Antonio Boccuto,* Dipartimento di Matematica e Informatica, via Vanvitelli, 1, I-06123 Perugia, Italy.   email: `boccuto@yahoo.it,` `boccuto@dmi.unipg.it`

Xenofon Dimitriou, Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece.   email: `xenofon11@gmail.com,` `dxenof@math.uoa.gr`

Nikolaos Papanastassiou, Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece.   email: `npapanas@math.uoa.gr`

LIMIT THEOREMS IN (l)-GROUPS WITH RESPECT TO (D)-CONVERGENCE

Abstract

Some Schur, Vitali-Hahn-Saks and Nikodým convergence theorems for (l)-group-valued measures are given in the context of (D)-convergence. We consider both the σ-additive and the finitely additive case. Here the notions of strong boundedness, countable additivity and absolute continuity are formulated not necessarily with respect to a same regulator, while the pointwise convergence of the measures is intended relatively to a common (D)-sequence. Among the tools, we use the Fremlin lemma, which allows us to replace a countable family of (D)-sequence with one regulator, and the Maeda-Ogasawara-Vulikh representation theorem for Archimedean lattice groups.

1   Introduction.

The limit theorems for absolutely continuous, σ-additive and strongly bounded set functions (Schur Lemma, Vitali-Hahn-Saks, Nikodým convergence and Brooks-Jewett theorems, see [22, 31, 37, 47, 49, 52]) are objects of several

studies in the literature. A survey about these kinds of theorems and related topics, among which some applications to integration theory, can be found in [25].

These theorems were extended to the cases of Banach space- and topological group-valued measures, defined even in some domains more general than σ-algebras. For the literature we quote, for instance, [26] and its bibliography, [2, 19, 23, 24, 30, 34, 43, 48]. These kinds of theorems have several functional analytic applications, and they are related with matrix theorems, which are powerful tools to give some results both in measure theory and in the context of operators (see for instance [1, 21, 40, 51]). Some other applications to integration, control measures and signal processing can be found, for example, in [5, 20, 33, 38].

In this paper we deal with (l)-group-valued measures. Among the literature existing for limit theorems in order structures, we quote [3, 41]. In [9, 12, 13] the authors proved some limit theorems for measures taking values in Riesz spaces and (l)-groups. In [9] they considered positive measures converging pointwise to a regular measure and positive measures taking values in a subspace of the space $L^0(X, \mathcal{B}, \mu)$ of all measurable functions (up to sets of measure zero) with respect to a σ-finite and σ-additive extended real-valued positive measure. In [12, 13] they proved some limit theorems when the (l)-group-valued measures converge pointwise with respect to a common regulator, and also the concepts of (s)-boundedness, σ-additivity and absolute continuity are defined analogously. Some other versions of limit theorems for Riesz space-valued measures, defined in abstract structures, can be found in [4].

In this paper we prove some limit theorems for (l)-group-valued σ-additive measures and finitely additive set functions. The main used tools are the Maeda-Ogasawara-Vulikh representation theorem for Archimedean (l)-groups as sets of suitable continuous extended real-valued functions (see [6]) and the Fremlin lemma (see [45]), which allows us to replace a countable family of regulators with one regulator in the proof of limit theorems, without doing too restrictive hypotheses on the involved (l)-group. For this aim and for technical reasons we often use the notion of (D)-*convergence* in (l)-groups (see also [45]). Here only the pointwise convergence of the measures (and not the notions of strongly boundedness, countable additivity and absolute continuity) is considered with respect to a common regulator. Similar results were proved in [17] for (l)-group-valued σ-additive measures with respect to the relatively uniform convergence (see [39, 53]). Some related results on matrix theorems for (l)-group-valued measures were given in [15], while in [16] some similar limit theorems and Drewnowski-type theorems on relations between finite and

countable additivity were proved in slightly different settings and with other kinds of techniques.

The paper is structured as follows. In Section 2 we introduce the preliminary notions and the tools used in our setting. In Section 3 we prove the Schur Lemma, the Vitali-Hahn-Saks and the Nikodým convergence theorem.

Our thanks to the referee for his valuable comments, remarks and suggestions to improve and simplify some parts of the paper. We thank also Prof. D. Candeloro for his helpful suggestions.

2   Preliminaries.

In this section we introduce the preliminary notions in (l)-groups and the fundamental concepts of (s)-boundedness, finite and countable additivity and absolute continuity of (l)-group-valued set functions. Furthermore, we give some examples and explain the tools and techniques, used to prove the main results of the paper.

We begin with introducing the order convergence and (D)-convergence in (l)-groups.

Definitions 2.1. An Abelian group $(R, +)$ is called (l)-*group* if it is a lattice and for any a, b, $c \in R$ we get $a + c \leq b + c$ whenever $a \leq b$.

From now on, we denote by $\vee$ and $\wedge$ the lattice supremum and infimum respectively.

An (l)-group R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

For the basic properties of (l)-groups, see [8] and [45]. Given an element $r \in R$, we call *absolute value* of r the element $|r|$ defined by setting $|r| = r \vee (-r)$. If we put $r^+ = r \vee 0$, $r^- = (-r) \vee 0$, it is not difficult to see that $r = r^+ - r^-$ and $|r| = r^+ + r^-$.

Let R be an (l)-group. We say that a sequence $(p_n)_n$ in R is an (o)-*sequence* if it is decreasing and $\wedge_n p_n = 0$. A sequence $(r_n)_n$ in R is said to be *order-convergent* (or (o)-*convergent*) to r if there exists an (o)-sequence $(p_n)_n$ in R with $|r_n - r| \leq p_n$ for all $n \in \mathbb{N}$, and in this case we write $(o) \lim_n r_n = r$. If Λ is any nonempty set, $(r_n^{(\lambda)})_n$ are sequences in R and $r^{(\lambda)} \in R$ for all $\lambda \in \Lambda$, we say that $(o) \lim_n r_n^{(\lambda)} = r^{(\lambda)}$ *uniformly with respect to* $\lambda \in \Lambda$ if there exists an (o)-sequence $(q_n)_n$ in R with $|r_n^{(\lambda)} - r^{(\lambda)}| \leq q_n$ for all $n \in \mathbb{N}$ and $\lambda \in \Lambda$. A sequence $(r_n)_n$ in R is (o)-*Cauchy* if $(o) \lim_n (r_n - r_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

We now introduce (D)-convergence (for its main properties, see [18, 45]).

A bounded double sequence $(a_{i,l})_{i,l}$ in R is called (D)-*sequence* or *regulator* if the sequence $(a_{i,l})_l$ is an (o)-sequence for all $i \in \mathbb{N}$. A sequence $(r_n)_n$ in R is said to be (D)-*convergent* to $r \in R$ (and we write $(D) \lim_n r_n = r$) if there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $n_0 \in \mathbb{N}$ with $|r_n - r| \leq \bigvee\limits_{i=1}^{\infty} a_{i,\varphi(i)}$ whenever $n \geq n_0$. If Λ is as above, $(r_n^{(\lambda)})_n$ are sequences in R and $r^{(\lambda)} \in R$ for all $\lambda \in \Lambda$, we say that $(D) \lim_n r_n^{(\lambda)} = r^{(\lambda)}$ *uniformly with respect to* $\lambda \in \Lambda$ if there is a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that $|r_n^{(\lambda)} - r^{(\lambda)}| \leq \bigvee\limits_{i=1}^{\infty} a_{i,\varphi(i)}$ whenever $n \geq n_0$ and $\lambda \in \Lambda$. The sequence $(r_n)_n$ is said to be (D)-*Cauchy* if $(D) \lim_n (r_n - r_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

We say that an (l)-group is (o)-*complete* if every (o)-Cauchy sequence is (o)-convergent, and (D)-*complete* if every (D)-Cauchy sequence is (D)-convergent. We recall that every Dedekind complete (l)-group is (o)-complete and (D)-complete (see also [18, Chapter 2]).

An (l)-group R is said to be *weakly σ-distributive* if for every (D)-sequence $(a_{i,l})_{i,l}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \Big) = 0.$$

Remark 2.2. In general, the (D)-limit of a sequence is not unique. However, (o)-convergence of sequences implies always (D)-convergence; moreover, if R is weakly σ-distributive, then a sequence is (D)-convergent if and only if it is (o)-convergent, and in this case the limit is unique (see [10] and [32, Proposition 1]).

If R is a Dedekind complete not weakly σ-distributive (l)-group (such groups do exist, see [35, 54]), then there exist a (D)-sequence $(a_{i,j})_{i,j}$ and a positive element $b \in R$, for which

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \Big) = b \neq 0.$$

We now prove that every sequence $(r_n)_n$ with $-b \leq r_n \leq b$ for all $n \in \mathbb{N}$ (D)-converges to 0, though it can be not (o)-convergent. Indeed, choose arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$: for each $n \in \mathbb{N}$ we get

$$|r_n| \leq b = \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \Big) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)},$$

that is $(D) \lim_n r_n = 0$.

