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ON LAPLACE CONTINUITY

Abstract

Some properties of Laplace continuous functions and Laplace derivable functions are studied.

1 Introduction.

The Laplace derivative was introduced by Ralph Svetic [3]. Properties of this derivative were studied in [2] where the notion of Laplace continuity was given. In this article we study some properties of Laplace continuous functions and we prove that the space of all bounded Laplace continuous functions on an interval $[a, b]$ is a Banach space with respect to the usual sup norm.

2 Preliminaries.

In what follows we shall use the special Denjoy integral which is equivalent to the Perron integral and the Henstock integral (see [1]). The integration by parts formula for this integral which will be used is stated here for convenience. For a proof see [1, p. 194, Theorem 12.19].

Theorem 2.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be special Denjoy integrable and let $F(x) = \int_a^x f(t)dt$ for each $x \in [a, b]$. If $G : [a, b] \to \mathbb{R}$ has Riemann integrable derivative g in $[a, b]$ then,*

$$\int_a^b fG = FG/_a^b - \int_a^b Fg.$$

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Definitions. If f is special Denjoy integrable we simply write that f is integrable. Let f be integrable in a neighborhood of x. Then f is said to be Laplace continuous at x if for a fixed $\delta > 0$

$$\lim_{s\to\infty} s\int_0^\delta e^{-st}[f(x+t)-f(x)]dt \quad\text{and}\quad \lim_{s\to\infty} s\int_0^\delta e^{-st}[f(x-t)-f(x)]dt$$

exist and are equal to zero.

The function f is called Laplace derivable at x if

$$\lim_{s\to\infty} s^2\int_0^\delta e^{-st}[f(x+t)-f(x)]dt \quad\text{and}\quad \lim_{s\to\infty} (-s^2)\int_0^\delta e^{-st}[f(x-t)-f(x)]dt$$

exists and are equal. The common value is called the Laplace derivative of f at x and is denoted by $LD_1f(x)$.

Let $LD_1f(x)$ exist. If

$$\lim_{s\to\infty} s^3\int_0^\delta e^{-st}[f(x+t)-f(x)-tLD_1f(x)]dt$$

and

$$\lim_{s\to\infty} s^3\int_0^\delta e^{-st}[f(x-t)-f(x)+tLD_1f(x)]dt$$

exist and are equal, then the common value is called the second order Laplace derivative of f at x and is denoted by $LD_2f(x)$. Replacing "lim" by "lim inf" in the above four limits we get the definitions of

$$\underline{LD}_1^+f(x),\quad \underline{LD}_1^-f(x),\quad \underline{LD}_2^+f(x) \quad\text{and}\; \underline{LD}_2^-f(x)$$

respectively. Also we define

$$\underline{LD}_if(x) = \min[\underline{LD}_i^+f(x), \underline{LD}_i^-f(x)]$$

for $i = 1, 2$. The definitions of $\overline{LD}_1^+f(x)$ etc are similar.

A function f is said to be Baire*-1 on $[a, b]$ if every non-empty perfect set contained in $[a, b]$ contains a portion on which the restriction of f is continuous. If a function f is Baire*-1, Baire-1, or Darboux on a set $[a, b]$ then we write $f \in B_1^*[a, b]$, $f \in B_1[a, b]$ or $f \in D[a, b]$ respectively.

It is clear that if $LD_1f(x)$ exists at x then f is Laplace continuous at x. The next Theorem is proved in [2] and [3] but for completeness we give a proof here.

Theorem 2.2. *The above definitions do not depend on δ.*

Proof. Take $0<\delta_1<\delta_2$ such that $x+\delta_1$, $x+\delta_2$ is in neighborhood of x. Using Theorem 2.1

$$\int_{\delta_1}^{\delta_2} e^{-st} f(x+t)dt = e^{-s\delta_2}\int_{\delta_1}^{\delta_2} f(x+t)dt + s\int_{\delta_1}^{\delta_2}\left(e^{-st}\int_{\delta_1}^{t} f(x+\xi)\,d\xi\right)dt.$$

Now, $\int_{\delta_1}^{\delta_2} f(x+t)dt$ is bounded and $\int_{\delta_1}^{t} f(x+\xi)d\xi$ is bounded for $\delta_1<t<\delta_2$, let M be the bound of both. So

$$\left|s\int_{\delta_1}^{\delta_2} e^{-st} f(x+t)dt\right| \le sMe^{-s\delta_2} + s(e^{-s\delta_1}-e^{-s\delta_2})M \to 0 \quad \text{as } s\to\infty.$$

This completes the proof. □

It is easy to verify that, if f is continuous, then f is Laplace continuous. The next example shows that the converse is not true.

