RESEARCH

Real Analysis Exchange
Vol. 37(2), 2011/2012, pp. 325–332

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THE STRUCTURE OF ARITHMETIC SUMS OF AFFINE CANTOR SETS

Abstract

In this paper we describe the structure of the arithmetic sum of two affine Cantor sets. These are self-similar sets which are part of the dynamically defined Cantor sets.

Let $\mathbf{C_1}, \mathbf{C_2}$ be affine Cantor sets with $[0, s]$ and $[0, r]$ as intervals of step 0. We explicit a generic family of these self-similar sets for which the structure of $\mathbf{C_1} + \mathbf{C_2}$ is of one of the following five types: **(i)** an M-Cantorval, **(ii)** an R-Cantorval, **(iii)** an L-Cantorval, or there exist $\lambda, \eta > 0$ and intervals I, $\tilde{I}$ of the construction of $\mathbf{C_1}$ and $\mathbf{C_2}$, respectively, such that **(iv)** $\lambda\mathbf{C_1}+\eta\mathbf{C_2} = \mathbf{C_1}\cap I+\mathbf{C_2}\cap\tilde{I}-\min I-\min\tilde{I} = [0, \lambda s+\eta r]$, or **(v)** $\lambda\mathbf{C_1}+\eta\mathbf{C_2} = \mathbf{C_1}\cap I+\mathbf{C_2}\cap\tilde{I}-\min I-\min\tilde{I}$ is homeomorphic to the Cantor ternary set. This result generalizes the description obtained by Mendes and Oliveira for the case of homogeneous Cantor sets and the one obtained by the first two authors for semi-homogeneous Cantor sets. It also provides a suitable framework in which a question of Mendes and Oliveira admits a positive answer.

Mathematical Reviews subject classification: Primary: 28A80; Secondary: 11B05
Key words: Cantor set, sums of sets
Received by the editors March 29, 2011
Communicated by: Ursula Molter
*The first and third author were supported in part by NSERC grants and the second author was supported by an NSERC USRA

1 Introduction and preliminaries

The study of measure-theoretical and topological properties of arithmetic sums of Cantor sets is of interest in many different settings, for example in connection with homoclinic bifurcations of dynamical systems ([9]) or in the study of continued fractions in number theory, as initiated by Hall [3]. In the former context, Palis asked the question of whether the difference (or sum) of two dynamically defined Cantor sets is either of Lebesgue measure zero or contains an interval. Although this is false in full generality ([2]), it is generically true as shown by Moreira and Yoccoz [8]. They proved that, generically, if the sum of the Hausdorff dimensions of two regular Cantor sets $\mathbf{C_1}$ and $\mathbf{C_2}$ is larger than 1 then $\mathbf{C_1}+\mathbf{C_2}$ contains intervals. On the other hand, it is known that if the sum of the Hausdorff dimensions of $\mathbf{C_1}$ and $\mathbf{C_2}$ is less than 1 then the arithmetic sum $\mathbf{C_1}+\mathbf{C_2}$ has Lebesgue measure zero, thus providing a positive answer to Palis's conjecture.

The question remains open when one concentrates on the particular class of affine Cantor sets, which form a collection of dynamically defined Cantor sets with strong self-similarity properties: a Cantor set $\mathbf{C}$ is *affine* if it is obtained from an interval $[0,s]$ by removing at the first step a finite number of disjoint open intervals, resulting in a finite union of non-trivial closed intervals, and then continuing (at scale) with the same pattern of removing intervals at each subsequent step of the construction. The interval $[0,s]$ is called the interval of step 0 and the closed intervals obtained at each step n of the construction ($n \geq 1$) are called intervals of step n.

In this paper we are interested in studying the arithmetic sum of two affine Cantor sets from the topological point of view.

This direction was initiated by Mendes and Oliveira [4] who described the topological structure of the arithmetic sum of two homogeneous sets when *both* have $[0,1]$ as interval of step 0. An affine Cantor set is *homogeneous* provided it has all its intervals of step 1 of equal lengths; consequently, for each $n \geq 1$ the intervals of step n have equal lengths as well.