From now on we assume that R is a Dedekind complete weakly σ-distributive (l)-group. Note that weak σ-distributivity guarantees the uniqueness of the (D)-limit and is used to prove σ-additivity of the limit measure in Theorem 3.1.

As in the classical definition of series, given a sequence $(r_n)_n$ in R, we denote by $\sum_{n=1}^{\infty} r_n$ the limit $(o) \lim_n \sum_{i=1}^{n} r_i = (D) \lim_n \sum_{i=1}^{n} r_i$, if it exists in R.

The following proposition will be useful in the sequel, in defining some suitable measures and proving some of its properties.

Proposition 2.3. *If* $(D) \lim_n \sum_{i=1}^{n} |r_i|$ *exists in* R, *then the limit*

$$(D) \lim_n \sum_{i \in A, i=1,\ldots,n} r_i$$

exists in R *uniformly with respect to* $A \subset \mathbb{N}$.

PROOF. First of all note that $(D) \lim_n \sum_{i=1}^{n} |r_i| = \bigvee_n \sum_{i=1}^{n} |r_i|$, since the corresponding sequence of partial sums is increasing. It is not difficult to deduce that the following equations hold in R:

$$(D) \lim_n \sum_{i \in A, i=1,\ldots,n} |r_i| = \bigvee_n \sum_{i \in A, i=1,\ldots,n} |r_i|,$$

$$(D) \lim_n \sum_{i \in A, i=1,\ldots,n} r_i^+ = \bigvee_n \sum_{i \in A, i=1,\ldots,n} r_i^+,$$

$$(D) \lim_n \sum_{i \in A, i=1,\ldots,n} r_i^- = \bigvee_n \sum_{i \in A, i=1,\ldots,n} r_i^-,$$

uniformly with respect to $A \subset \mathbb{N}$. Since $r_i = r_i^+ - r_i^-$ for all $i \in \mathbb{N}$, then it follows that

$$(D) \lim_n \sum_{i \in A, i=1,\ldots,n} r_i \tag{1}$$

exists in R uniformly with respect to $A \subset \mathbb{N}$, that is the assertion. $\qquad \square$

We denote by $\sum_{n \in A} r_n$ the (D)-limit in (1), when it exists.

The following result (Fremlin lemma, [36, Lemma 1C], see also [45, Theorem 3.2.3, pp. 42-45], [46, Proposition 2.1]) allows us to replace a countable family of (D)-sequences with one regulator, and its technique will be useful in the proof of the main results of the paper.

Lemma 2.4. *Let $(a_{i,l}^{(n)})_{i,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R such that:*

$$u \bigwedge \Big(\sum_{n=1}^{q} \bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \Big) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $q \in \mathbb{N}$ and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$.

We now recall the famous Maeda-Ogasawara-Vulikh representation theorem in its version for (l)-groups (see [6, Theorem 6]). This theorem allows us to identify any Archimedean Riesz space with a suitable space of continuous extended real-valued functions. Note that every Dedekind complete (l)-group is Archimedean (see also [8, Lemma XIII.5]). Here we denote by sup and inf the pointwise supremum and infimum respectively.

Theorem 2.5. *Given a Dedekind complete (l)-group R, there exists a compact extremely disconnected topological space Ω, unique up to homeomorphisms, such that R can be embedded as a solid subgroup of $\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{\mathbb{R}}^{\Omega} : f$ is continuous, and $\{\omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_\lambda)_{\lambda \in \Lambda}$ is any family such that $a_\lambda \in R$ for all $\lambda \in \Lambda$, and $a = \vee_\lambda a_\lambda \in R$ (where the supremum is taken with respect to R), then $a = \vee_\lambda a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$, and the set $\{\omega \in \Omega : (\vee_\lambda a_\lambda)(\omega) \neq \sup_\lambda [a_\lambda(\omega)]\}$ is meager in Ω.*

We now introduce the finitely additive and σ-additive set functions and their main properties.

Definitions 2.6. *Let G be any infinite set and $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra. A set function $m : \mathcal{A} \to R$ is bounded if there is $w \in R$, $w \geq 0$, with $|m(A)| \leq w$ for all $A \in \mathcal{A}$. The set functions $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are equibounded there exists an element $u \in R$, $u \geq 0$, such that*

$$|m_j(A)| \leq u \tag{2}$$

for every $j \in \mathbb{N}$ and for all $A \in \mathcal{A}$.

Given a finitely additive bounded set function $m : \mathcal{A} \to R$, we define m^+, m^-, $v(m)$, $\|m\|_{\mathcal{A}} : \mathcal{A} \to R$ by setting, for every $A \in \mathcal{A}$:

$$m^+(A) = \bigvee(m(B) : B \in \mathcal{A} \text{ with } B \subset A),$$

$$m^-(A) = \bigvee(-m(B) : B \in \mathcal{A} \text{ with } B \subset A),$$

$$v(m)(A) = m^+(A) + m^-(A),$$

$$\|m\|_{\mathcal{A}}(A) = \bigvee(|m(B)| : B \in \mathcal{A} \text{ with } B \subset A).$$

The quantities $m^+, m^-, v(m), \|m\|_{\mathcal{A}}$ are called *positive part, negative part, variation* and *semivariation* on $\mathcal{A}$ of m respectively. Analogously as in the real case, we have clearly

$$\|m\|_{\mathcal{A}}(A) \le v(m)(A) \le 2\|m\|_{\mathcal{A}}(A), \quad \text{for all } A \in \mathcal{A} \tag{3}$$

(see also [7]).

A set function $m : \mathcal{A} \to R$ or $m : \mathcal{A} \to [0, +\infty]$ is *finitely additive* if $m(A \cup B) = m(A) + m(B)$ whenever A, B are two disjoint elements of $\mathcal{A}$ (with the convention that $(+\infty) + a = +\infty$ for all $a \in \widetilde{\mathbb{R}}$). A finitely additive set function is said to be (s)-*bounded* if for every disjoint sequence $(H_n)_n$ in $\mathcal{A}$ we have:

$$(D) \lim_n \|m\|_{\mathcal{A}}(H_n) = 0.$$

We say that the finitely additive set functions $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are *uniformly (s)-bounded* if $(D) \lim_n [\vee_j \|m_j\|_{\mathcal{A}}(H_n)] = 0$ whenever $(H_n)_n$ is a sequence of pairwise disjoint elements of $\mathcal{A}$.

We now prove that, in the context of (l)-groups, every bounded finitely additive set function is (s)-bounded too. Differently than in the real case, the converse is in general not true (see [50, Example 3]).

Theorem 2.7. *Every bounded finitely additive set function $m : \mathcal{A} \to R$ is (s)-bounded.*

PROOF. Let R be a subgroup of $\mathcal{C}_\infty(\Omega)$, where Ω is as in 2.5. By Theorem 2.5 there is a nowhere dense set $N_0 \subset \Omega$ such that the real-valued set functions $m(\cdot)(\omega)$, $\omega \in \Omega \setminus N_0$, are finitely additive and bounded. By virtue of the classical results (see [7, Corollary 2.1.7]), they are (s)-bounded on $\mathcal{A}$.

Fix now an arbitrary disjoint sequence $(H_n)_n$ in $\mathcal{A}$. Then by Theorem 2.5 there is a meager set N, depending on $(H_n)_n$, without loss of generality with

$N \supset N_0$ and such that

$$[(D)\lim_n \|m\|_{\mathcal{A}}(H_n)](\omega) = \Big[\bigwedge_n \bigvee_{s \geq n} \|m\|_{\mathcal{A}}(H_s)\Big](\omega)$$

$$= \Big[\bigwedge_n \bigvee_{s \geq n} \bigvee\Big(|m(A)| : A \in \mathcal{A} \text{ with } A \subset H_s\Big)\Big](\omega)$$

$$= \inf_n \sup_{s \geq n} \sup\Big(|m(A)(\omega)| : A \in \mathcal{A} \text{ with } A \subset H_s\Big)$$

$$= \inf_n \sup_{s \geq n} \|m(\cdot)(\omega)\|_{\mathcal{A}}(H_s) = \lim_n \|m(\cdot)(\omega)\|_{\mathcal{A}}(H_n) = 0$$

for all $\omega \in \Omega \setminus N$. By a density argument we obtain $(D)\lim_n \|m\|_{\mathcal{A}}(H_n) = 0$, and hence we get the assertion. $\square$

Remark 2.8. Observe that, in the context of Banach spaces, the relations between boundedness and (s)-boundedness of finitely additive set functions are substantially different than in (l)-groups. First of all, note that every (s)-bounded Banach space-valued set function defined in an algebra is bounded too (see [44]). Let l^∞ and c_0 be the spaces of all real sequences and of the real sequences convergent to 0 respectively, endowed with the supremum norm. It is known that a Banach space X has the property that every finitely additive and bounded X-valued set function defined in an algebra (resp. σ-algebra) is (s)-bounded if and only if X does not contain isomorphically the space c_0 (resp. l^∞) (see [27, 28, 29]).

