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B. László and J. T. Tóth[*], Department of Mathematics, University of Education, Farská 19, 949 74 Nitra, Slovakia, e-mail: toth@@unitra.sk

RELATIVELY (R)-DENSE UNIVERSAL SEQUENCES FOR CERTAIN CLASSES OF FUNCTIONS

Abstract

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $(a_n)_{n=1}^{\infty}$ be a sequence of positive reals. We will say that $(a_n)_{n=1}^{\infty}$ is relatively (R)-dense for f provided that for every $x, y \in \mathbb{R}^+$ with $f(x) < f(y)$ there exists $n, m \in \mathbb{N}$ such that $f(x) < \frac{f(a_n)}{f(a_m)} < f(y)$. Sufficient conditions are given for a sequence of positive reals to be relatively (R)-dense for certain functions.

Introduction

Denote by $\mathbb{R}^+$ and $\mathbb{N}$ the set of all positive real numbers and the natural numbers, respectively. Let $R(A, B) = \{\frac{a}{b};\ a \in A,\ b \in B\}$ be the ratio set of $A, B \subset \mathbb{R}^+$ and put $R(A) = R(A, A)$ for any $A \subset \mathbb{R}^+$ (cf. [2], [3], [4], [5]). Note here that $R(A, B) \neq R(B, A)$ in general, however $R(A, B)$ is dense in $\mathbb{R}^+$ if and only if $R(B, A)$ is dense in $\mathbb{R}^+$.

Following [2] and [4] we call a set $A = \{a_1, a_2, \dots, a_n, \dots\} \subset \mathbb{R}^+$ (R)-dense, provided $R(A)$ is dense in $\mathbb{R}^+$. Occasionally we will work with a sequence $A = (a_n)_{n=1}^{\infty}$ rather than a set A. Sequences of real numbers that are relatively dense for a function f were introduced and investigated in [1]. A straightforward analogue for (R)-density is as follows: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $(a_n)_{n=1}^{\infty}$ be a sequence of positive reals. We will say that $(a_n)_{n=1}^{\infty}$ is relatively (R)-dense for f provided that for every $x, y \in \mathbb{R}^+$ with $f(x) < f(y)$ there

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exists $n, m \in \mathbb{N}$ such that $f(x) < \frac{f(a_n)}{f(a_m)} < f(y)$ (the choice $f(x) = x$ clearly yields the (R)-density). Evidently, $(a_n)_{n=1}^{\infty}$ is relatively (R)-dense for f, if the sequence $f(A) = (f(a_n))_{n=1}^{\infty}$ is (R)-dense. Further $(a_n)_{n=1}^{\infty}$ will be called a relatively (R)-dense universal sequence for the class of functions $\mathcal{M}$, if $(a_n)_{n=1}^{\infty}$ is relatively (R)-dense for every $f \in \mathcal{M}$. In what follows $A(x)$ will stand for the counting function of the set $A = \{a_1 < a_2 < \cdots < a_n < \dots\} \subset \mathbb{R}^+$ with $\lim_{n\to\infty} a_n = +\infty$, i.e. $A(x) = \sum_{a \le x, a \in A} 1$.

It is the purpose of this paper to study relatively (R)-dense universal sequences for a certain class of increasing functions, thus extending several results of [1], [2], [3], [4] concerning sets $R(A, B)$.

Main Results

Denote by $\mathcal{F}$ the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following properties:

1. $f(x \cdot y) \ge f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}^+$,
2. f is increasing and unbounded on $\mathbb{R}^+$,
3. f is continuous at $x = 1$ and $f(1) = 1$.

We have

Theorem 1 *Suppose that the sequence $(a_n)_{n=1}^{\infty}$ of positive reals contains an unbounded subsequence $(c_n)_{n=1}^{\infty}$ such that*

$$\limsup_{n\to\infty} \frac{c_{n+1}}{c_n} = 1 \tag{1}$$

Then $(a_n)_{n=1}^{\infty}$ is a relatively (R)-dense universal sequence for $\mathcal{F}$.

