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ON FUBINI-TYPE THEOREMS

Abstract

We discuss some questions concerning the strengthened version of the Kuratowski-Ulam theorem obtained by Ceder. In particular, we refute Ceder's conjecture that the measure analogue of his result holds. Further we consider mixed product σ-ideals $\mathbb{K}\times\mathbb{L}$ and $\mathbb{L}\times\mathbb{K}$ in $\mathbb{R}^2$ where $\mathbb{K}$ and $\mathbb{L}$ denote the families of meager and of Lebesgue null sets in $\mathbb{R}$. For a set $A \in \mathbb{K}\times\mathbb{L}$ (or $A \in \mathbb{L}\times\mathbb{K}$) we find large sets P and Q such that $P\times Q$ misses A. The proof is based on similar properties of $\mathbb{K}\times\mathbb{K}$ and $\mathbb{L}\times\mathbb{L}$ obtained by Ceder, Brodskiĭ and Eggleston. A parametrized version of a Fubini-type theorem is also given.

Ceder [C, Th. 12] presents the following strengthened form of the Kuratowski-Ulam theorem (see [O]). (For $A \subseteq X \times Y$, $A_x = \{y : \langle x, y\rangle \in A\}$ and $A^y = \{x : \langle x, y\rangle \in A\}$.)

Theorem 1 *If $A \subseteq \mathbb{R}^2$ is a meager set, then there exists a residual G_δ set $B \subseteq \mathbb{R}$ such that for each set $P \subseteq B$ of type F_σ the set $\bigcup_{x\in P} A_x$ is meager.*

Ceder asks whether the F_σ requirement here is essential. The following example shows that it is.

Example 1. Let A be the graph of the projection from $\mathbb{R}^2$ to $\mathbb{R}$, i.e., $A = \{\langle\langle y, z\rangle, y\rangle \in \mathbb{R}^2 \times \mathbb{R} : y, z \in \mathbb{R}\}$. Then A is meager in $\mathbb{R}^2\times\mathbb{R}$. Given a residual

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G_δ set $B \subseteq \mathbb{R}^2$ we can find a meager G_δ set $P \subseteq B$ such that $\bigcup_{\langle y,z\rangle \in P} A_{\langle y,z\rangle}$ is residual. Indeed, since B is residual in $\mathbb{R}^2$ there exists $z \in \mathbb{R}$ with B^z residual. Let $P = B^z \times \{z\}$. Clearly $P \subseteq B$ and P is a meager G_δ subset of $\mathbb{R}^2$. However

$$\bigcup_{\langle y,t\rangle \in P} A_{\langle y,t\rangle} = \bigcup_{y \in B^z} A_{\langle y,z\rangle} = B^z$$

is residual in $\mathbb{R}$.

To get an example in $\mathbb{R} \times \mathbb{R}$ rather than in $\mathbb{R}^2 \times \mathbb{R}$ proceed as follows. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Borel isomorphism such that for all $X \subseteq \mathbb{R}$, X is meager in $\mathbb{R}$ iff $f^{-1}[X]$ is meager in $\mathbb{R}^2$. There exist residual G_δ sets $G \subseteq \mathbb{R}^2$ and $H \subseteq \mathbb{R}$ such that f restricted to G is a homeomorphism of G and H. Let $A = \{\langle f(y,z), y\rangle : y, z \in \mathbb{R}\}$. Then A is a Borel subset of $\mathbb{R}^2$ such that for all $x \in \mathbb{R}$, A_x is a singleton. Hence A is meager. Given a residual G_δ set $B \subseteq \mathbb{R}$, $C = f^{-1}[B \cap H]$ is a residual G_δ subset of $\mathbb{R}^2$, so there exists $z \in \mathbb{R}$ with C^z residual in $\mathbb{R}$. The set $C^z \times \{z\}$ is a meager G_δ subset of $\mathbb{R}^2$, so $P = f[C^z \times \{z\}]$ is a meager G_δ subset of B. However

$$\bigcup_{x \in P} A_x = \bigcup_{y \in C^z} A_{f(y,z)} = C^z$$

is residual in $\mathbb{R}$.

The next example shows that we cannot claim the set B in Theorem 1 to be open, even for A closed with all vertical sections null (hence also nowhere dense).