We now give the following example of a finitely additive bounded set function, which is not (s)-bounded. Let Σ be the σ-algebra of all Lebesgue measurable subsets of $[0,1]$, ν be the Lebesgue measure, $X = L^\infty([0,1], \Sigma, \nu)$ be the space of all essentially bounded functions (with identification up to sets of Lebesgue measure zero), endowed with the essential supremum norm $\|\cdot\|_\infty$. We define $m : \Sigma \to X$ by setting $m(A) = \chi_A$, for all $A \in \Sigma$, where χ_A is the characteristic function associated to A. Then m is obviously a finitely additive and bounded set function, since $\|m(A)\|_\infty = \|\chi_A\|_\infty \leq 1$ for every $A \in \mathcal{A}$. But m is not (s)-bounded, since for the disjoint sequence (in Σ) $E_n = \Big[\dfrac{1}{n+1}, \dfrac{1}{n}\Big)$, $n \in \mathbb{N}$, we have that $\|m(E_n)\|_\infty = 1$ for all $n \in \mathbb{N}$.

We now introduce the concept of σ-additivity.

Definitions 2.9. Let $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra. A finitely additive set function $m : \mathcal{A} \to R$ is called σ-*additive measure* on $\mathcal{A}$ if, for every decreasing

sequence $(H_n)_n$ in $\mathcal{A}$ with $\bigcap\limits_{n=1}^{\infty} H_n = \emptyset$,

$$(D)\lim_n \|m\|_{\mathcal{L}}(H_n) = 0,$$

where $\mathcal{L}$ is the σ-algebra generated by the H_n's in H_1.

The σ-additive measures $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are *uniformly σ-additive* if for each decreasing sequence $(H_n)_n$ in $\mathcal{A}$ with $\bigcap\limits_{n=1}^{\infty} H_n = \emptyset$,

$$(D)\lim_n [\vee_j \|m_j\|_{\mathcal{L}}(H_n)] = 0.$$

Our definition looks weaker than the classical one, in which the semivariation on $\mathcal{A}$ is considered rather than the semivariation on $\mathcal{L}$. In general, it is still an open problem to find conditions under which these two kinds of σ-additivity are equivalent. However, when $R = \mathbb{R}$, it is not difficult to see that a measure is σ-additive according to 2.9 if and only if it is σ-additive in the classical sense (see also [7]).

It is natural to deal with the notion of σ-additivity given in 2.9, since in the sequel we will examine in detail the case $\mathcal{A} = \mathcal{P}(\mathbb{N})$. Indeed, the following result holds:

Proposition 2.10. *A finitely additive measure $m : \mathcal{P}(\mathbb{N}) \to R$ is σ-additive on $\mathcal{P}(\mathbb{N})$ if and only if*

$$(D)\lim_n \|m\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) \tag{4}$$
$$= \bigwedge_n \|m\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = 0.$$

PROOF. The necessary part is straightforward.

We turn to the sufficient part. Let $(C_n)_n$ be any decreasing sequence in $\mathcal{P}(\mathbb{N})$ with $\bigcap\limits_{n=1}^{\infty} C_n = \emptyset$, and $\mathcal{L}$ be the σ-algebra generated by the C_n's in C_1. Without loss of generality we can assume that $C_n \subset \{n, n+1, n+2, \ldots\}$ for any $n \in \mathbb{N}$. Hence

$$v_{\mathcal{L}}(m)(C_n) \leq v_{\mathcal{P}(\mathbb{N})}(m)(\{n, n+1, n+2, \ldots\}),$$

and thus the sufficient part is proved. $\qquad\square$

Remark 2.11. Note that an analogous version of Proposition 2.10 holds even for uniform σ-additivity of a sequence of measures $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$.

We now turn to the concept of absolute continuity.

Definitions 2.12. Let $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra and $\nu : \mathcal{A} \to [0, +\infty]$, $m : \mathcal{A} \to R$ be two finitely additive set functions. We say that $m : \mathcal{A} \to R$ is *ν-absolutely continuous,* if for each decreasing sequence $(E_n)_n$ in $\mathcal{A}$, with $\lim_n \nu(E_n) = 0$, we get $(D) \lim_n \|m\|_{\mathcal{L}}(E_n) = 0$, where $\mathcal{L}$ is the σ-algebra generated by $(E_n)_n$ and

$$\|m\|_{\mathcal{L}}(E_n) = \bigvee(|m(B)| : B \in \mathcal{L} \text{ with } B \subset E_n)$$

for each $n \in \mathbb{N}$.

The finitely additive set functions $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are said to be *uniformly ν-absolutely continuous* if

$$(D) \lim_n [\vee_j \|m_j\|_{\mathcal{L}}(E_n)] = 0$$

whenever $(E_n)_n$ is a decreasing sequence in $\mathcal{A}$ such that $\lim_n \nu(E_n) = 0$.

Remark 2.13. Observe that, in our definition of ν-absolute continuity, we consider only decreasing sequences in $\mathcal{A}$, while in the classical setting all sequences in $\mathcal{A}$ ν-convergent to zero are considered. If R is a topological group, the definition of ν-absolute continuity given in 2.12 coincides with the classical one, but in general this is not true in Riesz spaces (see [9]). Indeed, let Σ the σ-algebra of all measurable subsets of $[0,1]$ and $\nu : \Sigma \to \mathbb{R}$ be the Lebesgue measure. The space $R = L^0([0,1], \Sigma, \nu)$ of all measurable functions on $[0,1]$, with identification up to subsets of Lebesgue measure zero, is Dedekind complete and has the Egorov property (see [39]). By [11, Theorem 3.1], R is weakly σ-distributive (see also [12]). Let us define $m : \Sigma \to R$ as follows: $m(A) = \chi_A$, for all $A \in \Sigma$, where χ_A is the characteristic function associated to A. Then $\nu(A_n) \to 0$ if and only if $\chi_{A_n} \to 0$ in L^1, but $(o) \lim_n \chi_{A_n} = 0 \Leftrightarrow \chi_{A_n} \to 0$ ν-almost everywhere. So, in general, the implication

$$\nu(A_n) \to 0 \Rightarrow (o) \lim_n m(A_n) = 0 \tag{5}$$

does not hold, and m is not ν-absolutely continuous in the classical sense. However, it is easy to see that (5) holds whenever $(A_n)_n$ is any decreasing sequence in Σ. Thus m is ν-absolutely continuous according the definition given in 2.12 (see also [9, Remark 1.13.1]).

We now formulate the concept of pointwise (D)-convergence for set functions with respect to a common regulator, which will be our hypothesis in all of our versions of limit theorems.

Definitions 2.14. Let $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra. Given a sequence of finitely additive set functions $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, we say that the m_j's (RD)-*converge* (or *converge pointwise with respect to a same regulator*) to m_0, or shortly $(RD) \lim_j m_j = m_0$, if there exists a (D)-sequence $(b_{i,l})_{i,l}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $A \in \mathcal{A}$ there is an integer j_0 such that

$$|m_j(A) - m_0(A)| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}, \quad \text{for all } j \geq j_0. \tag{6}$$

We say that $(D) \lim_j m_j = m_0$ *uniformly,* or in short $(U) \lim_j m_j = m_0$, if there exists a (D)-sequence $(c_{i,l})_{i,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $j_0 \in \mathbb{N}$ with

$$|m_j(A) - m_0(A)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}, \quad \text{for all } A \in \mathcal{A} \text{ and for every } j \geq j_0, \tag{7}$$

that is if and only if $(D) \lim_j m_j(A) = m_0(A)$ uniformly with respect to $A \in \mathcal{A}$.

The following definitions are useful in order to present some examples.

Definitions 2.15. Let $u \in R$, $u \geq 0$. We say that u has the *Egorov property* if, for every regulator $(a_{i,l})_{i,l}$ bounded from above by u, there exist an (o)-sequence $(b_n)_n$ and a sequence $(\varphi_n)_n$ of elements of $\mathbb{N}^{\mathbb{N}}$, such that

$$\bigvee_{i=1}^{\infty} a_{i,\varphi_n(i)} \leq b_n$$

for all $n \in \mathbb{N}$. We say that R has the *Egorov property* if each positive element $u \in R$ has the Egorov property (see also [39, pp. 458, 467]).

Remark 2.16. In [12] the authors introduced some concepts of σ-additivity, (s)-boundedness and absolute continuity with respect to a common regulator. However, there are sequences of measures (even uniformly) σ-additive according to our definition, but not with respect to the same regulator (see [50, Example 5]).

