,z)) \cap \omega[n] = \varnothing$ and if $V_\sigma \subseteq \operatorname{int} B(y, \rho(y,z))$ then $V_\sigma \in \mathcal{K}_\omega$. By above, we have that $\mu(V(\omega \restriction n, z) \setminus \bigcup \mathcal{K}_\omega) \leq \Gamma l^{-t}$ and so $\mu(V(\omega \restriction n, z) \leq 2\mu(V(\omega \restriction n, z) \cap \bigcup \mathcal{K}_\omega)$. Hence, we get

$$\mu H_n(\omega) \leq \frac{2}{l^t} \cdot \operatorname{Card}(\mathcal{K}_\omega), \tag{19}$$

and so if $H_n(\omega) \geq \epsilon$, then

$$\operatorname{Card}(\mathcal{K}_\omega) \geq \frac{\epsilon l^t}{2} \geq m, \tag{20}$$

where $m = \lfloor \frac{\epsilon l^t}{2} \rfloor$. Again, as before, we get that $\{\omega \in \Omega : \mu H_n(\omega) \geq \epsilon\} \subseteq \{\omega \in \Omega : \operatorname{Card}(\mathcal{K}) \geq m\}$ and so

$$\lambda\{\omega \in \Omega : \operatorname{Card}(\mathcal{K}) \geq m\} \leq \sum_{\mathcal{K} \in [\mathcal{V}_t]^m} \lambda\{\omega : \omega[n] \text{ does not meet} \cup \mathcal{K}\}. \tag{21}$$

Claim 3.4. $\operatorname{Card} \mathcal{K} \leq l^t / \beta^s$.

PROOF. This follows because

$$1 \geq \mu(\cup \mathcal{K}) = \sum_{V_\sigma \in \mathcal{K}} \mu(V_\sigma) \geq \sum_{V_\sigma \in \mathcal{K}} (\beta\zeta)^s \geq \sum_{V_\sigma \in \mathcal{K}} \frac{\beta^s}{l^t} = \operatorname{Card} \mathcal{K} \frac{\beta^s}{l^t}. \tag{22}$$

□

Let $M = \beta^{-1}(1 + \ln(10\lceil\beta^{-s}\rceil/\epsilon))$. Then,

$$\begin{aligned}
\lambda\{\omega \in \Omega : \mathrm{Card}(\mathcal{K}) \geq m\} &\leq \binom{\lceil l^t/\beta^s\rceil}{m}\left(1 - \frac{m\beta}{l^{t/s}}\right)^n && (23)\\
&\leq \frac{\left(\lceil \frac{l^t}{\beta^s}\rceil\right)^m}{m!}\left(1 - \frac{m\beta}{l^{t/s}}\right)^n && (24)\\
&\leq \frac{\lceil \frac{l^t}{\beta^s}\rceil^m}{m!}\left(1 - \frac{m\beta}{l^{t/s}}\right)^{Ml^{t/s}} && (25)\\
&\leq \frac{e^m\lceil\beta^{-s}l^t\rceil^m}{m^m}\left(1 - \frac{1}{l^{t/s}}\right)^{Mm\beta l^{t/s}} && (26)\\
&\leq \frac{e^m\lceil\beta^{-s}l^t\rceil^m}{m^m}\left(\frac{1}{e}\right)^{Mm\beta} && (27)\\
&= \left(\frac{e\lceil\beta^{-s}l^t\rceil}{me^{M\beta}}\right)^m && (28)\\
&= \left(\frac{e\lceil\beta^{-s}l^t\rceil}{\lfloor\frac{\epsilon l^t}{2}\rfloor e^{M\beta}}\right)^m && (29)\\
&\leq \frac{1}{2^m} && (30)\\
&\leq \frac{1}{2^{\epsilon l^t/2}}. && (31)
\end{aligned}$$

By (11), we have that $l^t \geq n/Ml$ and so

$$\frac{1}{2^{\epsilon l^t/2}} \leq \left(\frac{1}{2^{\epsilon/2Ml}}\right)^n. \tag{32}$$