Theorem 1.1. *([4]) Let $\mathbf{C_1}, \mathbf{C_2}$ be homogeneous Cantor sets with $[0,1]$ as interval of step 0. Then $\mathbf{C_1}+\mathbf{C_2}$ is either:* **(i)** *an M-Cantorval,* **(ii)** *an R-Cantorval,* **(iii)** *an L-Cantorval,* **(iv)** *the interval $[0,2]$, or* **(v)** *homeomorphic to the Cantor ternary set.*

Definition 1.2. Let K be a compact subset of $\mathbb{R}$. A gap of K is a bounded connected component of $\mathbb{R} \setminus K$; an interval of K is a non-trivial connected component of K and a point of K is a trivial one. A perfect subset of $\mathbb{R}$, such that any gap is accumulated on each side by infinitely many intervals and gaps is called an M-Cantorval. A perfect subset of $\mathbb{R}$, such that any gap has an interval adjacent on its right and is accumulated on the left by infinitely many intervals and gaps is called an L-Cantorval. The definition of an R-Cantorval is analogous.

We refer the reader to [4] for specific examples of arithmetic sums which produce these types of Cantorval structures.

It was noticed in [1, Example 1.5] that the description from Theorem 1.1 does not hold for arithmetic sums of general homogeneous Cantor sets. Nevertheless, it is generically true as shown by the first two authors recently [1, Corollary 2.6].

Theorem 1.3. *([1]) Let* $\mathbf{C_1}, \mathbf{C_2}$ *be homogeneous Cantor sets. Then, generically,* $\mathbf{C_1} + \mathbf{C_2}$ *is either:* **(i)** *an M-Cantorval,* **(ii)** *an R-Cantorval,* **(iii)** *an L-Cantorval,* **(iv)** *a finite union of closed intervals, or* **(v)** *homeomorphic to the Cantor ternary set.*

If one looks outside the class of homogeneous Cantor sets, the topological structure of the arithmetic sum of two affine Cantor sets is more complex. On one hand, the arithmetic sum $\mathbf{C_1} + \mathbf{C_2}$ behaves very nicely when the sum of the Hausdorff dimensions of $\mathbf{C_1}$ and $\mathbf{C_2}$ is slightly larger than 1, in which case $\mathbf{C_1} + \mathbf{C_2}$ is an M-Cantorval. This is a generic result which was obtained by Moreira and Muñoz in [6, Theorem 1] and describes a phenomenon that is true not only for affine Cantor sets, but also for dynamically defined Cantor sets.

On the other hand, Moreira, Muñoz and Rivera-Letelier exhibited in [7] a construction of two affine Cantor sets $\mathbf{C_1}$ and $\mathbf{C_2}$ for which $\mathbf{C_1} + \mathbf{C_2}$ is persistently an LR-Cantorval, which is defined as a perfect subset of $\mathbb{R}$ with infinitely many intervals and gaps so that any gap has intervals adjacent to its left and right.

Theorem 1.4. *([7]) Given $\epsilon > 0$, let K_ϵ be the Cantor set defined by affine increasing functions ψ_i satisfying $\psi_i(I_i) = [0, 25]$, $1 \leq i \leq 6$, where*

$$I_1 = [0, 10] \qquad I_2 = [11, 11 + \epsilon^2/2] \qquad I_3 = [12, 12 + 2\epsilon^2]$$

$$I_4 = [13, 13 + \epsilon^2/2] \qquad I_5 = [14, 14 + 2\epsilon^2] \qquad I_6 = [15, 25].$$

If $0 < \epsilon < 1/100$ then there is a neighborhood $\mathcal{U}$ of K_ϵ in the C^{1+} topology such that if $(K_1, K_2) \in \mathcal{U} \times \mathcal{U}$ and $\lambda \in (\epsilon/3, 3\epsilon)$ then $K_1 + \lambda K_2$ has infinitely many connected components and all of its gaps are of the type LR.