In this paper we will present some versions of limit theorems with respect to (D)-convergence. The use of (D)-convergence could seem apparently quite difficult, but it utilizes only Dedekind completeness and weak σ-distributivity of the involved (l)-group, and often it simplifies the proofs and allows us to replace a countable family of regulators with one (D)-sequence without assuming further additional hypotheses on the involved (l)-group, differently than in the contexts of relatively uniform convergence $((r)$-convergence), where

we require also some suitable regularity property (see [39, 53]) or of order convergence, where we often require super Dedekind completeness (see also [14]).

Some similar versions of the Schur and Nikodým theorems were proved in [17] with respect to relatively uniform convergence (or (r)-convergence). Note that in general (D)-convergence is weaker than (r)-convergence, and there are some cases in which (r)-convergence is strictly stronger than (D)-convergence. For example, in the space l^∞ of all bounded real sequence endowed with the usual coordinatewise ordering, (o)-convergence is strictly weaker than (r)-convergence (see [39, Theorem 16.3, p. 80 and p. 479]). Moreover, note that l^∞ is an *ideal* in the space $\mathbb{R}^\mathbb{N}$ of all real sequences: indeed, given $x \in l^\infty$ and $y \in \mathbb{R}^\mathbb{N}$ with $|y| \le |x|$, we get clearly $y \in l^\infty$ (see [53, Definition III.9.1]). Therefore, since l^∞ is an ideal of $\mathbb{R}^\mathbb{N}$, we get that l^∞ is Dedekind complete (see also [53, p. 156 and Theorem VI.2.2, p. 157]). Furthermore, observe that l^∞ has the Egorov property (see [39, p. 465]), and hence, by [11, Theorem 3.1], l^∞ is weakly σ-distributive. Thus, order and (D)-convergences coincide, but they are different from (r)-convergence.

3   Limit theorems.

In this section we prove some versions of the Schur lemma, Vitali-Hahn-Saks and Nikodým convergence theorem with respect to (D)-convergence. Similar versions in the context of relatively uniform convergence were proved in [17, Theorems 3.4 and 3.5]. Note that in our context only the pointwise convergence of the involved measures, and not σ-additivity or absolute continuity, is intended with respect to a same regulator, while in [12, 13] all concepts are formulated relatively to a common (D)-sequence.

We begin with stating the Schur lemma, the Vitali-Hahn-Saks theorem and Nikodým convergence theorem.

Theorem 3.1. (Schur lemma) *Let $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of equibounded σ-additive measures, and assume that there exists a set function $m_0 : \mathcal{P}(\mathbb{N}) \to R$ such that $(RD)\lim_j m_j = m_0$. Then*

$$(D) \lim_j \Big(\sum_{n=1}^\infty |m_j(\{n\}) - m_0(\{n\})| \Big) = 0.$$

Moreover m_0 is σ-additive, $(U)\lim_j m_j = m_0$ and the m_j's are uniformly σ-additive.

Theorem 3.2. (Vitali-Hahn-Saks theorem) *Let G be any infinite set, $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra, $\nu : \mathcal{A} \to [0, +\infty]$ be a finitely additive set function,*

$m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, be a sequence of equibounded ν-absolutely continuous finitely additive set functions. Assume that there exists $m_0 : \mathcal{A} \to R$ with $(RD) \lim_j m_j = m_0$. Then m_0 is ν-absolutely continuous and the m_j's are uniformly ν-absolutely continuous.

Theorem 3.3. (Nikodým convergence theorem) *Let G, $\mathcal{A}$ be as in Theorem 3.2, assume that $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, is a sequence of equibounded σ-additive measures, and suppose that there is a set function $m_0 : \mathcal{A} \to R$ such that $(RD) \lim_j m_j = m_0$. Then m_0 is σ-additive and the m_j's are uniformly σ-additive.*

In order to prove Theorems 3.1, 3.2 and 3.3, let us introduce some preliminary definitions and results.

Definitions 3.4. We denote by $l^1(R)$ the set of all sequences $(a_j)_j$ of R, such that $\sum_{j=1}^{\infty} |a_j|$ exists in R.

It is not difficult to check that $l^1(R)$, endowed with the coordinatewise ordering, is an (l)-group.

A sequence $(a^{(n)})_n$ of elements of $l^1(R)$, where $a^{(n)} = (a_j^{(n)})_j$, $n \in \mathbb{N}$, is said to be *convergent in $l^1(R)$* to $a = (a_j)_j \in l^1(R)$ if

$$(D) \lim_n \left(\sum_{j=1}^{\infty} |a_j^{(n)} - a_j| \right) = 0. \tag{8}$$

We say that the sequence $(a^{(n)})_n$ is *Cauchy in $l^1(R)$* if

$$(D) \lim_n \left(\sum_{j=1}^{\infty} |a_j^{(n)} - a_j^{(n+p)}| \right) = 0 \tag{9}$$

uniformly with respect to $p \in \mathbb{N}$.

The set $l^1(R)$ satisfies the following completeness condition:

Proposition 3.5. *Every Cauchy in $l^1(R)$ sequence of elements of $l^1(R)$ is convergent in $l^1(R)$.*

PROOF. The proof is similar to [17, Proposition 2.3] and is a consequence of the fact that, in weakly σ-distributive Dedekind complete (l)-groups, order convergence of sequences coincides with (D)-convergence. □

The following lemma will be useful in the sequel ([17, Lemma 3.2]).

Lemma 3.6. *Let $m : \mathcal{P}(\mathbb{N}) \to R$ be a finitely additive bounded set function and $\mathcal{X}$ be a finite subset of $\mathbb{N}$. Then the following inequality holds:*

$$\sum_{l \in \mathcal{X}} |m(\{l\})| \leq 2 \bigvee_{A \subset \mathcal{X}} |m(A)|.$$

In order to prove our version of the Schur lemma, we will use the following technical result.

Lemma 3.7. *Let $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive equibounded measures, and $m_0 : \mathcal{P}(\mathbb{N}) \to R$ be a set function, with the property that $(RD)\lim_j m_j = m_0$. Then there exists a regulator $(c_{i,l})_{i,l}$ in R such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and for all sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for every $s \in \mathbb{N}$ there is $\overline{s} \in \mathbb{N}$ such that*

$$|m_{j_s}(A) - m_{j_s + p_s}(A)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \tag{10}$$

for any $s \geq \overline{s}$ and for all $A \subset \mathbb{N}$.

Proof. First of all, note that for all positive sequences $(a_i)_i$ and $(b_i)_i$ in R the following equations hold:

$$2 \bigvee_{i=1}^{\infty} a_i = \bigvee_{i=1}^{\infty} 2\, a_i; \tag{11}$$

$$\bigvee_{i=1}^{\infty} a_i + \bigvee_{j=1}^{\infty} b_j \leq \bigvee_{i=1}^{\infty} 2(a_i + b_i). \tag{12}$$

Observe that (12) is an easy consequence of (11).

Let now u be as in (2), and for each $j \in \mathbb{N}$ let $(a_{i,l}^{(j)})_{i,l}$ be a (D)-sequence related with the necessary and sufficient condition for σ-additivity of m_j given in Proposition 2.10. For every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$, let us define $\xi_j : \mathbb{N} \to \mathbb{N}$ as follows: $\xi_j(n) = \varphi(n + j)$, for all $n \in \mathbb{N}$. So, for every $j \in \mathbb{N}$, the regulator $(a_{i,l}^{(j)})_{i,l}$ is such that in correspondence with ξ_j there exists $\overline{n} \in \mathbb{N}$ with

$$|m_j(A)| \leq \bigvee_{i=1}^{\infty} a_{i,\xi_j(i)}^{(j)} = \bigvee_{i=1}^{\infty} a_{i,\varphi(i+j)}^{(j)} \tag{13}$$

for all $A \subset \{\overline{n}, \overline{n}+1, \overline{n}+2, \ldots\}$. By (2) and (13) it follows that

$$|m_j(A)| \leq u \bigwedge \left(\sum_{j=1}^{q} \bigvee_{i=1}^{\infty} a_{i,\varphi(i+j)}^{(j)} \right) \tag{14}$$

for all $A \subset \{\overline{n}, \overline{n}+1, \overline{n}+2, \ldots\}$ and for each $q \in \mathbb{N}$.

