INROADS

Real Analysis Exchange
Vol. 21(1), 1995–96, pp. 363–364

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AN EXTENSION OF FATOU'S LEMMA

Abstract

We give an extension of Fatou's lemma to tempered distributions $\mathcal{S}'(G)$, where G is a homogeneous group.

A *homogeneous group* G is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_r : r > 0\}$. Here, dilations $\{\delta_r\}$ on algebra $\mathfrak{g}$ mean a family of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r = \exp(A \log r)$, where A is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues. The number $Q \equiv \text{trace}(A)$ will be called the *homogeneous dimension* of G. If G is a homogeneous group, the maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphisms of G. We shall denote them also by δ_r and call them dilations on G, and write rx instead of $\delta_r x$ for $r > 0, x \in G$. Sometimes we even write x/r for $\delta_{1/r}x$. Under this notation, the distributive law becomes $r(xy) = (rx)(ry)$. For more details about homogeneous groups, we refer the reader to [1].

The classic Fatou's lemma is written as: *If $\{f_n\}$ is a sequence of nonnegative measurable functions on $E \subseteq \mathbb{R}^n$, then*

$$\int_E \liminf_{n\to\infty} f_n(x)\,dx \le \liminf_{n\to\infty} \int_E f_n(x)\,dx.$$

This result still holds for homogeneous groups. In this note, furthermore, we will extend the sequence of measurable functions to another class, tempered distributions.

Let G be a homogeneous group with homogeneous dimension Q. The Hardy space $H^p(G)$ is defined either in terms of maximal functions or in terms of atomic decompositions (*cf.* [1]). Namely, let ϕ be a function in

Key Words: Fatou's lemma, Hardy spaces, homogeneous groups
Mathematical Reviews subject classification: Primary: 42B30
Received by the editors May 30, 1995

$\mathcal{S}(G)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_G \phi(x)\,dx = 1$. Define

$$\phi_r(x) = r^{-Q}\phi(x/r)$$

and the maximal function f^* by

$$f^*(x) = \sup_{r>0} |f * \phi_r(x)|,$$

where the convolution operator is given by

$$f * g(x) = \int_G f(xy^{-1})g(y)\,dy = \int_G f(y)g(y^{-1}x)\,dy.$$

We say a tempered distribution $f \in \mathcal{S}'(G)$ is in $H^p(G)$ if f^* is in $L^p(G)$. The quasi-norm on H^p is $\|f\|_{H^p}^p \equiv \|f^*\|_{L^p}^p$, which satisfies $\|f+g\|_{H^p}^p \le \|f\|_{H^p}^p + \|g\|_{H^p}^p$ for $0 < p \le 1$. When $p > 1$, H^p and L^p are essentially the same because of the celebrated theorem of Hardy and Littlewood $\|f^*\|_{L^p} \le C_p\|f\|_{L^p}$; however, when $p \le 1$ the space H^p is much better adapted to problems arising in the theory of harmonic analysis. We now are ready to give an extension of Fatou's lemma to $H^p(G)$.

Theorem 1 *If f_n $(n = 1, 2, 3, \cdots), f \in \mathcal{S}'(G)$ and $\{f_n\}$ converges to f in $\mathcal{S}'(G)$, then $\|f\|_{H^p} \le \liminf_{n\to\infty} \|f_n\|_{H^p}$.*

PROOF. Since the bilinear map $(f, \phi_r) \mapsto f * \phi_r$ is separately continuous from $\mathcal{S}' \times \mathcal{S}$ into C^∞ (*cf.* [1, page 38]), for each $r > 0$,

$$|f * \phi_r(x)| = \lim_{n\to\infty} |f_n * \phi_r(x)| \le \liminf_{n\to\infty} \sup_{r>0} |f_n * \phi_r(x)| = \liminf_{n\to\infty} f_n^*(x).$$

The above inequality holds for all $r > 0$, so we have the pointwise inequality

$$f^*(x) = \sup_{r>0} |f * \phi_r(x)| \le \liminf_{n\to\infty} f_n^*(x).$$

We apply the classic Fatou's lemma to get

$$\int_G f^*(x)^p\,dx \le \int_G \liminf_{n\to\infty} f_n^*(x)^p\,dx \le \liminf_{n\to\infty} \int_G f_n^*(x)^p\,dx$$

which means $\|f\|_{H^p}^p = \|f^*\|_{L^p}^p \le \liminf_{n\to\infty} \|f_n^*\|_{L^p}^p = \liminf_{n\to\infty} \|f_n\|_{H^p}^p$. □

References

[1] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Mathematical Notes, **28** (1982), Princeton Univ. Press, Princeton, NJ.