This result provides a negative answer to the question of weather Theorem 1.1 (or, more precisely, Theorem 1.3) holds generically for affine Cantor sets, a question that was raised by Mendes and Oliveira in [4, Question 1].

It should be observed that the construction in Theorem 1.4 uses *semi-homogeneous* Cantor sets: these are affine Cantor sets for which the leftmost interval of step 1 has the same length as the rightmost interval of step 1. For semi-homogeneous Cantor sets it was shown in [1] that a weaker version of Theorem 1.3 is nonetheless true.

Theorem 1.5. *([1]) Let* $\mathbf{C_1}, \mathbf{C_2}$ *be semi-homogeneous Cantor sets with* $[0, s]$ *and* $[0, r]$ *as intervals of step 0. Then, generically, the structure of* $\mathbf{C_1} + \mathbf{C_2}$ *is of one of the following five types:* **(i)** *an M-Cantorval,* **(ii)** *an R-Cantorval,* **(iii)** *an L-Cantorval, or there exist* $\lambda, \eta > 0$ *and intervals* I*,* $\tilde{I}$ *of the construction of* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *such that* **(iv)** $\lambda\mathbf{C_1} + \eta\mathbf{C_2} = \mathbf{C_1} \cap I + \mathbf{C_2} \cap \tilde{I} - \min I - \min \tilde{I} = [0, \lambda s + \eta r]$*, or* **(v)** $\lambda\mathbf{C_1} + \eta\mathbf{C_2} = \mathbf{C_1} \cap I + \mathbf{C_2} \cap \tilde{I} - \min I - \min \tilde{I}$ *is homeomorphic to the Cantor ternary set.*

This result generalizes the description obtained by Mendes and Oliveira since it is actually *equivalent* to the statement of Theorem 1.3 when one only considers homogeneous Cantor sets $\mathbf{C_1}$ and $\mathbf{C_2}$. This is a consequence of the fact that, for homogeneous sets, $\mathbf{C_1} + \mathbf{C_2}$ is a finite union of translations of $\mathbf{C_1} \cap I + \mathbf{C_2} \cap \tilde{I}$ for any intervals I and $\tilde{I}$ of the construction of $\mathbf{C_1}$ and $\mathbf{C_2}$; as such, the conditions **(iv)** and **(v)** above imply the corresponding ones about $\mathbf{C_1} + \mathbf{C_2}$ in Theorem 1.3.

In light of these facts it is natural to conjecture that the phenomenon from Theorem 1.5 is more general and extends (generically) to the class of affine Cantor sets. The aim of the present note is to show that this is indeed true. As such, the main result of the paper (Theorem 2.2) provides a suitable framework in which the mentioned question of Mendes and Oliveira [4, Question 1] admits a positive answer.

The proof of this result follows the main steps from [1], however the arguments need to overcome the lack of symmetry which was essential in the semi-homogeneous case.

2 The main result

An important role in our argument will be played by the following result of Moreira, which is part of a more general result involving regular Cantor sets ([5], Theorem IV.1). It was also used in other papers which deal with the topological structure of the arithmetic sum of regular Cantor sets ([1], [6], [7]).

Theorem 2.1. *([5]) Let* $\mathbf{C_1}, \mathbf{C_2}$ *be affine Cantor sets with* $[0, s]$ *and* $[0, r]$ *as intervals of step 0. Let* J *and* $\tilde{J}$ *be the leftmost intervals of step 1 for* $\mathbf{C_1}$ *and*

$\mathbf{C_2}$*, respectively, and assume that*

$$\frac{\ln(|J|/s)}{\ln\left(|\tilde{J}|/r\right)} \notin \mathbb{Q}.$$

Then $\mathbf{C_1} + \mathbf{C_2}$ *does not contain an interval* $[0, \delta]$*, for all* $\delta > 0$*, if and only if there exist* $\mu > 0$ *and* $t \in \mathbb{R}$ *such that*

$$(-\mu\mathbf{C_1} + t) \cap \mathbf{C_2} = \emptyset \quad \textit{and} \quad (-\mu\mathbf{C_1} + t) \to \mathbf{C_2}.$$

The last condition means that the convex hulls $[c, d]$ and $[e, f]$ of $-\mu\mathbf{C_1} + t$ and $\mathbf{C_2}$, respectively, satisfy $c < e < d < f$.