By virtue of (14) and Lemma 2.4 there exists a (D)-sequence $(a_{i,l})_{i,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is $\overline{n} \in \mathbb{N}$ with

$$|m_j(A)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{15}$$

for each $A \subset \{\overline{n}, \overline{n}+1, \overline{n}+2, \ldots\}$, that is

$$\|m_j\|_{\mathcal{P}(\mathbb{N})}(\{\overline{n}, \overline{n}+1, \overline{n}+2, \ldots\}) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{16}$$

Let $(b_{i,l})_{i,l}$ satisfy the condition of (RD)-convergence of the m_j's to m_0. For each $\varphi \in \mathbb{N}^{\mathbb{N}}$, $n \in \mathbb{N}$ and for all sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for every s, there is $s_0 \in \mathbb{N}$ with

$$\bigvee_{A \subset \{1, \ldots, n\}} |\nu_s(A)| \leq 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \tag{17}$$

for all $s \geq s_0$, where $\nu_s = m_{j_s} - m_{j_s + p_s}$. From (17) and Lemma 3.6 it follows that

$$|\nu_s(\{1\})| + \ldots + |\nu_s(\{n\})| \leq 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \tag{18}$$

for all $s \geq s_0$. Set now

$$c_{i,l} = 8\, a_{i,l} + 12\, b_{i,l} \tag{19}$$

for all $i, l \in \mathbb{N}$. We prove that the (D)-sequence $(c_{i,l})_{i,l}$ defined in (19) satisfies condition (10). Otherwise there are an element $\varphi \in \mathbb{N}^{\mathbb{N}}$ and two sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$, with $j_s \geq s$ for each $s \in \mathbb{N}$, and such that for all $s \in \mathbb{N}$ there is a set $A_s \subset \mathbb{N}$ with

$$|\nu_s(A_s)| \not\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \tag{20}$$

Arguing analogously as in (13-16), since the ν_s's are equibounded and satisfy condition (4) of Proposition 2.10 with respect to the regulator $(2a_{i,l})_{i,l}$ and by virtue of Lemma 2.4, in correspondence with the function φ satisfying (20) and ν_1 it is possible to find a natural number n_1 such that

$$\|\nu_1\|_{\mathcal{P}(\mathbb{N})}(\{n_1+1, n_1+2, \ldots\}) \leq 2 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{21}$$

By (RD)-convergence of the ν_j's to 0 with respect to the regulator $(2\,b_{i,l})_{i,l}$, which is an easy consequence of (RD)-convergence of $(m_j)_j$ to m_0 with respect to the (D)-sequence $(b_{i,l})_{i,l}$, proceeding analogously as in (17) and (18), in correspondence with n_1 there exists $s_1 \in \mathbb{N}$ such that

$$|\nu_s(\{1\})| + \ldots + |\nu_s(\{n_1\})| \le 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \tag{22}$$

for all $s \ge s_1$. Proceeding similarly as above, it is possible to associate to ν_{s_1} a natural number $n_2 > n_1$ such that

$$\|\nu_{s_1}\|_{\mathcal{P}(\mathbb{N})}(\{n_2 + 1, n_2 + 2, \ldots\}) \le 2 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}, \tag{23}$$

and to find $s_2 > s_1$ such that

$$|\nu_s(\{1\})| + \ldots + |\nu_s(\{n_2\})| \le 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \tag{24}$$

for all $s \ge s_2$. Proceeding by induction, we get the existence of two strictly increasing sequences $(n_h)_h$ and $(s_h)_h$ in $\mathbb{N}$ such that for all $h \in \mathbb{N}$ we have:

$$\|\nu_{s_h}\|_{\mathcal{P}(\mathbb{N})}(\{n_h + 1, n_h + 2, \ldots\}) \le 2 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{25}$$

and

$$|\nu_s(\{1\})| + \ldots + |\nu_s(\{n_h\})| \le 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \tag{26}$$

for every $s \ge s_h$. Set now $s_0 = 1$, $n_0 = 0$ and

$$A = \bigcup_{h \in \mathbb{N} \bigcup \{0\}} (A_{s_h} \cap \{n_h + 1, \ldots, n_{h+1}\}).$$

Notice that $A \cap \{1, \ldots, n_1\} = A_1 \cap \{1, \ldots, n_1\}$. From this and taking into account the finite additivity of ν_1 we have:

$$\nu_1(A) = \nu_1(A_1 \cap \{1, \ldots, n_1\}) + \nu_1(A \cap \{n_1 + 1, n_1 + 2, \ldots\}),$$

$$\nu_1(A_1) = \nu_1(A_1 \cap \{1, \ldots, n_1\}) + \nu_1(A_1 \cap \{n_1 + 1, n_1 + 2, \ldots\}).$$

Observe now that $A \cap \{n_h + 1, \ldots, n_{h+1}\} = A_{s_h} \cap \{n_h + 1, \ldots, n_{h+1}\}$ for all $h \in \mathbb{N}$. From this and the finite additivity of ν_{s_h} we get:

$$
\begin{aligned}
\nu_{s_h}(A) &= \nu_{s_h}(A \cap \{1, \ldots, n_h\}) + \nu_{s_h}(A_{s_h} \cap \{n_h + 1, \ldots, n_{h+1}\}) \\
&+ \nu_{s_h}(A \cap \{n_{h+1} + 1, n_{h+1} + 2, \ldots\}),
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\nu_{s_h}(A_{s_h}) &= \nu_{s_h}(A_{s_h} \cap \{1, \ldots, n_h\}) + \nu_{s_h}(A_{s_h} \cap \{n_h + 1, \ldots, n_{h+1}\}) \\
&+ \nu_{s_h}(A_{s_h} \cap \{n_{h+1} + 1, n_{h+1} + 2, \ldots\})
\end{aligned} \tag{28}
$$

for every $h \in \mathbb{N}$. From (25), (26), (27) and (28), for all $h \in \mathbb{N}$ we obtain:

$$
|\nu_{s_h}(A) - \nu_{s_h}(A_{s_h})| \leq 4 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + 4 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}. \tag{29}
$$

By (RD)-convergence of the ν_j's to 0 with respect to the regulator $(2\,b_{i,l})_{i,l}$, in correspondence with A there exists an integer h_0 such that for every $h \geq h_0$ we get

$$
|\nu_{s_h}(A)| \leq \bigvee_{i=1}^{\infty} 2b_{i,\varphi(i)}. \tag{30}
$$

From (29) and (30), taking into account (11) and (12), for all $h \geq h_0$ we have:

$$
\begin{aligned}
|\nu_{s_h}(A_{s_h})| &\leq |\nu_{s_h}(A)| + |\nu_{s_h}(A) - \nu_{s_h}(A_{s_h})| \\
&\leq 8 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + 12 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} = \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.
\end{aligned}
$$

This is a contradiction with (20). Thus the lemma is completely proved. $\square$

We are now in position to prove our Schur lemma.

Proof. Theorem 3.1: First of all we know that, thanks to Lemma 3.7, there exists a (D)-sequence $(c_{i,l})_{i,l}$ such that, for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and for all sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for any s, there is $\overline{s} \in \mathbb{N}$ with

$$
|m_{j_s}(A) - m_{j_s + p_s}(A)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \tag{31}
$$

for any $s \geq \overline{s}$ and $A \subset \mathbb{N}$. From (31) and Lemma 3.6 applied to the set functions $m_{j_s} - m_{j_s+p_s}$, $s \geq \overline{s}$, it follows that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$, for all sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for each s, there exists $\overline{s} \in \mathbb{N}$ such that

$$\sum_{n=1}^{q} |m_{j_s}(\{n\}) - m_{j_s+p_s}(\{n\})|$$

$$\leq \; 2 \bigvee_{S \in \mathcal{S}_{1,q}} |m_{j_s}(S) - m_{j_s+p_s}(S)| \leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \qquad (32)$$

for every $s \geq \overline{s}$ and $q \in \mathbb{N}$ (here $\mathcal{S}_{1,q}$ is the set of all subsets of $\{1, \ldots, q\}$ for all $q \in \mathbb{N}$). Taking in (32) the supremum as q varies in $\mathbb{N}$, we obtain that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, for all sequences of natural numbers $(j_s)_s$, $(p_s)_s$ with $j_s \geq s$ for every s, there is $\overline{s} \in \mathbb{N}$ such that, whenever $s \geq \overline{s}$,

$$\sum_{n=1}^{\infty} |m_{j_s}(\{n\}) - m_{j_s+p_s}(\{n\})| \leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \qquad (33)$$

From (33) it follows that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\overline{j}$ with the property that, for each $j \geq \overline{j}$ and $p \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} |m_j(\{n\}) - m_{j+p}(\{n\})| \leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \qquad (34)$$

Otherwise, there is $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that for any $s \in \mathbb{N}$ there are j_s, $p_s \in \mathbb{N}$ with $j_s \geq s$ and such that

$$\sum_{n=1}^{\infty} |m_{j_s}(\{n\}) - m_{j_s+p_s}(\{n\})| \not\leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)},$$

that is there exist $\varphi \in \mathbb{N}^{\mathbb{N}}$ and two sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for each $s \in \mathbb{N}$ and such that

$$\sum_{n=1}^{\infty} |m_{j_s}(\{n\}) - m_{j_s+p_s}(\{n\})| \not\leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

whenever $s \in \mathbb{N}$. This contradicts (33), and thus (34) is proved.

Set now, for all j, $n \in \mathbb{N}$, $a_n^{(j)} = m_j(\{n\})$, and for every $j \in \mathbb{N}$, put $a^{(j)} = (a_n^{(j)})_n$.