Example 2. Talagrand [T] shows that for every perfect set $E \subseteq \mathbb{R}$ and every interval I there is a perfect null set $P \subseteq I$ such that $E+P$ contains an interval. (Here $E + P = \{x + y : x \in E, y \in P\}$.)

Now, fix a perfect null set $E \subseteq \mathbb{R}$. Obviously, it is nowhere dense. Let $A = \{\langle x, y\rangle : y - x \in E\}$. Then A is nowhere dense null since it is closed and for each $x \in \mathbb{R}$ the section $A_x = E + x$ is nowhere dense null. In any fixed interval I we can choose a perfect null set $P \subseteq I$ such that $E + P$ contains an interval. We have

$$\bigcup_{x \in P} A_x = \bigcup_{x \in P} (E + x) = E + P.$$

Hence $\bigcup_{x \in P} A_x$ contains an interval.

In view of the well known duality between measure and category, Ceder conjectured the following.

Conjecture 1 ([C, p. 284]) *If $A \subseteq \mathbb{R}^2$ is a null set, then there exists an F_σ set $B \subseteq \mathbb{R}$ of full measure such that for each null set $P \subseteq B$ of type G_δ, the set $\bigcup_{x\in P} A_x$ is null.*

Our next example refutes this conjecture.

Example 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Borel isomorphism such that for all $X \subseteq \mathbb{R}$, X is null in $\mathbb{R}$ iff $f^{-1}[X]$ is null in $\mathbb{R}^2$. Let $A = \{\langle f(y,z), y\rangle : y, z \in \mathbb{R}\}$. Then A is a Borel subset of $\mathbb{R}^2$ such that for all $x \in \mathbb{R}$, A_x is a singleton. Hence A is null. Given a set $B \subseteq \mathbb{R}$ of full measure, $C = f^{-1}[B]$ has full measure in $\mathbb{R}^2$, so there exists $z \in \mathbb{R}$ such that C^z has full measure in $\mathbb{R}$. Let $D \subseteq C^z$ be compact of positive measure such that $f(\cdot, z)$ is continuous on D. The set $D \times \{z\}$ is null in $\mathbb{R}^2$, so $P = f[D \times \{z\}]$ is compact and null in $\mathbb{R}$. However

$$\bigcup_{x\in P} A_x = \bigcup_{y\in D} A_{f(y,z)} = D$$

has positive measure in $\mathbb{R}$.

Theorem 1 yields the following folklore fact.

Theorem 2 *If $A \subseteq \mathbb{R}^2$ is meager, then there exist a c-dense F_σ set $P \subseteq \mathbb{R}$ and a residual G_δ set $Q \subseteq \mathbb{R}$ such that $P \times Q$ pmisses A.*

The measure analogue is a theorem of Brodskiĭ [Br] and Eggleston [E].

Theorem 3 *If $A \subseteq \mathbb{R}^2$ is a null set, then there exist a c-dense F_σ set $P \subseteq \mathbb{R}$ and an F_σ set $Q \subseteq \mathbb{R}$ of full measure such that $P \times Q$ pmisses A.*

Remarks.

(1) Theorems 1 and 2 are of course true for a wide class of spaces, in particular they are true if $\mathbb{R}^2$ is replaced by $T \times \mathbb{R}$, where T is a perfect subset of $\mathbb{R}$ considered with the relative topology.

(2) The Lebesgue measure can be defined for the classical Cantor set C (e.g., via the Cantor function from C onto $[0,1]$ which is one-to-one when the end-points of connected components of $[0,1] \setminus C$ are ignored). Similarly, for an arbitrary perfect set $T \subseteq \mathbb{R}$, we can define a natural "Lebesgue measure" on T. Then Theorem 3 holds when $\mathbb{R}^2$ is replaced by $T \times \mathbb{R}$ with the product measure.