Example 2.3. *Let*

$$f(x)=\begin{cases}\sin\frac{1}{x^2} & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

Then clearly $f(x)$ is not continuous at $x=0$. We shall show that f is Laplace continuous at $x=0$.
Let $g(x)=x^3 cos\frac{1}{x^2}$ for $x\neq 0$ and set $g(0)=0$. Then $g'(0)=0$ and for $x\neq 0$,

$$g'(x)=3x^2\cos\frac{1}{x^2}+2\sin\frac{1}{x^2}=3x^2\cos\frac{1}{x^2}+2f(x).$$

Therefore

$$f(x)=\frac{1}{2}g'(x)-\frac{3}{2}x^2\cos\frac{1}{x^2}.$$

Let $\epsilon>0$ be given. Then there is $\delta>0$ such that

$$\left|t^2\cos\frac{1}{t^2}\right|<\frac{\epsilon}{3} \qquad \text{whenever} \quad 0<t<\delta.$$

Hence

$$\left|s\int_0^{\delta} e^{-st}t^2\cos\frac{1}{t^2}dt\right| \le \frac{\epsilon}{3}s\int_0^{\delta} e^{-st}dt$$

$$=\frac{\epsilon}{3}+o(1) \quad as \quad s\to\infty.$$

Now

$$\left|s\int_0^\delta e^{-st}g'(t)dt\right| \leq \left|se^{-s\delta}g(\delta)+s^2\int_0^\delta e^{-st}g(t)dt\right|$$

$$\leq \left|se^{-s\delta}g(\delta)\right| + \frac{\epsilon}{3}s^2\int_0^\delta te^{-st} = \frac{\epsilon}{3}+o(1) \quad as \quad s\to\infty.$$

Hence

$$\left|s\int_0^\delta e^{-st}f(t)dt\right| \leq \frac{1}{2}\left|s\int_0^\delta e^{-st}g'(t)dt\right| + \frac{3}{2}\left|s\int_0^\delta e^{-st}t^2\cos\frac{1}{t^2}dt\right|$$

$$\leq \frac{1}{2}\frac{\epsilon}{3}+\frac{3}{2}\frac{\epsilon}{3}+o(1) \quad as \quad s\to\infty.$$

So letting $s\to\infty$, we get

$$\lim_{s\to\infty}\sup\left|s\int_0^\delta e^{-st}f(t)dt\right| < \epsilon.$$

Since ϵ is arbitrary

$$\lim_{s\to\infty} s\int_0^\delta e^{-st}f(t)dt = 0,$$

verifying that f is Laplace continuous at zero.

Remark 2.4. *Let f be the function in previous example. Then, in a similar way, it can be shown that $\lim\limits_{s\to\infty} s^2\int_0^\delta e^{-st}f(t)dt = 0$. So $LD_1f(0) = 0$.*

3 Main results.

Lemma 3.1. *Let $\delta > 0$ be fixed and f be integrable in $[x-\delta, x+\delta]$. Then for all $s > 0$,*

$$\int_0^\delta\int_0^t e^{-st}f(x+z)\,dzdt = \frac{1}{s}\left[\int_0^\delta e^{-st}f(x+t)dt - e^{-s\delta}\int_0^\delta f(x+t)\,dt\right]$$

and

$$\int_0^\delta\int_0^t e^{-st}f(x-z)\,dzdt = \frac{1}{s}\left[\int_0^\delta e^{-st}f(x-t)dt - e^{-s\delta}\int_0^\delta f(x-t)\,dt\right].$$

Proof. We shall prove the first identity; the proof of the second is similar. Using Theorem 2.1 we get

$$\int_0^\delta e^{-st} f(x+t)dt = e^{-s\delta}\int_0^\delta f(x+t)dt + s\int_0^\delta \left[e^{-st}\int_0^t f(x+z)dz\right]dt.$$

Hence

$$s\int_0^\delta\int_0^t e^{-st} f(x+z)dzdt = \int_0^\delta e^{-st} f(x+t)dt - e^{-s\delta}\int_0^\delta f(x+t)dt.$$

This completes the proof. □

Theorem 3.2. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be integrable and $F(t) = \int_a^t f(\xi)d\xi$. If f is Laplace continuous at a point $x \in [a,b]$ then $LD_1F(x)$ exists and is equal to $f(x)$.*