This theorem is a consequence of the following stronger statement:

Theorem 2 *Suppose that $A = (a_n)_{n=1}^{\infty}$ fulfills the conditions of Theorem 1 and $B = (b_n)_{n=1}^{\infty}$ is an unbounded sequence of positive reals. Suppose that $\lim_{x\to\infty} g(x) = +\infty$ where $g : \mathbb{R}^+ \to \mathbb{R}^+$. Then $R(f(A), g(B))$ is dense in $\mathbb{R}^+$ for each $f \in \mathcal{F}$.*

Proof. According to (1) $\liminf_{n\to\infty} \frac{c_n}{c_{n+1}} = 1$ thus by properties of $\mathcal{F}$ $\liminf_{n\to\infty} f(\frac{c_n}{c_{n+1}}) = 1$, so $\limsup_{n\to\infty} \frac{1}{f(\frac{c_n}{c_{n+1}})} = 1$. Let $0 < a < b$. Then

$$\frac{1}{f(\frac{c_n}{c_{n+1}})} < \frac{b}{a}$$

whenever $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Consequently by the first property of f

$$\frac{f(c_{n+1})}{f(c_n)} < \frac{b}{a} \quad \text{for all} \quad n \geq n_0\,. \tag{2}$$

The sequences $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are unbounded so there exists an $i \in \mathbb{N}$ with $f(c_{n_o})b \leq g(b_i)$ further the number $j = \min\{k \in \mathbb{N};\ k > n_0 \text{ and } f(c_k)b > g(b_i)\}$. Then $j-1 \geq n_0$ and $f(c_j)b > g(b_i) \geq f(c_{j-1}) \cdot b$. Hence from (2) for $n = j-1$ we get that $f(c_{j-1})b > f(c_j)a$, so by the previous inequalities $f(c_j)b > g(b_i) > f(c_j)a$ which yields the density of $R(g(B), f(A))$ thus also of $R(f(A), g(B))$ in $\mathbb{R}^+$ since $c_j \in A$. □

Remark 1 *The preceding theorem generalizes Theorem* 2.1 *in* [1] *stating that* $R(A,B)$ *is dense in* $\mathbb{R}^+$ *for every couple of unbounded sequences* A, B *of positive reals such that* $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

Proof of Theorem 1. Let $f \in \mathcal{F}$. It suffices to choose $b_n = a_n$ $(n = 1, 2, \dots)$ and $g(x) = f(x)$ in Theorem 2. □

Remark 2 *Let* $0 < c \leq 1$ *and* $\alpha \in \mathbb{R} \setminus \{0\}$. *The function* $f(x) = cx^{\alpha}$, $x \in \mathbb{R}^+$ *belongs to* $\mathcal{F}$. *Further* $\mathcal{F}$ *also contains functions which are not continuous or strictly increasing, respectively on* $\mathbb{R}^+$; *e.g.*

$$f(x) = \begin{cases} c, & \text{if } x \in [1, x_0) \\ cx^{\alpha}, & \text{if } x \in \mathbb{R}^+ \setminus [1, x_0) \text{ where } x_0 > 1. \end{cases}$$

It is proved in [4] (Satz 5) that if $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$, $B = \{b_1 < b_2 < \cdots < b_n < \dots\} \subset \mathbb{N}$ and at least one of A, B has positive asymptotic density (i.e. $\lim\limits_{n\to\infty} \frac{A(n)}{n} > 0$ or $\lim\limits_{n\to\infty} \frac{B(n)}{n} > 0$) then $R(A,B)$ is dense in $\mathbb{R}^+$. The pertinent theorem for one set is proved in (3, Theorem 4). The following theorem generalizes these results for positive real numbers. Before stating it introduce the class $\mathcal{F}^*$ of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which:

1. $f(x \cdot y) \geq f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}^+$,
2. $f(x) > 1$ for all $x > 1$.

It is not hard to prove that members of $\mathcal{F}^*$ are increasing and unbounded on $\mathbb{R}^+$, so $\mathcal{F} \not\subset \mathcal{F}^*$ and $\mathcal{F}^* \not\subset \mathcal{F}$ but $\mathcal{F} \cap \mathcal{F}^*$ is an infinite set. We have

Theorem 3 *Suppose that* $B = \{b_1, b_2, \dots, b_n, \dots\} \subset \mathbb{R}^+$ *is unbounded as well as the set* $A = \{a_1 < a_2 < \cdots < a_n < \dots\} \subset \mathbb{R}^+$. *Suppose that*

$$\lim_{x\to\infty} \frac{f(A)(x)}{g(x)} > 0 \tag{3}$$

for some $g \in \mathcal{F}^*$ and a strictly increasing unbounded function $f : \mathbb{R}^+ \to \mathbb{R}^+$. Then $R(f(A), g(B))$ is dense in $\mathbb{R}^+$.