Theorems 2 and 3 are also true for mixed product ideals in $\mathbb{R}^2$. Let $\mathbb{K}$ be the σ-ideal of meager subsets of $\mathbb{R}$, $\mathbb{L}$ — the σ-ideal of Lebesgue null subsets

of $\mathbb{R}$.
For $\mathcal{I}, \mathcal{J} \in \{\mathbb{K}, \mathbb{L}\}$, let

$$\mathcal{I} \times \mathcal{J} = \{E \subseteq \mathbb{R}^2 : \big(\exists \text{ Borel } B \subseteq \mathbb{R}^2\big)(E \subseteq B \ \& \ \{x \in \mathbb{R} : B_x \notin \mathcal{J}\} \in \mathcal{I})\}$$

The family $\mathcal{I} \times \mathcal{J}$ is called the Fubini product of ideals $\mathcal{I}$ and $\mathcal{J}$. By the Fubini theorem and the Kuratowski-Ulam theorem (see [O]), $\mathbb{L} \times \mathbb{L}$ and $\mathbb{K} \times \mathbb{K}$ are exactly the σ-ideals of Lebesgue null and of meager sets in $\mathbb{R}^2$. The families $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ also form σ-ideals. They were studied in [Me, G, CP, F, BH]. Note that there is no inclusion between any two of the families $\mathbb{K} \times \mathbb{K}$, $\mathbb{L} \times \mathbb{L}$, $\mathbb{K} \times \mathbb{L}$, $\mathbb{L} \times \mathbb{K}$ (cf. [Me]).

Theorem 4 *If $A \subseteq \mathbb{R}^2$ is in $\mathbb{K} \times \mathbb{L}$, then there exist a c-dense F_σ set $P \subseteq \mathbb{R}$ and an F_σ set $Q \subseteq \mathbb{R}$ of full measure such that $P \times Q$ misses A.*

Proof. Without loss of generality we can assume that A is Borel. Let $D \subseteq \mathbb{R}$ be a meager set such that for all $x \in \mathbb{R} \setminus D$, A_x is null. Fix a countable base $\{U_n\}_{n=1}^{\infty}$ of nonempty open sets in $\mathbb{R}$ and choose a sequence $\{T_n\}_{n=1}^{\infty}$ of pairwise disjoint nowhere dense perfect sets such that $T_n \subseteq U_n \setminus D$ for every n. Now, for every n, apply the version of Theorem 3 discribed in Remark (2) to the set $A_n = A \cap (T_n \times \mathbb{R})$ in the space $T_n \times \mathbb{R}$ with the appriopriate product measure. Then we can find a perfect set $P_n \subseteq T_n$ and a set $Q_n \subseteq \mathbb{R}$ of full measure such that $P_n \times Q_n$ misses A_n. Put $P = \bigcup_n P_n$ and $Q = \bigcap_n Q_n$. □

Theorem 5 *If $A \subseteq \mathbb{R}^2$ is in $\mathbb{L} \times \mathbb{K}$, then there exist a c-dense F_σ set $P \subseteq \mathbb{R}$ and a residual G_δ set $Q \subseteq \mathbb{R}$ such that $pP \times Q$ misses A.*

The proof is analogous to that of Theorem 4.

Finally, we give a parametrized Fubini-type theorem for $\mathbb{K} \times \mathbb{L}$, $\mathbb{L} \times \mathbb{K}$, $\mathbb{L} \times \mathbb{L}$ and $\mathbb{K} \times \mathbb{K}$ (see [Mi] for similar versions of other known theorems).

Theorem 6 *Let $\mathcal{I}, \mathcal{J} \in \{\mathbb{K}, \mathbb{L}\}$. Assume that $E \subseteq \mathbb{R}^3$ is a Borel set and $T \subseteq \mathbb{R}$ is a perfect set such that $E_t \in \mathcal{I} \times \mathcal{J}$ for each $t \in T$. Then there exists a perfect set $P \subseteq T$ and a set $H \in \mathcal{I}$ such that $E_{\langle t,x \rangle} \in \mathcal{J}$ for each $\langle t, x \rangle \in P \times (\mathbb{R} \setminus H)$.*

Proof. The set

$$A = \{\langle t, x \rangle \in T \times \mathbb{R} : E_{\langle t,x \rangle} \notin \mathcal{J}\}$$

is Borel (see e.g. [G, Th. 2.1, 2.2]) and $A_t = \{x \in \mathbb{R} : E_{\langle t,x \rangle} \notin \mathcal{J}\} \in \mathcal{I}$ for each $t \in T$. By Theorems 2 and 3 (where $T \times \mathbb{R}$ is considered instead of $\mathbb{R}^2$), we find a perfect set $P \subseteq T$ and a set $H \in \mathcal{I}$ such that $P \times (\mathbb{R} \setminus H)$ misses A. Then for each $\langle t, x \rangle \in P \times (\mathbb{R} \setminus H)$ we have $E_{\langle t,x \rangle} \in \mathcal{J}$. □

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