From (34) we get that the sequence $(a^{(j)})_j$ is Cauchy in $l^1(R)$. By Proposition 3.5, $(a^{(j)})_j$ is convergent in $l^1(R)$, and thus there exist an element

$a \in l^1(R)$, $a = (a_n^{(0)})_n$, and a (D)-sequence $(\beta_{i,l})_{i,l}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $\bar{j} \in \mathbb{N}$ with

$$\sum_{n=1}^{\infty} |m_j(\{n\}) - a_n^{(0)}| \leq \bigvee_{l=1}^{\infty} \beta_{i,\varphi(i)} \tag{35}$$

for every $j \geq \bar{j}$. Note that convergence of $(a^{(j)})_j$ in $l^1(R)$ implies (D)-convergence of $(m_j(\{n\}))_j$ to $a_n^{(0)}$ for all $n \in \mathbb{N}$, and this limit is unique, thanks to weak σ-distributivity of R. Thus we get $a_n^{(0)} = m_0(\{n\})$ for every $n \in \mathbb{N}$.

We now prove that m_0 is σ-additive. For all $A \subset \mathbb{N}$ set

$$m^*(A) = (D)\lim_q \Big(\sum_{n \in A, n=1,\ldots,q} m_0(\{n\}) \Big) = \sum_{n \in A} m_0(\{n\}). \tag{36}$$

By Proposition 2.3 the limit in (36) exists in R uniformly with respect to $A \subset \mathbb{N}$. This also implies that

$$\sum_{n \in A, n \geq q} m_0(\{n\}) = (D)\lim_l \Big(\sum_{n \in A, n=q,q+1,\ldots,q+l} m_0(\{n\}) \Big)$$

exists in R and that

$$(D)\lim_q \sum_{n \in A, n \geq q} m_0(\{n\}) = 0 \tag{37}$$

uniformly with respect to $A \subset \mathbb{N}$.

We now claim that

$$m^*(A) = m_0(A), \quad \text{for all } A \subset \mathbb{N}. \tag{38}$$

Indeed, we have:

$$m_0(A) - m^*(A) = (D)\lim_j [m_j(A) - m^*(A)] \tag{39}$$

$$= (D)\lim_j \Big[(D)\lim_q \Big(\sum_{n \in A, n=1,\ldots,q} (m_j(\{n\}) - m_0(\{n\})) \Big) \Big]$$

for each $A \subset \mathbb{N}$; moreover from (35) it follows that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ a natural

number $\bar{j}$ can be associated, with the property that, for all $j \geq \bar{j}$ and $A \subset \mathbb{N}$,

$$
\begin{aligned}
& \left| (D) \lim_q \Big(\sum_{n \in A, n=1,\ldots,q} (m_j(\{n\}) - m_0(\{n\})) \Big) \right| \\
\leq \quad & \sum_{n \in A} |m_j(\{n\}) - m_0(\{n\})| \qquad\qquad\qquad (40) \\
\leq \quad & \sum_{n=1}^{\infty} |m_j(\{n\}) - m_0(\{n\})| \leq \bigvee_{i=1}^{\infty} \beta_{i,\varphi(i)}.
\end{aligned}
$$

From (39), (40) and weak σ-distributivity of R we get:

$$
0 \leq |m_0(A) - m^*(A)| \leq \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{i=1}^{\infty} \beta_{i,\varphi(i)} \Big) = 0
$$

for all $A \subset \mathbb{N}$; thus we get (38).

From (36), (37) and (38) we have, uniformly with respect to $A \subset \mathbb{N}$:

$$
\begin{aligned}
(D) \lim_q |m^*(A \cap \{q, q+1, \ldots\})| \;&=\; (D) \lim_q \Big| \sum_{n \in A, n \geq q} m^*(\{n\}) \Big| \qquad (41) \\
&=\; (D) \lim_q \Big| \sum_{n \in A, n \geq q} m_0(\{n\}) \Big| = 0.
\end{aligned}
$$

From (41) it follows easily that $(D) \lim_q \bigvee(|m^*(B)| : B \subset \{q, q+1, \ldots\}) = 0$, namely

$$
(D) \lim_q \|m^*\|_{\mathcal{P}(\mathbb{N})}(\{q, q+1, \ldots\}) = 0. \qquad (42)
$$

From (42) and Proposition 2.10 it follows that m^* is σ-additive on $\mathcal{P}(\mathbb{N})$, and hence m_0 is σ-additive too, since m_0 coincides with m^*.

From σ-additivity of m_0 and (40) it follows that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $\bar{j} \in \mathbb{N}$ such that, for any $j \geq \bar{j}$ and $A \subset \mathbb{N}$, we have

$$
\begin{aligned}
& |m_j(A) - m_0(A)| \\
=\quad & \left| (D) \lim_q \Big(\sum_{n \in A, n=1,\ldots,q} (m_j(\{n\}) - m_0(\{n\})) \Big) \right| \leq \bigvee_{i=1}^{\infty} \beta_{i,\varphi(i)},
\end{aligned}
$$

and hence

$$
(U) \lim_j m_j = m_0. \qquad (43)
$$

Uniform σ-additivity of the m_j's is a consequence of (43) and σ-additivity of m_0. Indeed, by virtue of (43), there exists a (D)-sequence $(h_{i,l})_{i,l}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ a positive integer $\bar{j}$ can be associated, with

$$|m_j(A) - m_0(A)| \leq \bigvee_{i=1}^{\infty} h_{i,\varphi(i)} \tag{44}$$

whenever $j \geq \bar{j}$ and $A \subset \mathbb{N}$. Moreover, by virtue of σ-additivity of m_0 and the m_j's, their equiboundedness and Lemma 2.4, arguing analogously as in (13-15), there exists a (D)-sequence $(f_{i,l})_{i,l}$ such that, for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N} \bigcup \{0\}$, there exists $\bar{n} = \bar{n}(\varphi, j) \in \mathbb{N}$ with

$$|m_j(A)| \leq \bigvee_{i=1}^{\infty} f_{i,\varphi(i)}, \quad \text{for all } A \subset \{\bar{n}, \bar{n}+1, \bar{n}+2, \ldots\}. \tag{45}$$

Fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$, and let $\bar{j}$ be as in (44). In correspondence with φ and $j = 0, 1, \ldots, \bar{j}-1$, there exist $\bar{n}_0, \bar{n}_1, \ldots, \bar{n}_{\bar{j}-1}$ as in (45). Set $n^* = \max(\bar{n}_0, \bar{n}_1, \ldots, \bar{n}_{\bar{j}-1})$: we have

$$|m_j(A)| \leq \bigvee_{i=1}^{\infty} f_{i,\varphi(i)}, \quad \text{for all } A \subset \{n^*, n^*+1, n^*+2, \ldots\}. \tag{46}$$

Moreover, for every $j \geq \bar{j}$ and $A \subset \{n^*, n^*+1, n^*+2, \ldots\}$, we get

$$|m_j(A)| \leq |m_j(A) - m_0(A)| + |m_0(A)| \leq \bigvee_{i=1}^{\infty} h_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} f_{i,\varphi(i)}. \tag{47}$$

Uniform σ-additivity of the m_j's follows from (46) and (47). $\qquad\square$

The next step is to prove our version of the Vitali-Hahn-Saks-type theorem. In order to do it, we first formulate the following technical lemma.

Lemma 3.8. *Let R be a Dedekind complete weakly σ-distributive (l)-group, G be any infinite set, $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra, $m : \mathcal{A} \to R$ be a finitely additive set function. Let $(E_n)_n$ be any decreasing sequence of $\mathcal{A}$, put $F = \bigcap_{n=1}^{\infty} E_n$ and suppose that $m(F) = 0$. Set $B_n = E_n \setminus E_{n+1}$ for all $n \in \mathbb{N}$, and denote by $\mathcal{K}$ and $\mathcal{L}$ the σ-algebras generated by the B_n's in E_1 and by the E_n's in E_1 respectively. Then for all $n \in \mathbb{N}$ we get*

$$\|m\|_{\mathcal{L}}(E_n) = \|m\|_{\mathcal{K}}\left(\bigcup_{l=n}^{\infty} B_l\right). \tag{48}$$

Proof. First of all notice that $\mathcal{L} = \mathcal{K} \cup \{X \cup F : X \in \mathcal{K}\}$. For all $n \in \mathbb{N}$ let

$$\mathcal{K}_n := \{X \in \mathcal{K} : X \subset \bigcup_{l=n}^{\infty} B_l\}.$$

The following equalities hold for all $n \in \mathbb{N}$:

$$v_{\mathcal{L}}(m)(E_n) = v_{\mathcal{L}}(m)\Big(F \cup \Big(\bigcup_{l=n}^{\infty} B_l\Big)\Big) = \bigvee_{X \in \mathcal{K}_n} [|m(X)| \vee |m(X \cup F)|]$$

$$= \bigvee_{X \in \mathcal{K}_n} |m(X)| \vee |m(X) + m(F))|, \quad \text{because } m \text{ is finitely additive}$$

$$= \bigvee_{X \in \mathcal{K}_n} |m(X)|, \quad \text{as } m(F) = 0 \text{ by hypothesis}$$

$$= v_{\mathcal{K}}(m)\Big(\bigcup_{l=n}^{\infty} B_l\Big).$$

This ends the proof. $\square$

Finally we are ready to give our version of the Vitali-Hahn-Saks theorem for ν-absolutely continuous set functions (not necessarily with respect to a common regulator, differently than in [12]).