PROOF. Let $0 < a < b$ and $x \in \mathbb{R}^+$. Then by the 1. property of $g \in \mathcal{F}^*$ we get:

$$\frac{f(A)(bx)}{f(A)(ax)} = \frac{\frac{f(A)(bx)}{g(bx)}}{\frac{f(A)(ax)}{g(ax)}} \cdot \frac{g(bx)}{g(ax)} \geq \frac{\frac{f(A)(bx)}{g(bx)}}{\frac{f(A)(ax)}{g(ax)}} \cdot g\left(\frac{b}{a}\right).$$

Then by (3) and the 2nd property of g, $\lim\limits_{x\to\infty} \frac{f(A)(bx)}{f(A)(ax)} \geq g\left(\frac{b}{a}\right) > 1$. Accordingly $\frac{f(A)(bx)}{f(A)(ax)} > 1$ whenever $x \geq x_0$ for some $x_0 \in \mathbb{R}^+$, hence $f(A)(bx) - f(A)(ax) > 0$ thus for all $x \geq x_0$ there exists $i = i(x) \in \mathbb{N}$ such that $ax < f(a_i) < bx$. Now $\lim\limits_{x\to\infty} g(x) = +\infty$ and B is unbounded so $g(b_j) > x_0$ for some $j \in \mathbb{N}$. It means by previous considerations that $g(b_j)a < f(a_i) < g(b_j)b$ for some $i = i(g(b_j)) \in \mathbb{N}$ and the density of $R(f(A), g(B))$ in $\mathbb{R}^+$ follows. □

If X^d denotes the set of all limit points of the set X then we have:

Theorem 4 *If $A = \{a_1 < a_2 < \dots a_n < \dots\} \subset \mathbb{R}^+$ is relatively (R)-dense for some $f \in \mathcal{F}^*$, then $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.*

PROOF. Let $f \in \mathcal{F}^*$ and suppose that $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = c > 1$, where $c \in \mathbb{R}^+$. It follows from properties of f that

$$\liminf_{n\to\infty} f\left(\frac{a_{n+1}}{a_n}\right) \geq f(c) > 1 \quad \text{and} \quad \frac{f(a_{n+1})}{f(a_n)} \geq f\left(\frac{a_{n+1}}{a_n}\right),$$

thus $\liminf\limits_{n\to\infty} \frac{f(a_{n+1})}{f(a_n)} \geq f(c)$. If $c = +\infty$ then put $f(c) = +\infty$.

Now choose $t \in R(f(A))^d$ and $t > 1$. Then $\frac{f(a_{m_k})}{f(a_{n_k})} \to t$ (as $k \to \infty$) for some sequences $(m_k)_{k=1}^\infty$, $(n_k)_{k=1}^\infty$ of natural numbers such that $m_k > n_k$ $(k = 1, 2, \dots)$. Clearly $\frac{f(a_{m_k})}{f(a_{n_k})} \geq \frac{f(a_{n_k+1})}{f(a_{n_k})}$ so

$$t \geq \liminf_{k\to\infty} \frac{f(a_{n_{k+1}})}{f(a_{n_k})} \geq \liminf_{n\to\infty} \frac{f(a_{n+1})}{f(a_n)} \geq f(c).$$

Consequently $R(f(A))^d \cap (1, f(c)) = \emptyset$. Then properties of f easily yield numbers $1 < x < y < c$ such that $1 < f(x) < f(y) < f(c)$ and $(f(x), f(y)) \cap R(f(A)) = \emptyset$, whence A is not relatively (R)-dense for f. □

Remark 3 *Functions $f(x) = x^\alpha$, $x \in \mathbb{R}^+$, $\alpha \in \mathbb{R} \setminus \{0\}$ belong to $\mathcal{F} \cap \mathcal{F}^*$, so these functions satisfy Theorem 1 and Theorem 4.*

An argument similar to that of in Theorem 4 justifies that $t \leq \frac{1}{f(c)}$ if $t \in R(f(A))^d$ and $t < 1$. Hence we have the following generalization of Proposition 3 of [6]:

Corollary 1 *Let $f \in \mathcal{F}^*$ and $A = \{a_1 < a_2 < \dots a_n < \dots\} \subset \mathbb{R}^+$ such that $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = c > 1$. Then $R(f(A))^d \cap (\frac{1}{f(c)}, f(c)) = \emptyset$.*

Remark 4 *If $c = +\infty$ in the preceding Corollary, then $R(f(A))$ consists of isolated points in $\mathbb{R}^+$.*

Remark 5 *It was already mentioned in the introduction that (R)-density of $f(A)$ implies relative (R)-density of $A = (a_n)_{n=1}^{\infty}$ for a function f. It is evident that the reverse implication also holds for surjective $f : \mathbb{R}^+ \to \mathbb{R}^+$.*

In connection with this it would be interesting to characterize functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which $f(A)$ is (R)-dense if and only if the sequence $A = (a_n)_{n=1}^{\infty}$ is relatively (R)-dense for f.

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