Proof. Define $f(x) = f(b)$ for all $x > b$. Let δ be a fixed positive number. We have, for a fixed $x \in [a,b)$,

$$F(x+t) - F(x) = \int_x^{x+t} f(\xi)d\xi = \int_0^t f(z+x)\,dz.$$

Now by Lemma 3.1

$$\int_0^\delta e^{-st}[F(x+t) - F(x)]dt = \int_0^\delta e^{-st}\left(\int_0^t f(z+x)dz\right)dt$$

$$= \frac{1}{s}\left[\int_0^\delta e^{-st} f(x+t)dt - e^{-s\delta}\int_0^\delta f(x+t)dt\right].$$

Hence

$$\lim_{s\to\infty} s^2\int_0^\delta e^{-st}[F(x+t) - F(x)]dt$$

$$= \lim_{s\to\infty} s\int_0^\delta e^{-st} f(x+t)dt - \lim_{s\to\infty}\frac{s}{e^{s\delta}}\int_0^\delta f(x+t)dt.$$

Since f is Laplace continuous at x and $\int_0^\delta f(x+t)dt$ is finite

$$\lim_{s\to\infty} s^2\int_0^\delta e^{-st}[F(x+t) - F(x)]dt = f(x).$$

Thus $LD_1^+F(x)$ exists and equals to $f(x)$. Again take $x \in (a,b]$ then by Lemma 3.1,

$$\int_0^\delta e^{-st}[F(x-t)-F(x)]dt = -\int_0^\delta \int_0^t e^{-st} f(x-z)dzdt$$

$$= -\frac{1}{s}\left[\int_0^\delta e^{-st}f(x-t)dt - e^{-s\delta}\int_0^\delta f(x-t)dt\right].$$

So,

$$\lim_{s\to\infty}(-s^2)\int_0^\delta e^{-st}[F(x-t)-F(x)]dt = f(x).$$

Hence $LD_1^-F(x)$ exists and equal to $f(x)$. This completes the proof. □

Corollary 3.3. *If f is Laplace continuous on $[a,b]$ then $f \in D[a,b]$ and $f \in B_1[a,b]$.*

Proof. By Theorem 3.2 $LD_1F(x) = f(x)$ for all $x \in [a,b]$ where $F(x) = \int_a^x f(t)dt$. Since F is continuous on $[a,b]$, by Theorem 8 and Corollary 4 of [2], $LD_1F \in D[a,b]$ and $LD_1F \in B_1[a,b]$. So the result follows. □

Lemma 3.4. *Under the hypotheses of Theorem 3.2*

$$\underline{LD}_2F(x) = \underline{LD}_1 f(x) \quad \textit{for all} \quad x \in [a,b].$$

Proof. For any $\delta > 0$ and using Lemma 3.1 we get

$$\int_0^\delta e^{-st}[F(x+t)-F(x)-tLD_1F(x)]\,dt$$

$$= \int_0^\delta \int_0^t e^{-st}[f(z+x)-f(x)]\,dzdt$$

$$= \frac{1}{s}\left[\int_0^\delta e^{-st}[f(x+t)-f(x)]dt - e^{-s\delta}\int_0^\delta [f(x+t)-f(x)]dt\right].$$

Now multiplying by s^3 and taking " lim inf " as $s \to \infty$, we get

$$\underline{LD}_2^+F(x) = \underline{LD}_1^+ f(x).$$

Again

$$\int_0^\delta [F(x-t)-F(x)+tLD_1F(x)]e^{-st}dt$$

$$= -\int_0^\delta \int_0^t e^{-st}[f(x-z)-f(x)]dzdt$$

$$= -\frac{1}{s}\left[\int_0^\delta e^{-st}[f(x-t)-f(x)]dt - e^{-s\delta}\int_0^\delta [f(x-t)-f(x)]dt\right].$$

So

$$\underline{LD_2^-}F(x) = \underline{LD_1^-}f(x).$$

This completes the proof. $\square$

Lemma 3.5. *If $LD_1 f$ exists on $[a,b]$, then $LD_1 f$ has the Darboux property and the Baire 1 property.*

Proof. $LD_1 f$ exists on $[a,b]$ so f is Laplace continuous on $[a,b]$. So, as in Lemma 3.4, we get that $LD_2 F$ exists on $[a,b]$ and that $LD_2 F = LD_1 f$. Since F is continuous, by Theorem 17 and Corollary 14 of [2], $LD_2 F \in D[a,b]$ and $LD_2 F \in B_1[a,b]$. Hence, $LD_1 f \in D[a,b]$ and $LD_1 f \in B_1[a,b]$. $\square$