It is easy to see that the conclusion of Theorem 2.1 can be restated as follows, which is how it will be used later in the arguments: there exists $\delta > 0$ such that $\mathbf{C_1} + \mathbf{C_2} \supset [0, \delta]$ if and only if $\mu\mathbf{C_1} + \mathbf{C_2} \supset [0, \min\{\mu s, r\}]$ for all $\mu > 0$.

We can now state the main result of the paper which describes the structure of the arithmetic sum of two affine Cantor sets. In what will follow, the implicit topology in the class of affine Cantor sets is induced from the Hausdorff metric on compact sets.

Theorem 2.2. *Let* $\mathcal{A}$ *be the class of all affine Cantor sets* $\mathbf{C}$ *with* $\min \mathbf{C} = 0$*. Then there is a residual set* $\mathcal{R} \subset \mathcal{A} \times \mathcal{A}$ *such that for* $(\mathbf{C_1}, \mathbf{C_2}) \in \mathcal{R}$ *the structure of* $\mathbf{C_1} + \mathbf{C_2}$ *is of one of the following types:* **(i)** *an M-Cantorval,* **(ii)** *an R-Cantorval,* **(iii)** *an L-Cantorval, or there exist* $\lambda,\ \eta > 0$ *and intervals* $I, \tilde{I}$ *of the construction of* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *such that* **(iv)** $\lambda\mathbf{C_1} + \eta\mathbf{C_2} = \mathbf{C_1} \cap I + \mathbf{C_2} \cap \tilde{I} - \min I - \min \tilde{I} = [0, \lambda s + \eta r]$*, or* **(v)** $\lambda\mathbf{C_1} + \eta\mathbf{C_2} = \mathbf{C_1} \cap I + \mathbf{C_2} \cap \tilde{I} - \min I - \min \tilde{I}$ *is homeomorphic to the Cantor ternary set, where* $s = \max \mathbf{C_1}$*,* $r = \max \mathbf{C_2}$*.*

The residual set $\mathcal{R}$ *consists of the pairs* $(\mathbf{C_1}, \mathbf{C_2})$ *satisfying two properties. On one hand,*

$$\frac{\ln(|J|/s)}{\ln\left(|\tilde{J}|/r\right)} \notin \mathbb{Q} \quad \textit{and} \quad \frac{\ln(|K|/s)}{\ln\left(|\tilde{K}|/r\right)} \notin \mathbb{Q}, \tag{1}$$

where J *and* K *are the leftmost and rightmost intervals of step 1 for* $\mathbf{C_1}$*, and* $\tilde{J}$ *and* $\tilde{K}$ *are the leftmost and rightmost intervals of step 1 for* $\mathbf{C_2}$*. On the other hand, for any* c, d *extremes of gaps of* $\mathbf{C_1}$ *and* e, f *extremes of gaps of* $\mathbf{C_2}$ *we have* $c + e \neq d + f$*.*

PROOF. Let $(\mathbf{C_1}, \mathbf{C_2}) \in \mathcal{R}$. We start by looking at the gaps of $\mathbf{C_1} + \mathbf{C_2}$. Any left (respectively, right) extreme of a gap of $\mathbf{C_1} + \mathbf{C_2}$ is always of the form $c + d$, with $c \in \mathbf{C_1}$ and $d \in \mathbf{C_2}$ left (respectively, right) extremes of gaps in $\mathbf{C_1}$ and $\mathbf{C_2}$. This gives us countably many intervals of the construction of $\mathbf{C_1}$ and