Note that

$$\frac{1}{2^{\epsilon/2Ml}} < 1, \tag{33}$$

and so, letting $\gamma = \frac{1}{2^{\epsilon/2Ml}}$, we have that

$$\lambda\{\omega \in \Omega : \mathrm{Card}(\mathcal{K}) \geq m\} < \sum_{n=1}^{\infty}\gamma^n. \tag{34}$$

Let n_0 be such that for all $n \geq n_0$, (11) holds. Then,

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu(H_n(\omega)) \geq \epsilon\} \leq n_0 + \sum_{n=n_0}^{\infty} \lambda\{\omega : \mu H_n(\omega) \geq \epsilon\} \tag{35}$$

$$\leq n_0 + \sum_{n=n_0}^{\infty} \gamma^n < \infty. \tag{36}$$

Of course, the definition of moderated Voronoi tessellations is that for every $\epsilon > 0$ there exists $M \geq 0$ such that

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V(\omega\restriction n, z) : z \in \omega[n], \mu(V(\omega\restriction n, z)) \geq M/n\}\right) \geq \epsilon\} < \infty, \tag{37}$$

but this is true if and only if

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V(\omega\restriction n, z) : z \in \omega[n], \mu(V(\omega\restriction n, z)) \geq 2\Gamma l^{-t}\}\right) \geq \epsilon\} < \infty. \tag{38}$$

Note that if we call

$$A_n = \{\omega : \mu\left(\bigcup\{V(\omega\restriction n, z) : z \in \omega[n], \mu(V(\omega\restriction n, z)) \geq M/n\}\right) \geq \epsilon\} \tag{39}$$

and

$$B_{t(n)} = \{\omega : \mu\left(\bigcup\{V(\omega\restriction n, z) : z \in \omega[n], \mu(V(\omega\restriction n, z)) \geq 2\Gamma l^{-t(n)}\}\right) \geq \epsilon\}, \tag{40}$$

then for every $n \in \mathbb{N}$, we have that $B_{t(n)} \subseteq A_n$ since $M/n \leq 2\Gamma l^{-t(n)}$ by (11), and it follows that if $\sum \lambda(A_n) < \infty$, then $\sum \lambda(B_{t(n)}) < \infty$.

Now suppose that $\sum \lambda(B_{t(n)}) < \infty$. By (11), we also have that $A_n \subseteq B_{t(n)}$ since $2\Gamma Ml/n \geq 2\Gamma l^{-t(n)}$ (let M in (39) be replaced by $M' = 2\Gamma Ml$). It follows that $\sum \lambda(A_n) < \infty$. It follows that μ has moderated Voronoi tessellations. □

Example. Let (X, ρ, μ) be defined as follows: Let X be the two-dimensional unit square, let ρ be the Euclidean metric, and let μ be the self-similar measure described above and which concentrates its mass on the the two-dimensional Cantor set (with similarity dimension = Hausdorff dimension = $\ln 4/\ln 3$). In this case, the similitudes are $\phi_1(x, y) = (x/3, y/3)$, $\phi_2(x, y) = ((x+2)/3, y/3)$, $\phi_3(x, y) = (x/3, (y+2)/3)$, and $\phi_4(x, y) = ((x+2)/3, (y+2)/3)$. Note that the contraction ratios are obviously all the same and are equal to 1/3. As the above theorem implies, this measure is Mycielski-regular.

3.1 Conclusion

The question as to which measures are Mycielski-regular was posed for arbitrary Radon probability measures on $\mathbb{R}^d$ for $d \geq 1$. Fremlin proved it for Lebesgue measure on the unit cube. Actually, for the one-dimensional case, he has proved it for all Radon probability measures [2]. In this paper, we have shown that it is true for those self-similar measures which up to a constant are equal to Hausdorff measure in its dimension on an invariant subset of $\mathbb{R}^d$.

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