Theorem 3.6. *If f is Laplace continuous on $[a,b]$ and $\underline{LD_1} f > -\infty$ in $[a,b]$, then $f \in B_1^*[a,b]$.*

Proof. Let $F(x) = \int_a^x f(t)dt$. So by Theorem 3.2 and Lemma 3.4 $LD_1 F = f$ and $\underline{LD}_2 F = \underline{LD}_1 f$ in $[a,b]$. Since F is continuous on $[a,b]$ by Corollary 13 of [2], $LD_1 F \in B_1^*[a,b]$ and so $f \in B_1^*[a,b]$. $\square$

Theorem 3.7. *If f is Laplace continuous on $[a,b]$ and $\underline{LD}_1 f \geq 0$ on $[a,b]$ then f is nondecreasing on $[a,b]$.*

Proof. Let G be the set of all points x in $[a,b]$ such that there is a neighborhood of x in which f is continuous. By Theorem 3.6, $f \in B_1^*[a,b]$ and so $G \neq \emptyset$. We show that $G = [a,b]$. Let $P = [a,b] \sim G$. Suppose $P \neq \emptyset$. Since G is open in $[a,b]$, P is closed. Clearly P has no isolated point. For, if x_0 is an isolated point of P, then there are points α, β, $\alpha < x_0 < \beta$, such that

$$(\alpha, x_0) \cup (x_0, \beta) \subset G.$$

Since f is continuous in $(\alpha, x_0) \cup (x_0, \beta)$, by Theorem 6 of [2] f is nondecreasing in (α, x_0) and in (x_0, β). Since by Corollary 3.3, $f \in D[a,b]$, f is nondecreasing in $[\alpha, x_0]$ and in $[x_0, \beta]$ and so f is nondecreasing and continuous on $[\alpha, \beta]$. Thus $(\alpha, \beta) \subset G$, which contradicts $x_0 \in P$. So P is perfect. Since $f \in B_1^*[a,b]$ there is a portion $(c,d) \cap P$ on which f is continuous. Since f is nondecreasing in each contiguous interval, of $(c,d) \cap P$, f is continuous in (c,d) and so $(c,d) \subset G$ which is a contradiction. Therefore $P = \emptyset$ and so $G = [a,b]$. So f is continuous in $[a,b]$ and the proof follows from Theorem 6 of [2]. $\square$

Theorem 3.8. *If LD_1f exists in $[a,b]$ then for all $x, x+t \in [a,b]$ there is θ, $0 < \theta < 1$ such that*

$$f(x+t) - f(x) = tLD_1f(x+\theta t).$$

Proof. Let
$$g(z) = f(z) - \left[\frac{f(x+t) - f(x)}{t}\right](z-x).$$
We shall prove that there exists θ, $0 < \theta < 1$, such that $LD_1g(x+\theta t) = 0$. Suppose that $LD_1g(z) \geq 0$ for all $z \in [x, x+t]$. Then by Theorem 3.7, g is non decreasing in $[x, x+t]$. Since $g(x) = g(x+t)$ the existence of θ is proved. If $LD_1g(z) \leq 0$ the existence of θ is proved by a similar argument.

So suppose that there are ξ and η in $[x, x+t]$ such that $LD_1g(\xi) > 0$ and $LD_1g(\eta) < 0$. Since, by Lemma 3.5, $LD_1g \in D[x, x+t]$, there is θ, $0 < \theta < 1$, such that $LD_1g(x+\theta t) = 0$. This completes the proof. □

Theorem 3.9. *Let LD_1f exist and be bounded on $[a,b]$. Then LD_1f is Lebesgue integrable on $[a,b]$ and*

$$\int_a^x LD_1f(t)\,dt = f(x) - f(a)$$

for all $x \in [a,b]$.

Proof. By Lemma 3.5, $LD_1f \in B_1[a,b]$ and so LD_1f is measurable and, since it is bounded, then LD_1f is Lebesgue integrable in $[a,b]$ and the function $F(x) = \int_a^x LD_1f$ is absolutely continuous in $[a,b]$. Hence F' exists and $F' = LD_1f$ almost everywhere. Also

$$|f(x+t) - f(x)| = |tLD_1f(x+\theta t)| \leq M|t|$$

where $0 < \theta < 1$ and $M = \sup\{LD_1f(x) : x \in [a,b]\}$. Hence f is absolutely continuous and so f' exists almost everywhere. Hence $f' = LD_1f$ almost everywhere. So $\int_a^x LD_1f(t)dt = f(x) - f(a)$. □

Theorem 3.10. *If LD_1f exists and is bounded on $[a,b]$, then LD_1f is Laplace continuous on $[a,b]$.*