$\mathbf{C_2}$ which have c and d as their right (respectively, left) extremes. However, by the uniqueness of c and d, it is not hard to see that there exist intervals I, $\tilde{I}$ of the construction of $\mathbf{C_1}$ and $\mathbf{C_2}$ which have c and d as their extremes and, in addition, $\mathbf{C_1} \cap I + \mathbf{C_2} \cap \tilde{I}$ coincides with a neighborhood of $c+d$ in $\mathbf{C_1} + \mathbf{C_2}$. When $c+d$ is a right extreme of a gap of $\mathbf{C_1} + \mathbf{C_2}$ this implies that we can find $\lambda, \eta > 0$ such that $\lambda \mathbf{C_1} + \eta \mathbf{C_2} + (c+d) = \mathbf{C_1} \cap I + \mathbf{C_2} \cap \tilde{I}$ is a neighborhood of $c+d$ in $\mathbf{C_1} + \mathbf{C_2}$. The situation when $c+d$ is a left extreme of a gap can be treated in an analogous way, since $s + r - (c+d)$ is a right extreme of a gap of $(s - \mathbf{C_1}) + (r - \mathbf{C_2})$.

Assume that (**v**) is not satisfied, for every choice of $\lambda, \eta > 0$. Then any extreme $c+d$ of a gap of $\mathbf{C_1} + \mathbf{C_2}$ has intervals of $\mathbf{C_1} + \mathbf{C_2}$ arbitrarily close, and this includes the cases when $c = d = 0$ and $c = s$, $d = r$. By Theorem 2.1 the condition $\mathbf{C_1} + \mathbf{C_2} \supset [0, \delta]$, for some $\delta > 0$, is equivalent to the property that, given any $\lambda, \eta > 0$, there exists $\delta' > 0$ such that $\lambda \mathbf{C_1} + \eta \mathbf{C_2} \supset [0, \delta']$, and the same is true when we replace $\mathbf{C_1}$ and $\mathbf{C_2}$ with $s - \mathbf{C_1}$ and $r - \mathbf{C_2}$. Consequently, $\mathbf{C_1} + \mathbf{C_2}$ contains an interval $[0, \delta]$ (respectively, $[s+r-\delta, s+r]$) for some $\delta > 0$ if and only if $\mathbf{C_1} + \mathbf{C_2}$ contains an interval adjacent to $c+d$, whenever $c+d$ is a right (respectively, left) extreme of a gap of $\mathbf{C_1} + \mathbf{C_2}$. In conclusion, $\mathbf{C_1} + \mathbf{C_2}$ is either an M, R, L- Cantorval, or there is $\delta > 0$ such that

$$[0, \delta] \cup [s + r - \delta, s + r] \subset \mathbf{C_1} + \mathbf{C_2}. \tag{2}$$

We need to discuss now in more details the structure of the affine Cantor sets $\mathbf{C_1}$ and $\mathbf{C_2}$. Let $a := |J|/s$ and $\tilde{a} := |K|/s$. Then, for every $n \geq 1$, $J_n := [0, sa^n]$ is the leftmost interval of step n and $K_n := [s(1 - \tilde{a}^n), s]$ is the rightmost interval of step n of $\mathbf{C_1}$. If we denote by (sa, sw) the leftmost gap at step 1 of the construction of $\mathbf{C_1}$, for some $a < w < 1$, then $\left(sa^n, swa^{n-1}\right)$ is the leftmost gap at step n, for all $n \geq 1$. We can proceed in the same way with $\mathbf{C_2}$ and get $0 < b, \tilde{b}, \tilde{w}$ such that, for all $n \geq 1$, $\tilde{J}_n := [0, rb^n]$ and $\tilde{K}_n := [r(1 - \tilde{b}^n), r]$ are the leftmost and rightmost intervals of step n, while $\left(rb^n, r\tilde{w}b^{n-1}\right)$ is the leftmost gap at step n of the construction of $\mathbf{C_2}$. Note that the working conditions (1) translate to

$$\frac{\ln a}{\ln b} \notin \mathbb{Q} \quad \text{and} \quad \frac{\ln \tilde{a}}{\ln \tilde{b}} \notin \mathbb{Q}.$$