Proof. Theorem 3.2: Let $(H_n)_n$ be any decreasing sequence of elements of $\mathcal{A}$ such that $\lim_n \nu(H_n) = 0$, set $B_n = H_n \setminus H_{n+1}$ for every $n \in \mathbb{N}$, and let $\mathcal{L}$, $\mathcal{K}$ be the σ-algebras generated by the H_n's and by the B_n's in H_1 respectively. Put $F = \bigcap_{n=1}^{\infty} H_n$.

Choose arbitrarily $j \in \mathbb{N}$. Since m_j is ν-absolutely continuous, it follows that

$$\bigwedge_n \|m_j\|_{\mathcal{L}}(H_n) = \bigwedge_n \bigvee \Big(|m_j(C)| : C \in \mathcal{L} \text{ with } C \subset H_n\Big) = 0. \tag{49}$$

Since $0 \leq \|m_j\|_{\mathcal{L}}(F) \leq \|m_j\|_{\mathcal{L}}(H_n)$ for every $n \in \mathbb{N}$, from (49) it follows that $\|m_j\|_{\mathcal{L}}(F) = 0$, and a fortiori $m_j(F) = 0$.

Now, for every $A \subset \mathbb{N}$, set $\mu_j(A) = m_j\Big(\bigcup_{n \in A} B_n\Big)$. We claim that

$$\|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\})$$

$$= \bigvee \Big(|\mu_j(B)| : B \subset \{n, n+1, n+2 \ldots\}\Big) \tag{50}$$

$$= \bigvee \Big(|\mu_j(C)| : C \in \mathcal{L} \text{ with } C \subset H_n\Big) = \|m_j\|_{\mathcal{L}}(H_n)$$

for all $n \in \mathbb{N}$. Indeed, in order to prove that

$$\|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) \leq \|m_j\|_{\mathcal{L}}(H_n),$$

it is enough to associate to every $B \subset \{n, n+1, n+2, \ldots\}$ the set $C = \bigcup_{n \in B} B_n$, which is contained in H_n. Conversely, observe that by Lemma 3.8 we have

$$\|m_j\|_{\mathcal{L}}(H_n) = \|m_j\|_{\mathcal{K}}\Big(\bigcup_{l=n}^{\infty} B_l\Big) \tag{51}$$

for every $n \in \mathbb{N}$.

If $C \in \mathcal{K}$ and $C \subset \bigcup_{l=n}^{\infty} B_l$, then there is $B \subset \{n, n+1, n+2, \ldots, \}$ such that $C = \bigcup_{n \in B} B_n$. The equality (50) follows from this and (51). From (49) and (50) we obtain

$$\bigwedge_n \|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = 0. \tag{52}$$

By virtue of (52) and Proposition 2.10 we get that μ_j is a σ-additive measure on $\mathcal{P}(\mathbb{N})$.

Now, for each $A \subset \mathbb{N}$, set $\mu_0(A) = m_0\Big(\bigcup_{n \in A} B_n\Big)$. The equiboundedness of the μ_j's and (RD)-convergence of the μ_j's to μ_0 follow easily from the equiboundedness of the m_j's and (RD)-convergence of the m_j's to m_0 respectively. By the Schur lemma 3.1, the measures μ_j, $j \in \mathbb{N}$, are uniformly σ-additive on $\mathcal{P}(\mathbb{N})$. By Remark 2.11 we get

$$\bigwedge_n \bigvee_j \|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = 0. \tag{53}$$

From (50) and (53) we obtain that $\bigwedge_n \bigvee_j \|m_j\|_{\mathcal{L}}(H_n) = 0$ for all $j \in \mathbb{N}$, and thus the set functions m_j, $j \in \mathbb{N}$, are uniformly ν-absolutely continuous. From this it follows easily that m_0 is ν-absolutely continuous too. This ends the proof. $\qquad\square$

The proof of our version of the Nikodým convergence theorem is similar to the one of the Vitali-Hahn-Saks theorem and uses the Schur Lemma 3.1 too.

PROOF. Theorem 3.3: Let $(H_n)_n$ be any disjoint sequence of elements of $\mathcal{A}$, and for every $n \in \mathbb{N}$ put $E_n = \bigcup_{l=n}^{\infty} H_l$. Let $\mathcal{L}$ be the σ-algebra generated by the H_n's in the set $\bigcup_{n=1}^{\infty} H_n$.

Choose arbitrarily $j \in \mathbb{N}$. Since m_j is σ-additive, we have

$$\bigwedge_n \|m_j\|_{\mathcal{L}}(E_n) = \bigwedge_n \bigvee \Big(|m_j(C)| : C \in \mathcal{L} \text{ with } C \subset E_n \Big) = 0. \qquad (54)$$

Now, for every $A \subset \mathbb{N}$, set $\mu_j(A) = m_j \Big(\bigcup_{n \in A} H_n \Big)$. By arguing similarly as in (50), it follows that

$$\|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = \|m_j\|_{\mathcal{L}}(E_n) \qquad (55)$$

for all $n \in \mathbb{N}$. From (54) and (55) we obtain

$$\bigwedge_n \|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = 0. \qquad (56)$$

By virtue of (56) and Proposition 2.10 we obtain that μ_j is a σ-additive measure on $\mathcal{P}(\mathbb{N})$.

For each $A \subset \mathbb{N}$, put $\mu_0(A) = m_0 \Big(\bigcup_{n \in A} H_n \Big)$. The equiboundedness of the μ_j's and (RD)-convergence of the μ_j's to μ_0 are easy consequences of the equiboundedness of m_j's and (RD)-convergence of the m_j's to m_0 respectively. By the Schur lemma 3.1, the measures μ_j, $j \in \mathbb{N}$, are uniformly σ-additive and μ_0 is σ-additive. Now, proceeding analogously as in the final part of the proof of Theorem 3.2, we get that $\bigwedge_n \bigvee_j \|m_j\|_{\mathcal{L}}(E_n) = 0$ for all $j \in \mathbb{N}$. Thus the measures m_j, $j \in \mathbb{N}$, are uniformly σ-additive and m_0 is σ-additive too. $\quad\square$

Finally, we give the following result, which is a consequence of Lemma 3.8 and connects uniform σ-additivity and uniform absolute continuity. Observe that every ν-absolutely continuous finitely additive set function m defined on $\mathcal{A}$ has the property that $m(S) = 0$ whenever $S \in \mathcal{A}$ and $\nu(S) = 0$.

Theorem 3.9. *Let R, G and $\mathcal{A}$ be as in Lemma 3.8, $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, be a sequence of uniformly σ-additive measures. Let $\nu : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ be a finitely additive set function. If $\nu(S) = 0$ implies $m_j(S) = 0$ for each $j \in \mathbb{N}$ and each $S \in \mathcal{A}$, then the m_j's are uniformly ν-absolutely continuous.*

Proof. Let $(E_n)_n$ be any decreasing sequence of elements of $\mathcal{A}$ such that $\lim_n \nu(E_n) = 0$. Put $F = \bigcap_{n=1}^{\infty} E_n$. As $\nu(F) \leq \nu(E_n)$ for all $n \in \mathbb{N}$, it follows that $\nu(F) = 0$. Therefore $m_j(F) = 0$ for all $j \in \mathbb{N}$.

Put $B_n = E_n \setminus E_{n+1}$ for all $n \in \mathbb{N}$. Denote by $\mathcal{K}$ and $\mathcal{L}$ the σ-algebras generated by the B_n's in E_1 and by the E_n's in E_1 respectively. By Lemma

3.8 applied to m_j, for every j and $n \in \mathbb{N}$ we get:

$$\|m_j\|_{\mathcal{L}}(E_n) = \|m_j\|_{\mathcal{K}}\Big(\bigcup_{l=n}^{\infty} B_l\Big). \tag{57}$$

The m_j's are uniformly σ-additive, thus

$$(D)\lim_n \bigvee_j \|m_j\|_{\mathcal{K}}\Big(\bigcup_{l=n}^{\infty} B_l\Big) = \bigwedge_n \bigvee_j \|m_j\|_{\mathcal{K}}\Big(\bigcup_{l=n}^{\infty} B_l\Big) = 0.$$

Hence $(D)\lim_n \bigvee_j \|m_j\|_{\mathcal{L}}(E_n) = \bigwedge_n \bigvee_j \|m_j\|_{\mathcal{L}}(E_n) = 0$, therefore the m_j's are uniformly ν-absolutely continuous. □

Open problem: Find some necessary and/or sufficient conditions for which the semivariation of (l)-group-valued measures introduced in the paper is equal to the one with respect to the σ-algebra where the measures are defined.