Proof. Since LD_1f is bounded on $[a,b]$, by Theorem 3.9, for all x, $x+t \in [a,b]$

$$f(x+t) - f(x) = \int_x^{x+t} LD_1f(\xi)d\xi = \int_0^t LD_1f(x+u)du.$$

Now by Lemma 3.1

$$\int_0^\delta e^{-st}[f(x+t)-f(x)]dt = \int_0^\delta \left[e^{-st}\int_0^t LD_1 f(u+x)du\right] dt$$

$$= \frac{1}{s}\left[\int_0^\delta e^{-st} LD_1 f(t+x)dt - e^{-s\delta}\int_0^\delta LD_1 f(x+t)dt\right].$$

Multiplying by s^2 and letting $s \to \infty$ we get

$$LD_1 f(x) = \lim_{s\to\infty} s\int_0^\delta e^{-st} LD_1 f(x+t)dt.$$

Similarly we can show that

$$LD_1 f(x) = \lim_{s\to\infty} s\int_0^\delta e^{-st} LD_1 f(x-t)dt.$$

This completes the proof. □

Theorem 3.11. *If f' exists on $[a,b]$ then f' is Laplace continuous on $[a,b]$.*

Proof. Since f' exists on $[a,b]$, f' is integrable on $[a,b]$ and for every x and t such that $x, x+t \in [a,b]$

$$\int_0^t f'(x+\xi)d\xi = f(x+t) - f(x).$$

See [1, p. 108, Theorem 7.2]. Hence by Theorem 2.1, for all $x \in [a,b]$,

$$s\int_0^\delta e^{-st} f'(x+t)dt = se^{-s\delta}\int_0^\delta f'(x+t)dt + s^2\int_0^\delta \left[e^{-st}\int_0^t f'(x+\xi)d\xi\right] dt.$$

Since,

$$\lim_{s\to\infty}\left[se^{-s\delta}\int_0^\delta f'(x+t)dt\right] = 0$$

we get

$$\lim_{s\to\infty} s\int_0^\delta e^{-st} f'(x+t)dt = \lim_{s\to\infty} s^2\int_0^\delta e^{-st}[f(x+t)-f(x)]dt = LD_1 f(x).$$

Since $f'(x)$ exists, $LD_1 f(x)$ exists and $f'(x) = LD_1 f(x)$ [2, Remark 1]. This completes the proof. □

Theorem 3.12. *If $\{f_n\}$ converges uniformly to f in $[a,b]$ and if each f_n is Laplace continuous at $x_0 \in [a,b]$ then f is Laplace continuous at x_0.*

Proof. Let $\epsilon > 0$ and $\delta > 0$ be arbitrary. Since $\{f_n\}$ converges uniformly to f there is N such that

$$|f_N(x) - f(x)| < \epsilon \quad \text{for all } x \in [a,b]. \tag{1}$$

Let $x_0 \in [a,b]$. Since f_N is Laplace continuous at x_0 there is M such that

$$\left| s\int_0^\delta e^{-st}[f_N(x_0+t) - f_N(x_0)]dt \right| < \epsilon \;\; for\ s \geq M. \tag{2}$$

From (1) and (2)

$$\begin{aligned} &\left| s\int_0^\delta e^{-st}[f(x_0+t) - f(x_0)]dt \right| \\ &\leq \left| s\int_0^\delta e^{-st}[f(x_0+t) - f_N(x_0+t)]dt \right| \\ &+ \left| s\int_0^\delta e^{-st}[f_N(x_0+t) - f_N(x_0)]dt \right| + \left| s\int_0^\delta e^{-st}[f_N(x_0) - f(x_0)]dt \right| \\ &< \epsilon\left(1 - \frac{1}{e^{s\delta}}\right) + \epsilon + \epsilon\left(1 - \frac{1}{e^{s\delta}}\right) \quad \text{for } s \geq M. \end{aligned}$$

Letting $s \to \infty$, we get

$$\lim_{s\to\infty} s\int_0^\delta e^{-st}[f(x_0+t) - f(x_0)]dt \leq 3\epsilon.$$

Since ϵ is arbitrary, f is Laplace continuous at x_0. □

From this theorem one easily deduces the following corollary.

Corollary 3.13. *Let $LC[a,b]$ be the space of all bounded Laplace continuous function on [a,b]. For $f \in LC[a,b]$ define the usual sup norm*

$$||f|| = \sup\{|f(t)| : t \in [a,b]\}.$$

Then $LC[a,b]$ is a Banach space with respect to this norm.

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