Let p, q be arbitrary positive integers which satisfy $sa^p \leq rb^q$. As a consequence of (2) we get $[0, \delta] \subset (s - \mathbf{C_1}) + (r - \mathbf{C_2})$ and thus by applying Theorem 2.1 to the affine Cantor sets $s - \mathbf{C_1}$ and $r - \mathbf{C_2}$ we obtain

$$\left[0, \min\left\{\frac{a^p}{b^q} s, r\right\}\right] \subset \frac{a^p}{b^q}(s - \mathbf{C_1}) + (r - \mathbf{C_2}).$$

Therefore

$$[0, sa^p] = [0, \min\{sa^p, rb^q\}] \subset a^p(s - \mathbf{C_1}) + b^q(r - \mathbf{C_2})$$

which in turn implies

$$[rb^q, rb^q + sa^p] \subset a^p\mathbf{C_1} + b^q\mathbf{C_2} = \mathbf{C_1} \cap J_p + \mathbf{C_2} \cap \tilde{J}_q.$$

The conclusion of the theorem follows once we show that there exists p, q such that $sa^p \leq rb^q$ and $[0, rb^q] \subset \mathbf{C_1} \cap J_p + \mathbf{C_2} \cap \tilde{J}_q$.

Take p', q' satisfying $sa^{p'} + rb^{q'} \leq \delta$ and let $\alpha = (w - a)/(2a) > 0$. Since $\ln a / \ln b$ is irrational we can find θ irrational such that $a = b^\theta$. We can then find integers $p \geq p'$ and $q \geq q'$ such that

$$0 \leq \frac{r}{s}\frac{b^q}{a^p} - 1 \leq \alpha. \tag{3}$$

This implies that the leftmost gap at step p of $\mathbf{C_1}$ intersects the leftmost gap at step q of $\mathbf{C_2}$:

$$\left(sa^p, swa^{p-1}\right) \cap \left(rb^q, r\tilde{w}b^{q-1}\right) \neq \emptyset. \tag{4}$$

Indeed, as a consequence of (3) and the choice of α we have

$$sa^p \leq rb^q \leq (1 + \alpha)sa^p = (1/2)(w + a)sa^{p-1} < wsa^{p-1}.$$

Finally, we claim that

$$[0, rb^q] \subset \mathbf{C_1} \cap J_p + \mathbf{C_2} \cap \tilde{J}_q. \tag{5}$$

Taking into account (2) we can write $[0, rb^q] \subset [0, sa^{p'} + rb^{q'}] \subset [0, \delta] \subset \mathbf{C_1} + \mathbf{C_2}$. Together with (4) this gives

$$[0, rb^q] \subset \mathbf{C_1} \cap [0, sa^p] + \mathbf{C_2} \cap [0, rb^q] = \mathbf{C_1} \cap J_p + \mathbf{C_2} \cap \tilde{J}_q.$$

□

Remark 2.3. By Theorem 2.1 it is clear that the alternatives (**iv**) and (**v**) are mutually exclusive. Condition (**iv**) can be also rewritten as $\lambda\mathbf{C_1} + \eta\mathbf{C_2} = \lambda(s - \mathbf{C_1}) + \eta(r - \mathbf{C_2}) = [0, \lambda s + \eta r]$ and then the first part of the proof implies that any gap of $\mathbf{C_1} + \mathbf{C_2}$ has intervals adjacent to its left and right. This shows that the alternatives (**iv**) and (**i**), (**ii**), (**iii**) are also mutually exclusive. In the same way we can conclude that (**v**) and (**ii**), (**iii**) are mutually exclusive, but it is not clear whether or not (**v**) and (**i**) are mutually exclusive.

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