References

[1] P. Antosik and C. Swartz, *The Nikodým boundedness theorem and the uniform boundedness principle,* (Measure theory and its applications, Sherbrooke, Que., 1982), 36–42. Lecture Notes in Math., **1033**, Springer, Berlin, 1983.

[2] P. Antosik and C. Swartz, *Matrix Methods in Analysis,* Lecture Notes in Math., **1113**, Springer, Berlin-Heidelberg, 1985.

[3] P. Antosik and C. Swartz, *The Nikodým convergence theorem for lattice-valued measures,* Rev. Roumaine Math. Pures Appl., **37** (1992), 299–306.

[4] A. Avallone, S. Rinauro and P. Vitolo, *Boundedness and convergence theorems in effect algebras*, Tatra Mt. Math. Publ., **35** (2007), 159–174.

[5] C. Bardaro and I. Mantellini, *Approximation properties in abstract modular spaces for a class of general sampling-type operators,* Appl. Anal., **85(4)** (2006), 383–413.

[6] S. J. Bernau, *Unique representation of Archimedean lattice group and normal Archimedean lattice rings,* Proc. London Math. Soc., **15** (1965), 599–631.

[7] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges,* Academic Press Inc., London, 1983.

[8] G. Birkhoff, *Lattice theory,* Amer. Math. Soc., Providence, Rhode Island, 1967.

[9] A. Boccuto, *Vitali-Hahn-Saks and Nikodým theorems for means with values in Riesz spaces,* Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, **44** (1996), 157–173.

[10] A. Boccuto, *Integration in Riesz spaces with respect to (D)-convergence,* Tatra Mt. Math. Publ., **10** (1997), 33–54.

[11] A. Boccuto, *Egorov property and weak σ-distributivity in Riesz spaces,* Acta Math. (Nitra), **6** (2003), 61–66.

[12] A. Boccuto and D. Candeloro, *Uniform (s)-boundedness and convergence results for measures with values in complete (l)-groups,* J. Math. Anal. Appl., **265** (2002), 170–194.

[13] A. Boccuto and D. Candeloro, *Convergence and decompositions for (l)-group-valued set functions,* Comment. Math. Prace Mat., **44** (2004), 11–37.

[14] A. Boccuto and D. Candeloro, *Sobczyk-Hammer decompositions and convergence theorems for measures with values in (l)-groups,* Real Anal. Exchange, **33** (2008), 91–106.

[15] A. Boccuto, X. Dimitriou and N. Papanastassiou, *Basic matrix theorems for $\mathcal{I}$-convergence in Riesz spaces,* Math. Slovaca, (2012), to appear.

[16] A. Boccuto, X. Dimitriou and N. Papanastassiou, *Countably additive restrictions and limit theorems in (l)-groups,* Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, **57** (2010), 121–134; *Addendum to: "Countably additive restrictions and limit theorems in (l)-groups",* ibidem, (2012), to appear.

[17] A. Boccuto and N. Papanastassiou, *Schur and Nikodým convergence-type theorems in Riesz spaces with respect to the (r)-convergence,* Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, **55** (2007), 33–46.

[18] A. Boccuto, B. Riečan and M. Vrábelová, *Kurzweil-Henstock Integral in Riesz Spaces,* Bentham Science Publ., 2009.

[19] J. K. Brooks, *On the Vitali-Hahn-Saks and Nikodým theorems,* Proc. Natl. Acad. Sci. USA, **64** (1969), 468–471.

[20] J. K. Brooks, *Equicontinuous sets of measures and applications to Vitali's integral convergence theorem and control measures,* Adv. Math., **10** (1973), 165–171.

[21] J. K. Brooks and J. Mikusiński, *On some theorems in Functional Analysis,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **18(3)** (1970), 151–155.

[22] J. K. Brooks and R. S. Jewett, *On finitely additive vector measures,* Proc. Natl. Acad. Sci. USA, **67** (1970), 1294–1298.

[23] F. Cafiero, *Sulle famiglie di funzioni additive d'insieme uniformemente continue,* Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **12(8)** (1952), 155–162.

[24] D. Candeloro, *Sui teoremi di Vitali-Hahn-Saks, Dieudonné e Nikodým,* Rend. Circ. Mat. Palermo (2), **8** (1985), 439–445.

[25] J. R. Choksi, *Vitali's convergence theorem on term by term integration,* Enseign. Math., **47** (2001), 269–285.

[26] P. de Lucia and E. Pap, *Convergence Theorems for Set Functions,* in: E. Pap(Ed.), Handbook on Measure Theory, vol. I, North-Holland, Amsterdam (2002), 125–178.

[27] J. Diestel, *Applications of weak compactness and bases to vector measures and vectorial integration,* Rev. Roumaine Math. Pures Appl., **18** (1973), 211–224.

[28] J. Diestel, *On the essential uniqueness of an example of C. E. Rickart,* Comment Math. Prace Mat., **17** (1973), 263–264.

[29] J. Diestel and B. Faires, *On vector measures*, Trans. Amer. Math. Soc., **198** (1974), 253–271.

[30] L. Drewnowski, *Equivalence of Brooks - Jewett, Vitali - Hahn - Saks and Nikodým theorems,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **20** (1972), 725–731.

[31] Y. Dubrovsky, *On some properties of completely additive set functions*, Izv. Ross. Akad. Nauk Ser. Mat., **9** (1945), 311–320.

[32] M. Duchoň and B. Riečan, *On the Kurzweil-Stieltjes integral in ordered spaces,* Tatra Mt. Math. Publ., **8** (1996), 133–142.

[33] O. Duman, M. A. Őzarslan and E. Erkuş-Duman, *Rates of Ideal Convergence for Approximation Operators,* Mediterr. J. Math., **7** (2010), 111–121.

[34] B. Faires, *On Vital-Hahn-Saks-Nikodým type-theorems,* Ann. Inst. Fourier (Grenoble), **26** (1976), 99–114.

[35] E. E. Floyd, *Boolean algebras with pathological order properties,* Pacific J. Math., **5** (1955), 687–689.

[36] D. H. Fremlin, *A direct proof of the Matthes-Wright integral extension theorem,* J. London Math. Soc., **11(2)** (1975), 276–284.

[37] H. Hahn, *Űber Folgen linearer Operationen,* Monatsh. Math., **32** (1922), 3–88.

[38] R. B. Holmes, *Mathematical Foundations of Signal Processing,* SIAM Rev., **21(3)** (1979), 361–388.

[39] W. A. J. Luxemburg and A. C. Zaanen, Riesz Spaces, I, North-Holland Publishing Co., Amsterdam, 1971.

[40] J. Mikusiński, *A theorem on vector matrices and its applications in measure theory and functional analysis,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **18** (1970), 193–196.

[41] M. Nakamura, *Notes on Banach Spaces X: Vitali-Hahn-Saks' theorem and K-spaces*, Tôhoku Math. J. (2), **1** (1949), 100-108.

[42] O. M. Nikodým, *Sur les suites convergentes de fonctions parfaitement additives d'ensamble abstrait,* Monatsh. Math., **40** (1933), 427–432.

[43] D. Popa, *On some classical theorems in measure theory,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **40** (4) (1992), 255–263.

[44] C. E. Rickart, *Decompositions of additive set functions,* Duke Math. J., **10** (1943), 653–665.

[45] B. Riečan and T. Neubrunn, *Integral, Measure and Ordering*, Kluwer Acad. Publ./Ister Science, Dordrecht/Bratislava, 1997.

[46] B. Riečan and P. Volauf, *On a technical lemma in lattice ordered groups,* Acta Math. Univ. Comenian., **44-45** (1984), 31–35.

[47] S. Saks, *On some functionals,* Trans. Amer. Math. Soc., **35** (2) (1933), 549–556; *Addition to the note: On some functionals,* Trans. Amer. Math. Soc., **35** (4) (1933), 965–970.

[48] W. Schachermayer, *On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras,* Dissertationes Math. (Rozprawy Mat.), **214** (1982), 1–33.

[49] J. Schur, *Über linearer Transformationen in der Theorie der unendlichen Reichen,* J. Reine Angew. Math., **151** (1920), 79–111.

[50] C. Swartz, *The Nikodým boundedness Theorem for lattice-valued measures,* Arch. Math., **53** (1989), 390–393.

[51] C. Swartz, *The uniform boundedness principle for order bounded operators,* Int. J. Math. & Math. Sci., **12** (3) (1989), 487–492.

[52] G. Vitali, *Sull'integrazione per serie,* Rend. Circ. Mat. Palermo, **23** (1907), 137–155.

[53] B. Z. Vulikh, Introduction to the theory of partially ordered spaces, Wolters-Noordhoff Sci. Publ., Groningen, 1967.

[54] J. D. M. Wright, *The measure extension problem for vector lattices,* Ann. Inst. Fourier (Grenoble), **21** (1971), 65–85.